ESS 524 Class #3

Highlights from last Wednesday – Surabhi
Today's highlights on Wednesday – Erich

- Matlab Basics code is available under MATLAB CODE tab, (not under READING)
- HW #1 (Matlab) due on Wednesday. I have set up a Canvas site where you can turn it in.
- Week 1 journals due today. I have set up a Canvas site.

Today

- 2-minute initial ideas about projects.
- Derivation of Conservation Laws for linear momentum

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Next class we will move on to numerical methods
Finite Difference Method FDM
Finite Element Method FEM
Finite Volume MethodFVM

At start of class, let's talk about the readings

Under READING

- Kwon and Bang Ch 1 on Matlab
- Ed's Notes on Steady 1-D Diffusion with Finite Elements
- Huebner on Finite elements
- Reddy and Gartling on Finite elements
- Versteeg and Malalasekera, Chapter 4
- Patankar, Chapters 3 and 4

Velocity components are (u,v,w)
Force components are (F_x, F_y, F_z)

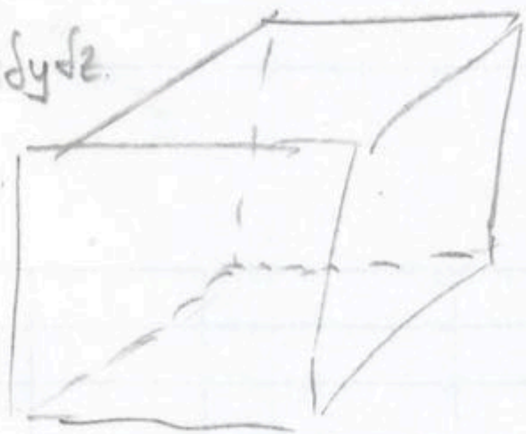
$\vec{F} = m\frac{d\vec{u}}{dt}$

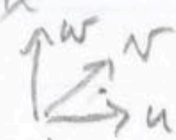
x Momentum

1

→ x component

$F_x = \frac{d}{dt}(\rho u)\,\delta x\,\delta y\,\delta z$





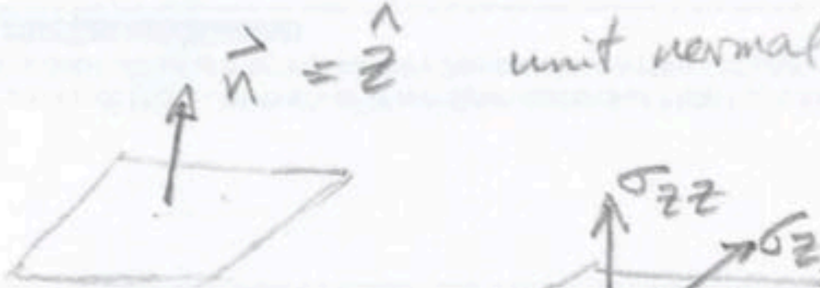
$\vec{F} = m\vec{a}$
$= m\frac{d\vec{v}}{dt}$

Find $\vec{F}$

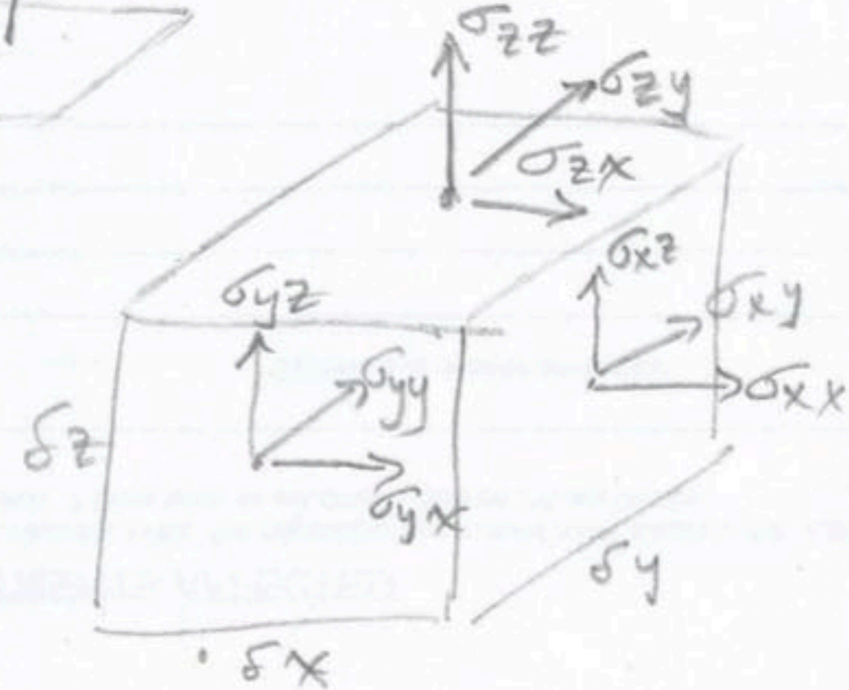
Newton's second law is a material derivative

Find $\vec{F}$

$$\text{Stress} = \frac{\text{force}}{\text{unit area}}$$



σ_{ij} i = normal to plane
j = dir$^{\underline{n}}$ of force



At center point p, σ_{ij}

So on x face at $\frac{\delta x}{2}$

$$X_{x+} = \left[\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x}\frac{\delta x}{2}\right]\delta y \delta z$$

at $-\frac{\delta x}{2}$

$$X_{x-} = -\left\{\left[\sigma_{xx} - \frac{\partial \sigma_{xx}}{\partial x}\frac{\delta x}{2}\right]\delta y \delta z\right\}$$

-ve because it's on -ve face

Body forces $\vec{b}$ $\frac{\text{force}}{\text{unit mass}}$

$\rho b_x \delta x \delta y \delta z$ = net force in x dir^{n}

On y faces at $\pm\frac{\delta y}{2}$

$$X_{y+} = \left(\sigma_{yx} + \frac{\partial \sigma_{yx}}{\partial y}\frac{\delta y}{2}\right)\delta x \delta z$$

$$X_{y-} = -\left\{\left(\sigma_{yx} - \frac{\partial \sigma_{yx}}{\partial y}\frac{\delta y}{2}\right)\delta x \delta z\right\}$$

On z faces

$$X_{z+} = \left(\sigma_{zx} + \frac{\partial \sigma_{zx}}{\partial z}\frac{\delta z}{2}\right)\delta x \delta y$$

$$X_{z-} = -\left\{\left(\sigma_{zx} - \frac{\partial \sigma_{zx}}{\partial z}\frac{\delta z}{2}\right)\delta x \delta y\right\}$$

Newton's second law is a material derivative

Find $m\vec{a}$

Rate of Change of x Momentum

$$\frac{d}{dt}(\rho u)\,dx\,dy\,dz$$

material derivative following mass in volume $dx\,dy\,dz$. (that's how $F = m\vec{a}$ works)

$$= \frac{d}{dt}(\rho u)\,dx\,dy\,dz + \left([(\rho u)u]_{\frac{dx}{2}} - [(\rho u)u]_{-\frac{dx}{2}} \right) dy\,dz$$

x momentum carried across x faces

$$+ \left([(\rho u)v]_{\frac{dy}{2}} - [(\rho u)v]_{-\frac{dy}{2}} \right) dx\,dz$$

across y faces

$$+ \left([(\rho u)w]_{\frac{dz}{2}} - [(\rho u)w]_{-\frac{dz}{2}} \right) dx\,dy$$

across z faces

values at sites by Taylor series around P

$$\ldots (\rho u u) + \frac{\partial}{\partial y}(\rho u v) + \frac{\partial}{\partial z}(\rho u w) \Big] dx\,dy\,dz$$

$$= \frac{\partial}{\partial t}(\rho u)\,dx\,dy\,dz + \left[\frac{\partial}{\partial x}(\rho u u) + \frac{\partial}{\partial y}(\rho u v) + \frac{\partial}{\partial z}(\rho u w)\right] dx\,dy\,dz$$

$$= \left[\rho \frac{\partial u}{\partial t} + u\frac{\partial \rho}{\partial t} + u\frac{\partial(\rho u)}{\partial x} + u\frac{\partial(\rho v)}{\partial y} + u\frac{\partial(\rho w)}{\partial z} + \rho u \frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} + \rho w \frac{\partial u}{\partial z}\right] dx\,dy\,dz$$

$$= \left[\rho \frac{\partial u}{\partial t} + u\frac{\partial \rho}{\partial t} + u\,\nabla\cdot(\rho\vec{u}) + \rho\vec{u}\cdot\nabla u\right] dx\,dy\,dz$$

$$= \left[\rho\left(\frac{\partial u}{\partial t} + \vec{u}\cdot\nabla u\right) + u\underbrace{\left(\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{u})\right)}_{=0 \text{ by mass conservation}}\right] dx\,dy\,dz$$

$$= \rho\frac{du}{dt}\,dx\,dy\,dz$$

Put all together $\sum X = \delta x \delta y \delta z \, \rho \frac{du}{dt}$

$$\left[\underbrace{\frac{\partial \sigma_{xx}}{\partial x}}_{\text{normal stress}} + \underbrace{\frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z}}_{\text{Shear Stress}} + \underbrace{\rho b_x}_{\text{body forces}}\right] = \rho \frac{du}{dt}$$

& Similarly $\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} + \rho b_y = \rho \frac{dv}{dt}$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + \rho b_z = \rho \frac{dw}{dt}$$

or in compact form

$$\frac{\partial \sigma_{ij}}{\partial x_i} + \rho b_j = \rho \frac{d u_j}{dt}$$

summation convention
repeated indices are
Summed $i = x, y, z$

$$\frac{\partial \sigma_{xj}}{\partial x} + \frac{\partial \sigma_{yj}}{\partial y} + \frac{\partial \sigma_{zj}}{\partial z} + \rho b_j = \rho \frac{du_j}{dt}$$

$j = x, y,$ or z
$\Rightarrow$ 3 equations

For a linear viscous material, Stokes Law (Newtonian Fluid)

$\tau_{ij} = \sigma_{ij} + p\delta_{ij}$ $\quad \dot{\epsilon}_{ij}$ = strain rate $= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$

τ_{ij} = deviatoric stress $\quad p = -\sigma_{ii}/3$ -mean stress

pressure negative mean stress

-ve sign because -ve stress is compressive

Summation convention

$$\sigma_{ij} = 2\mu\dot{\epsilon}_{ij} - p\delta_{ij} + \lambda\dot{\epsilon}_{kk}\delta_{ij}$$

μ = shear viscosity

λ = bulk parameter

$k = \lambda + \frac{2}{3}\mu$ bulk viscosity

for incompressible fluid

$\dot{\epsilon}_{kk} = 0$

$$\sigma_{ij} = 2\mu\dot{\epsilon}_{ij} - p\delta_{ij}$$

Kronecker delta = 1 if *i* equals *j*,
=0 if *i* not equal to *j*

In x momentum equation

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + \rho b_x = \rho\left(\frac{du}{dt}\right)$$

$$\frac{\partial}{\partial x}(2\mu\dot{\epsilon}_{xx} - p) + \frac{\partial}{\partial y}(2\mu\dot{\epsilon}_{yx}) + \frac{\partial}{\partial z}(2\mu\dot{\epsilon}_{zx}) + \rho b_x = \rho\frac{du}{dt}$$

$$2\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right) - \frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left(\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right) + \frac{\partial}{\partial z}\left(\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right) + \rho b_x = \rho\frac{du}{dt}$$

$$\rho\frac{du}{dt} = \rho\frac{\partial u}{\partial t} + \rho\vec{u}\cdot\nabla u = \frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(\mu\frac{\partial u}{\partial z}\right)$$

half goes here $\rightarrow$ $+ \frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial z}\left(\mu\frac{\partial w}{\partial x}\right) - \frac{\partial p}{\partial x} + \rho b_x$

Navier-Stokes equations

$$\rho \frac{\partial u}{\partial t} + \rho \vec{u}\cdot\nabla u - \nabla\cdot(\mu \nabla u) - \underbrace{\nabla\cdot\left(\mu \frac{\partial \vec{u}}{\partial x}\right)}_{S_{Mx}} + \frac{\partial p}{\partial x} - \rho b_x = 0 \qquad (1)$$

S_{Mx} as Versteeg calls it – a "Source" term

$$\rho \frac{\partial v}{\partial t} + \rho \vec{u}\cdot\nabla v - \nabla\cdot(\mu \nabla v) - S_{My} + \frac{\partial p}{\partial y} - \rho b_y = 0 \qquad (2)$$

$$\rho \frac{\partial w}{\partial t} + \rho \vec{u}\cdot\nabla w - \nabla\cdot(\mu \nabla w) - S_{Mz} + \frac{\partial p}{\partial z} - \rho b_z = 0 \qquad (3)$$

S_{Mx}, S_{My}, S_{Mz} couple the 3 equations together, as do $\vec{u}\cdot\nabla u$, $\vec{u}\cdot\nabla v$, $\vec{u}\cdot\nabla w$

Pressure p is a 4th variable.

$p(x,y,z)$ is the function that lets (1), (2), (3) satisfy mass conservation when solved for (u, v, w)