ESS 524 Class #4

Highlights from last Monday – Erich
Today's highlights report next Monday – Shashank

- Highlights reporters – please send me your report by email.
- Great Week 1 journals.
- HW #1 (Matlab) due today.

Today
- Discussion points about reading materials?
- Introduction of Finite Difference (FDM) formulas

ESS 524 Class #4

Now we will move on to numerical methods
Finite Difference Method FDM
Finite Element Method FEM
Finite Volume MethodFVM

At start of class, let’s talk about the readings

Under READING

- Kwon and Bang Ch 1 on Matlab
- Ed’s Notes on Steady 1-D Diffusion with Finite Elements
- Huebner on Finite elements
- Reddy and Gartling on Finite elements
- Versteeg and Malalasekera, Chapter 4
- Patankar, Chapters 3 and 4

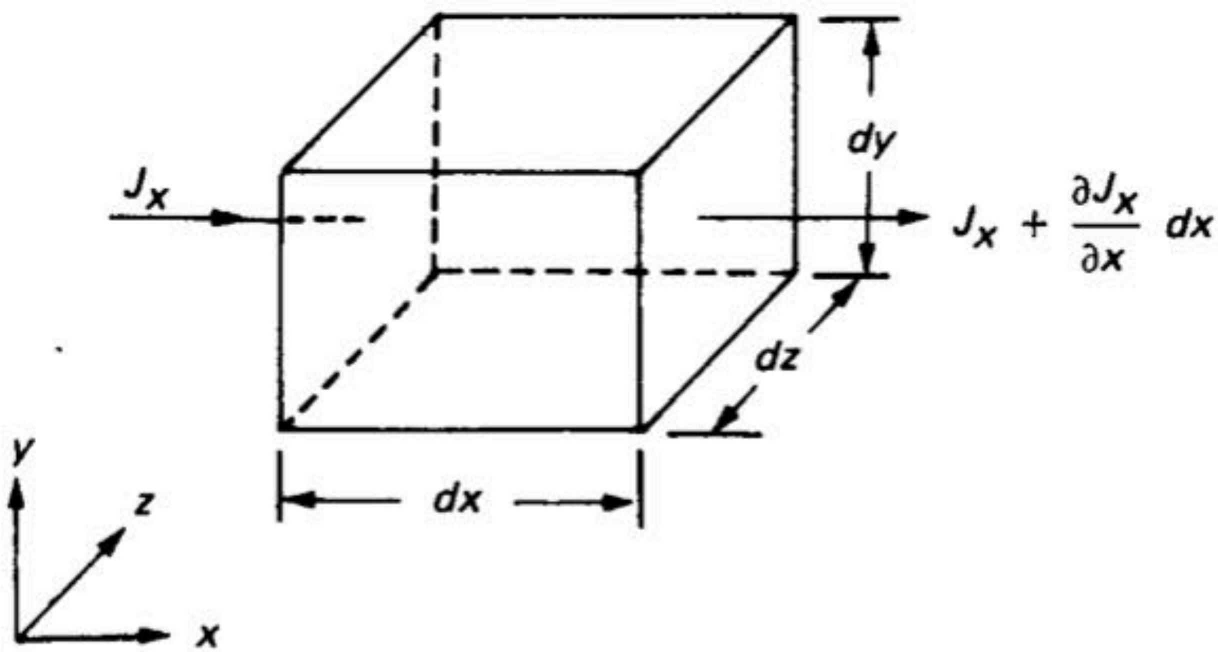


Figure 2.1 Flux balance over a control volume.

The terms in a differential equation of this type denote influences on a *unit-volume* basis. An example will make this clear. Suppose $\mathbf{J}$ denotes a flux influencing a typical dependent variable ϕ. Let us consider the control volume of dimensions dx, dy, dz shown in Fig. 2.1. The flux J_x (which is the x-direction component of $\mathbf{J}$) is shown entering one face of area $dy\, dz$, while the flux leaving the opposite face is shown as $J_x + (\partial J_x/\partial x)\, dx$. Thus, the *net* efflux is $(\partial J_x/\partial x)\, dx\, dy\, dz$ over the area of the face. Considering the contributions of the y and z directions as well and noting that $dx\, dy\, dz$ is the volume of the region considered, we have

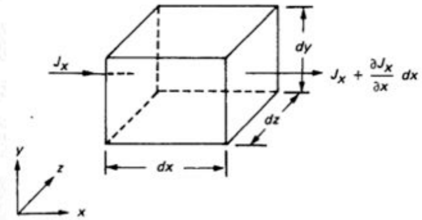


Figure 2.1 Flux balance over a control volume.

$$\text{Net efflux per unit volume} = \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z}$$

$$= \operatorname{div} \mathbf{J}\,. \tag{2.1}$$

This interpretation of div $\mathbf{J}$ will be particularly useful to us because, as we shall see later, our numerical method will be constructed by performing a balance over a control volume.

Another example of a term expressed on a unit-volume basis is the rate-of-change term $\partial(\rho\phi)/\partial t$. If ϕ is a specific property and ρ is the density, then $\rho\phi$ denotes the amount of the corresponding *extensive* property contained in a unit volume. Thus, $\partial(\rho\phi)/\partial t$ is the rate of change of the relevant property per unit volume.

A differential equation is a compilation of such terms, each representing an influence on a unit-volume basis, and all the terms together implying a balance or conservation. We shall now take as examples a few standard differential equations, to find a general form.

2.1-7 The General Differential Equation

This brief journey through some of the relevant differential equations has indicated that all the dependent variables of interest here seem to obey a generalized conservation principle. If the dependent variable is denoted by ϕ, the general differential equation is

$$\frac{\partial}{\partial t}(\rho\phi) + \mathrm{div}\,(\rho \mathbf{u}\phi) = \mathrm{div}\,(\Gamma\ \mathrm{grad}\ \phi) + S\,, \tag{2.13}$$

where Γ is the diffusion coefficient, and S is the source term. The quantities Γ and S are specific to a particular meaning of ϕ. (Indeed, we should have used the symbols Γ_ϕ and S_ϕ; this would, however, lead to too many subscripts in subsequent work.)

If $\phi = 1$, what equation do we get?

If $\phi = e$ (enthalpy), or c_pT, what equation do we get?

Conservation laws in a continuum

- mass, momentum, energy

ρ $\vec{v}$ T.

General equation for conservation laws for specific quantity ϕ

$$\underbrace{\frac{\partial(\rho\phi)}{\partial t}}_{\substack{\text{rate of}\\ \text{change}}} + \nabla\cdot\underbrace{(\rho\vec{u}\phi)}_{\substack{\text{advective}\\ \text{flux}}} = \nabla\cdot\underbrace{(\Gamma\nabla\phi)}_{\substack{\text{diffusive}\\ \text{flux}}} + \underbrace{S}_{\substack{\text{source}\\ \text{unit volume}}} \qquad (1)$$

$\nabla . f$ - divergence - tendency to accumulate or disperse from a unit volume

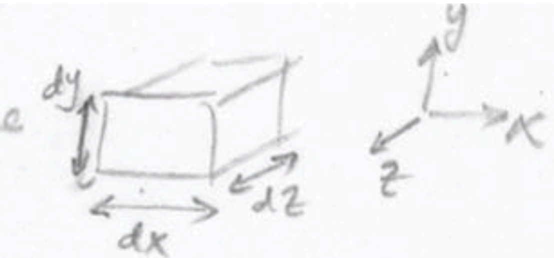
$\nabla . f$ - scalar - $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}$ in 3-D cartesian

∇f - gradient vector (slope) - $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$ "

ρ = density mass/vol

Γ - diffusivity

we get ① by considering a small volume



– rate of change of quantity Q in volume $dx\,dy\,dz$

$$\text{rate change in}\left[\underbrace{\frac{Q}{\text{unit mass}}}_{\phi}\times\underbrace{\frac{\text{mass}}{\text{unit volume}}}_{\rho}\times\underbrace{\text{vol}}_{dx\,dy\,dz}\right] = \frac{\rho(t+\Delta t)\,\phi(t+\Delta t)\,dx\,dy\,dz - \rho(t)\phi(t)\,dx\,dy\,dz}{\Delta t}$$

$$= V\,\frac{\partial(\rho\phi)}{\partial t}.$$

Finite Difference Method FDM

Fourier's Law

$$q = -k \nabla \phi$$

flux

conductivity

temprature

1 Fixed BC

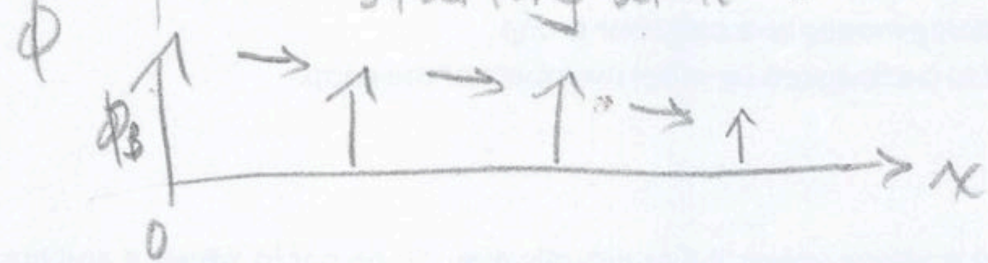
In 1-D $\quad q = -k \frac{d\phi}{dx} \qquad \phi(0) = \phi_B$

We could just solve

$$\frac{d\phi}{dx} = -q_0/k \qquad \phi_0 = \phi_B$$

Starting at $x=0$

$$\phi(\Delta x) = \phi_B - \frac{q_0}{k}\Delta x$$



$$\phi(2\Delta x) = \phi(\Delta x) - \frac{q_0}{k}\Delta x$$

This is just a 1-term Taylor expansion from each point in turn.

Finite Difference Method FDM

We also know that divergence of flux $\vec{q}$ causes a change in ϕ

$$\frac{\partial \phi}{\partial t} = -\nabla \cdot \vec{q}$$

$$\Rightarrow \quad \frac{\partial \phi}{\partial t} = -\nabla \cdot (-\Gamma \nabla \phi)$$

in 1-D: $\dfrac{\partial \phi}{\partial t} = \dfrac{\partial}{\partial x}\left(\Gamma \dfrac{\partial \phi}{\partial x}\right)$

if Γ = constant: $\dfrac{\partial \phi}{\partial t} = \Gamma \dfrac{\partial^2 \phi}{\partial x^2}$

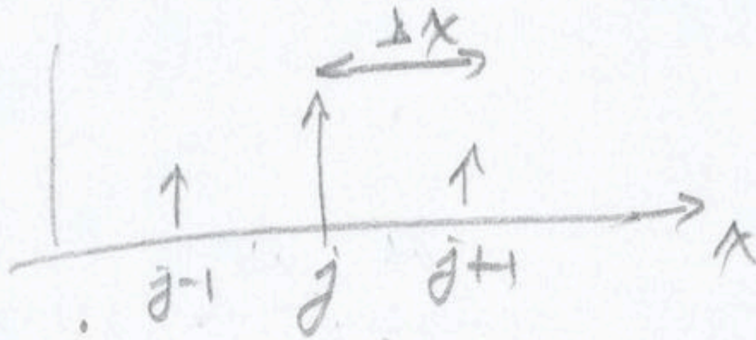
Need 2 BC

$$\phi(0) = \phi_B$$

$$\left.\frac{\partial \phi}{\partial x}\right|_{x_B} = -\frac{q_B}{\Gamma_B}$$

Finite Difference Method FDM

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right)$$



Between j & $j-1$

$$\left.\frac{\partial \phi}{\partial x}\right|_{j-1/2} \sim \frac{\phi_j - \phi_{j-1}}{\Delta x}$$

Between $j+1$ & j

$$\left.\frac{\partial \phi}{\partial x}\right|_{j+1/2} \sim \frac{\phi_{j+1} - \phi_j}{\Delta x}$$

$$\left.\frac{\partial^2 \phi}{\partial x^2}\right|_j \sim \frac{\left.\frac{\partial \phi}{\partial x}\right|_{j+1/2} - \left.\frac{\partial \phi}{\partial x}\right|_{j-1/2}}{\Delta x} = \frac{\phi_{j+1} - \phi_j - \phi_j + \phi_{j-1}}{\Delta x^2}$$

$$= \frac{\phi_{j+1} - 2\phi_j + \phi_{j-1}}{\Delta x^2}$$

Finite Difference Method FDM

In Steady State

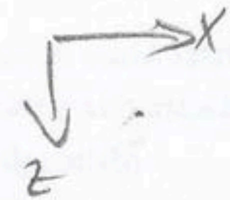
$$\frac{d^2\phi}{dx^2} = 0 \qquad \underline{2\ BC} \qquad \phi(0) = \phi_B \qquad \left.\frac{d\phi}{dx}\right|_{x_B} = -\frac{q_B}{\Gamma_B}$$

At generic point j

$$\frac{1}{\Delta x^2}\left[\phi_{j+1} - 2\phi_j + \phi_{j-1}\right] = 0$$

$$\begin{bmatrix} \cdots & 1 & -2 & 1 & 0 & \cdots & & \\ & \cdots & 1 & -2 & 1 & \cdots & & \\ 0 & 0 & & 1 & -2 & 1 & \cdots & \end{bmatrix} \begin{bmatrix} \phi_{j-1} \\ \phi_j \\ \phi_{j+1} \\ \vdots \end{bmatrix} = \begin{bmatrix} \leftarrow BC \\ 0 \\ \vdots \\ 0 \\ \leftarrow BC \end{bmatrix}$$

Finite Difference Method FDM



$$\frac{d}{dx}\left(K\frac{dT}{dx}\right) + S(x) = 0$$

$$K\frac{d^2T}{dx^2} + \frac{dK}{dx}\frac{dT}{dx} + S(x) = 0 \qquad S(x) = S_c + S_p T_p$$

$$K_j\frac{(T_{j+1} - 2T_j + T_{j-1})}{\Delta x^2} + \left.\frac{dK}{dx}\right|_j \frac{(T_{j+1} - T_{j-1})}{2\Delta x} + S_j = 0$$

$$K_j\left(\frac{T_{j+1}-2T_j+T_{j-1}}{\Delta x^2}\right)+\left.\frac{dK}{dx}\right|_j\left(\frac{T_{j+1}-T_{j-1}}{2\Delta x}\right)+S_j=0$$

$$T_{j-1}\left(\frac{K_j}{\Delta x^2}-\left.\frac{dK}{dx}\right|_j\frac{1}{2\Delta x}\right)+T_j\left(\frac{-2K_j}{\Delta x^2}\right)+T_{j+1}\left(\frac{K_j}{\Delta x^2}+\left.\frac{dK}{dx}\right|_j\frac{1}{2\Delta x}\right)$$

$$=-S_j$$

$$a_P=\left(-\frac{2K_j}{\Delta x^2}\right)\qquad a_W=\left(\frac{K_j}{\Delta x^2}-\left.\frac{dK}{dx}\right|_j\frac{1}{2\Delta x}\right)$$

$$a_E=\left(\frac{K_j}{\Delta x^2}+\left.\frac{dK}{dx}\right|_j\frac{1}{2\Delta x}\right)$$

$$a_U=\left(\frac{K_j}{\Delta z^2}-\left.\frac{dK}{dz}\right|_j\frac{1}{2\Delta z}\right)$$

$$a_D=\left(\frac{K_j}{\Delta z^2}+\left.\frac{dK}{dz}\right|_j\frac{1}{2\Delta z}\right)$$

Fixed T BC, $T_P = T_S$

$a_P = 1 \quad a_E = a_W = a_N = a_D = 0 \qquad RHS = T_S$

Fixed Flux BC

e.g. at East edge:

(fake gridpoint outside mesh.)

$$K_P\left(\frac{T_E - T_W}{2\Delta x}\right) = q_e$$

$$T_E = T_W + 2\Delta x \frac{q_e}{K_P}$$

Substitute this into (*)

$$\left(T_W + 2\Delta x \frac{q_e}{K_P}\right)\left(\frac{K_P}{\Delta x^2} + \left.\frac{dk}{dx}\right|_P \frac{1}{2\Delta x}\right) + T_P\left(-\frac{2K_P}{\Delta x^2} + S_P\right) + T_W\left(\frac{K_P}{\Delta x^2} - \left.\frac{dk}{dx}\right|_P \frac{1}{2\Delta x}\right)$$