Notes – Glacier Variations

I. Net Change of Glacier Length: A Simple Kinematic Approach

Consider a glacier with an x-axis that points along-flow and down-glacier. In its original steady state, the glacier spans from $x = 0$ to $x = L_0$. We assume that the terminus is wedge shaped and that the glacier width is uniform (see Figure 1 for sketch of glacier terminus). Because the glacier is in steady state with its specific balance profile $b_0(x)$, initially no ice passes through $x = L_0$. We will now consider what happens if the specific mass balance changes by amount $b_1(x)$, so that the new specific balance is $b_0(x) + b_1(x)$. To reach a new equilibrium length, the ice flux per unit width at $x = L_0$ must be equal to to the net change in upstream ice flux, or attains

$$\int_0^{L_0} b_1(x) dx = \langle b_1 \rangle L_0,$$

where $\langle b_1 \rangle$ is the spatial average of $b_1(x)$. The terminus must advance by an amount $L_1$ to allow the additional ice to be removed by ablation. We will examine how much a glacier will advance in this scenario.

Figure 1: Schematic of the relation between increased ice flux and advance of the glacier terminus (from Cuffey and Paterson, 2010).

We wish to derive a formula for the amount of glacier advance as a result of this change in mass balance. This problem is a similar mass flux conservation to those we have already investigated in class and in homework. If we set the two flowline (meaning we are not considering width variability in accumulation or ablation here or even that the glacier has geometric width variability) fluxes equal to each other and solve for $L_1$, we have:

$$Q_{in} = Q_{out}$$

$$\langle \dot{b}_1 \rangle L_0 = \dot{a}_0 L_1$$

$$L_1 = \frac{\langle \dot{b}_1 \rangle L_0}{\dot{a}_0}$$
Sample Problem – If $a_0 = 5 \text{ m yr}^{-1}$ and the specific mass balance increases by an amount $\langle b_1 \rangle = 0.5 \text{ m yr}^{-1}$, how far will the glacier advance? Give you answer in terms of the initial length $L_0$.

We simply plug into the above formula:

$$L_1 = \frac{\langle b_1 \rangle L_0}{a_0} = \frac{0.5 \text{ m yr}^{-1}}{5 \text{ m yr}^{-1}} L_0 = 0.1 L_0$$

Glacier Width Variability – You can account for glacier width variability by multiplying the relationship above by $\frac{Y}{Y_t}$, where $Y$ is the average width of the glacier and $Y_t$ is the width of the glacier at the terminus, or:

$$L_1 = L_0 \frac{\langle b_1 \rangle L_0}{a_0} \frac{Y}{Y_t}$$

Looking at this formula, we can gain some intuition for the glacier’s response based on it’s geometry. Especially large terminus length fluctuations are likely to occur if the glacier drains a wide basin into a narrow tongue ($\frac{Y}{Y_t}$ is large so fluctuations compared to $L_0$ are large). If we consider an ice sheet to have an unconstrained circular plan (a circle in map view), then $\frac{Y}{Y_t} = \frac{1}{2}$, so terminus fluctuations are suppressed because the length and width of the ablation zone change simultaneously. Note that in the discussion to this point, there has been no discussion of how long these changes will take to occur, just that there will be change of this magnitude in response to change in mass balance.

II. Equilibrium Response Time: A Kinematic Flux Conservation Approach

Consider the same glacier as in Figure 1. If we consider the entire length of such a glacier, rather than focusing on the terminus, we have the schematic shown in Figure 2. We also now consider a time-varying perturbation to the specific mass balance. The new mass balance is thus: $b(\hat{x}, t) = b_0(\hat{x}) + b_1(\hat{x}, t)$, where $t$ is an arbitrary variable for time. The volume change of the glacier will now depend on the the value of the specific mass balance perturbation in both time and space (here we still neglect width variability in specific mass balance for simplicity). Thus, the change in volume $V$ as a function of time is:

$$\frac{dV}{dt} = \int_0^{L_0} Y(x, t) b_1(\hat{x}, t) dx + \int_{L_0}^{L} Y(x, t) b(\hat{x}, t) dx$$

where $x$ is distance along-flow, $Y(x, t)$ is the spatiotemporally variable width of the glacier, and $\hat{b}$ is the specific mass balance (as defined above). The total volume of the glacier is given by the width $\times$ thickness, or:
\[ V(t) = \int_0^L Y(x,t)H(x,t)dx \]

where \( H(x,t) \) is the mean value of thickness over the a cross-section at position \( x \) and at arbitrary time \( t \). If we consider a glacier on a long sloping surface, except for near the termini, the glacier can be approximated as a parallel-sided slab of thickness \( H_0 \), which we will treat as constant in time for this problem. This situation is pictured schematically in Figure 2a.

Figure 2: Schematic model for advance of a glacier following an increase in mass balance (a) if thickness changes are negligible and (b) allowing for thickness changes (from Cuffey and Paterson, 2010).

In the initial steady state, \( \frac{dV}{dt} = 0 \). If the mass balance increases from \( b_0(x) \) to \( b_0(x) + \dot{b}_1 \), then the length will vary from \( L_0 \) to \( L_0 + L_1(t) \). In this situation we can rewrite, \( \frac{dV}{dt} \) from above as:

\[
\frac{dV}{dt} = \bar{Y}L_0\dot{b}_1 - Y_t\dot{a}_0L_1(t)
\]

As the glacier advances, its volume increases by \( Y_tH_0L_1(t) \). Taking the time derivative of this yields:

\[
\frac{dV}{dt} = Y_tH_0 \frac{dL_1(t)}{dt}
\]

If we equate these two expressions, we can derive a simple mathematical model for glacier response in terms of differential equaiton that has a known analytical solution. So we begin by setting these two volume change with respect to time expressions equal:

\[
\bar{Y}L_0\dot{b}_1 - Y_t\dot{a}_0L_1(t) = Y_tH_0 \frac{dL_1(t)}{dt}
\]

Rearranging to get all the length adjustment terms as a function of time on the same side of the equations gives:

\[
Y_tH_0 \frac{dL_1(t)}{dt} - Y_t\dot{a}_0L_1(t) = \bar{Y}L_0\dot{b}_1
\]
And with some more algebra:

\[
\frac{dL_1(t)}{dt} - \frac{a_0}{H} L_1(t) = \frac{b_1}{Y_t} \frac{L_0}{H}
\]

If we set \( t_r = -H/a_0 \) (this is termed the glacier’s response time), we have:

\[
\frac{dL_1(t)}{dt} + \frac{L_1(t)}{t_r} = \frac{b_1}{Y_t} \frac{L_0}{H}
\]

The solution for this equation for a step change in mass balance is:

\[
L_1(t) = L_0 \frac{b_1}{a_0} \frac{\bar{Y}}{Y_t} [1 - \exp(-t/t_r)]
\]

After time \( t_r \) (the response time) passes following a step change in mass balance, the glacier has advanced by:

\[
L_1(t_r) = L_0 \frac{b_1}{a_0} \frac{\bar{Y}}{Y_t} [1 - \exp(-t_r/t_r)] = 0.63 \times L_0 \frac{b_1}{a_0} \frac{\bar{Y}}{Y_t}
\]

That is the glacier’s response to this mass balance change is 63% complete. At time \( 2t_r \), the glacier’s response would be 86% complete. Thus \( t_r \) is an exponential response time scale for a glacier response to a change in mass balance.

**Sample Problem** – What are typical values of \( t_r \) for a mountain glacier (100 m thickness, 10 m yr\(^{-1}\) ablation rate at the terminus) and for an ice sheet (1500 m thickness, 1 m yr\(^{-1}\) ablation rate at the terminus)?

For a typical mountain glacier (such as one on Mt. Baker or Mt. Rainer):

\[
t_r = \frac{-H}{a_0} = \frac{-100 \text{ m}}{-10 \text{ m yr}^{-1}} = 10 \text{ yrs}
\]

For an ice sheet:

\[
t_r = \frac{-H}{a_0} = \frac{-1500 \text{ m}}{-1 \text{ m yr}^{-1}} = 1500 \text{ yrs}
\]