ESS 524 Class #13

Highlights from last Wednesday – Erich
Today's highlights report next Wednesday – Shashank

Today

- Project updates
- Another look at instabilities
  - Time stepping $2\Gamma \Delta t / \Delta x^2 > 1$
  - Flow $|P| > 1$
- Code – ice-sheet

Reading

- Patankar – Chapter 6: Calculation of the Flow Field
- Versteeg and Malalasekara – Chapter 6: coupled p and v
- Ed's notes on transient ice cap or glacier flow

Time steps with diffusion

Discretized PDE

$$\rho c \frac{\Delta x}{\Delta t}(T_P^1 - T_P^0) = f\left[\frac{k_e(T_E^1 - T_P^1)}{(\delta x)_e} - \frac{k_w(T_P^1 - T_W^1)}{(\delta x)_w}\right] + (1-f)\left[\frac{k_e(T_E^0 - T_P^0)}{(\delta x)_e} - \frac{k_w(T_P^0 - T_W^0)}{(\delta x)_w}\right] . \quad (4.35)$$

Terms collected

$$a_P T_P = a_E\,[f T_E + (1-f)T_E^0] + a_W\,[f T_W + (1-f)T_W^0] + [a_P^0 - (1-f)a_E - (1-f)a_W]\,T_P^0 , \quad (4.36)$$

Terms

$$a_E = \frac{k_e}{(\delta x)_e} , \quad (4.37a)$$

$$a_W = \frac{k_w}{(\delta x)_w} , \quad (4.37b)$$

$$a_P^0 = \frac{\rho c\ \Delta x}{\Delta t} , \quad (4.37c)$$

$$a_P = f a_E + f a_W + a_P^0 . \quad (4.37d)$$

Explicit Scheme f=0

Discretized PDE

$$\rho c \frac{\Delta x}{\Delta t}(T_P^1 - T_P^0) = f\left[\frac{k_e(T_E^1 - T_P^1)}{(\delta x)_e} - \frac{k_w(T_P^1 - T_W^1)}{(\delta x)_w}\right] + (1-f)\left[\frac{k_e(T_E^0 - T_P^0)}{(\delta x)_e} - \frac{k_w(T_P^0 - T_W^0)}{(\delta x)_w}\right]. \quad (4.35)$$

For the explicit scheme ($f = 0$), Eq. (4.36) becomes

Terms collected

$$a_P T_P = a_E T_E^0 + a_W T_W^0 + (a_P^0 - a_E - a_W)T_P^0 .$$

Terms

$$a_E = \frac{k_e}{(\delta x)_e},$$

$$a_W = \frac{k_w}{(\delta x)_w},$$

$$a_P^0 = \frac{\rho c\, \Delta x}{\Delta t},$$

$$a_p = a_p^0$$

$$\left(a_p^0 - a_E - a_W\right) > 0$$

$$\Delta t < \frac{\rho c(\Delta x)^2}{2k}$$

Diffusing a signal

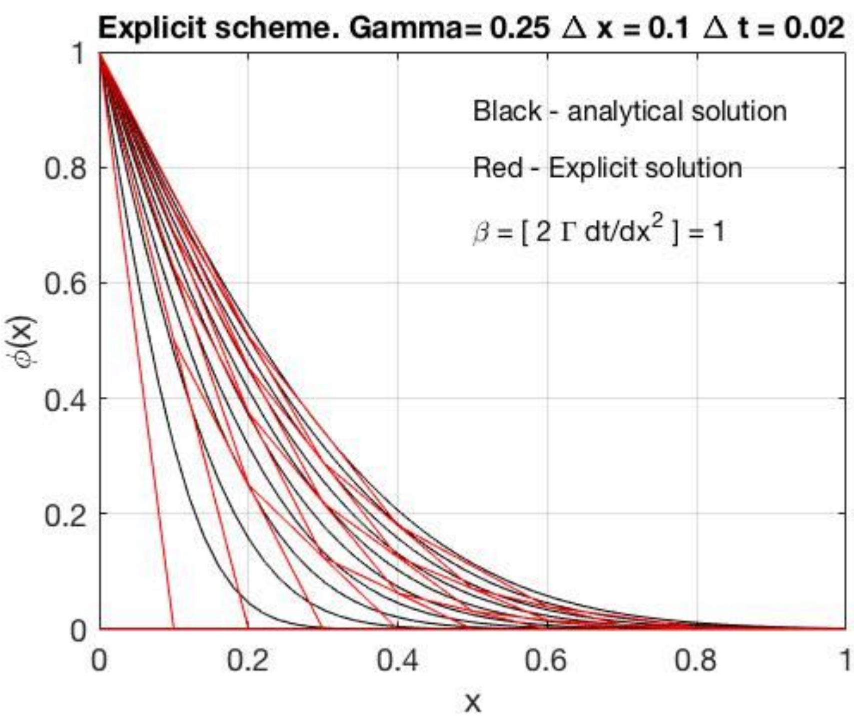
Temperature raised suddenly by ΔT at the boundary of an isothermal half-space with conductivity k, density ρ, and heat capacity c.

Analytical solution for temperature at distance x into half space

$$T(x,t) = T_0 + \Delta T \ \mathrm{erfc}\left(\frac{x}{\sqrt{4\Gamma t}}\right)$$

Time to raise temperature ~$\Delta T/3$ at a distance Δx into the half space

$$\Delta t > \frac{\Delta x^2}{2\Gamma}$$



Compare to the stability criterion for explicit scheme (f=0)

$$\Delta t < \frac{\rho c (\Delta x)^2}{2k}$$

Diffusing a signal

Analytical solution for temperature at distance *x* into half space after step change ΔT at boundary *x*=0.

$$T(x,t) = T_0 + \Delta T \ \mathrm{erfc}\left(\frac{x}{\sqrt{4\Gamma t}}\right) \qquad \Gamma = k/(\rho c)$$

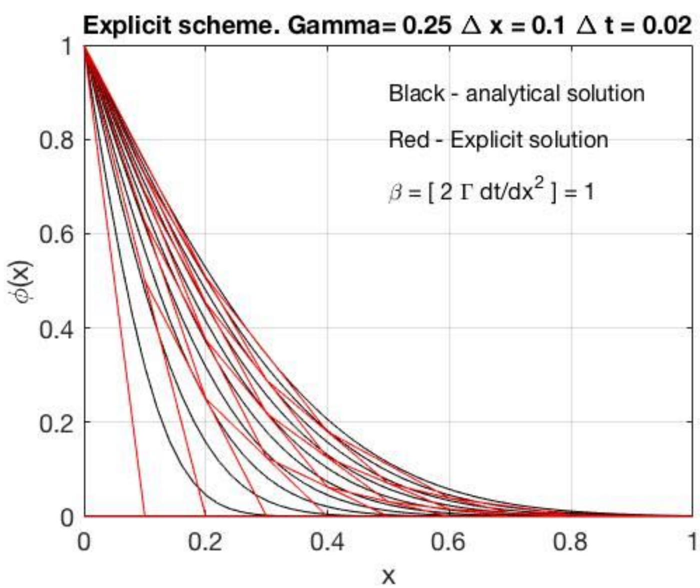
Time τ to change temperature *T* by ~$\Delta T/3$ at a distance *x* into the half space

$$\tau = \frac{x^2}{2\Gamma}$$

Compare τ to the stability criterion for explicit scheme (*f*=0)

$$\Delta t < \frac{\rho c (\Delta x)^2}{2k}$$

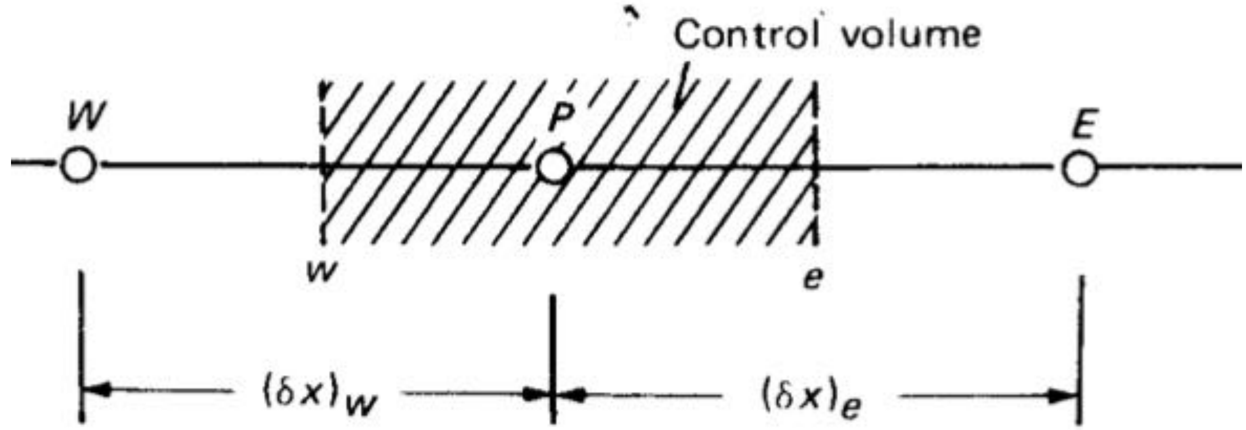
Recall that the explicit scheme can transfer information only Δx in a single time step Δt



If $\Delta t > \frac{\Delta x^2}{2\Gamma}$, significant information (temperature) needs to be sent beyond Δx in one time step Δt, but the explicit scheme can't push information beyond Δx.

- Signal squashes up, distorts, and we get the Linear Computational Instability.

Advection – central-difference scheme



$$\frac{d}{dx}(\rho u \phi) = \frac{d}{dx}\left(\Gamma \frac{d\phi}{dx}\right)$$

Integrate over volume

$$(\rho u \phi)_e - (\rho u \phi)_w = \left(\Gamma \frac{d\phi}{dx}\right)_e - \left(\Gamma \frac{d\phi}{dx}\right)_w$$

ϕ values at boundaries

$$\phi_e = \tfrac{1}{2}(\phi_E + \phi_P)$$

$$\phi_w = \tfrac{1}{2}(\phi_P + \phi_W)$$

Advection strength $F \equiv \rho u$

Diffusion conductance $D \equiv \dfrac{\Gamma}{\delta x}$

Discretization

$$a_P \phi_P = a_E \phi_E + a_W \phi_W$$

$$a_E = D_e - \frac{F_e}{2}$$

$$a_W = D_w + \frac{F_w}{2}$$

$$a_P = D_e + \frac{F_e}{2} + D_w - \frac{F_w}{2}$$

$$= a_E + a_W + (F_e - F_w)$$

a_E or a_W can become negative when advection $|F|$ is large –

$$\left|\frac{F}{2D}\right| > 1$$

Advecting a signal

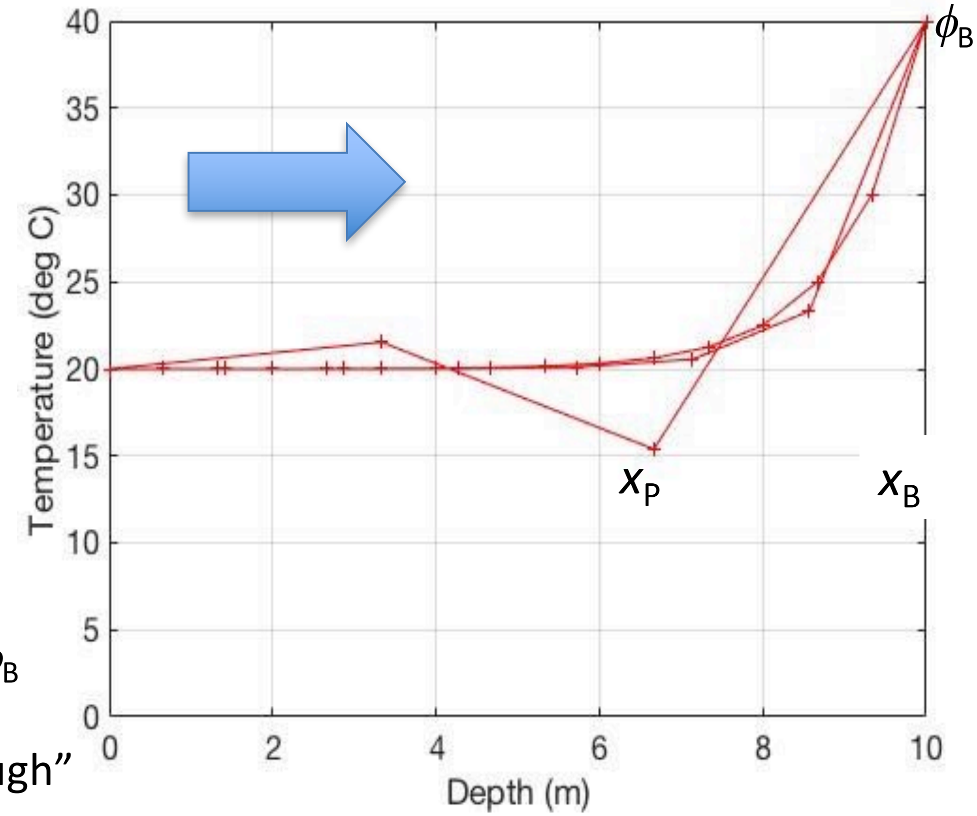
Time is still involved in this steady-state problem. At the grid scale

$$T_{diff} = \frac{\Delta x^2}{\Gamma} \qquad T_{adv} = \frac{\Delta x}{u}$$

- Diffusion time increases quadratically with increasing distances Δx.
- Advection time increases only linearly.

- $\phi = \phi_B$ is fixed at the right boundary.
- Adjacent node X_P needs to know about ϕ_B only if x_P is "close enough" to x_B.
- Setting $T_{adv} = T_{diff}$ shows that "close enough" means

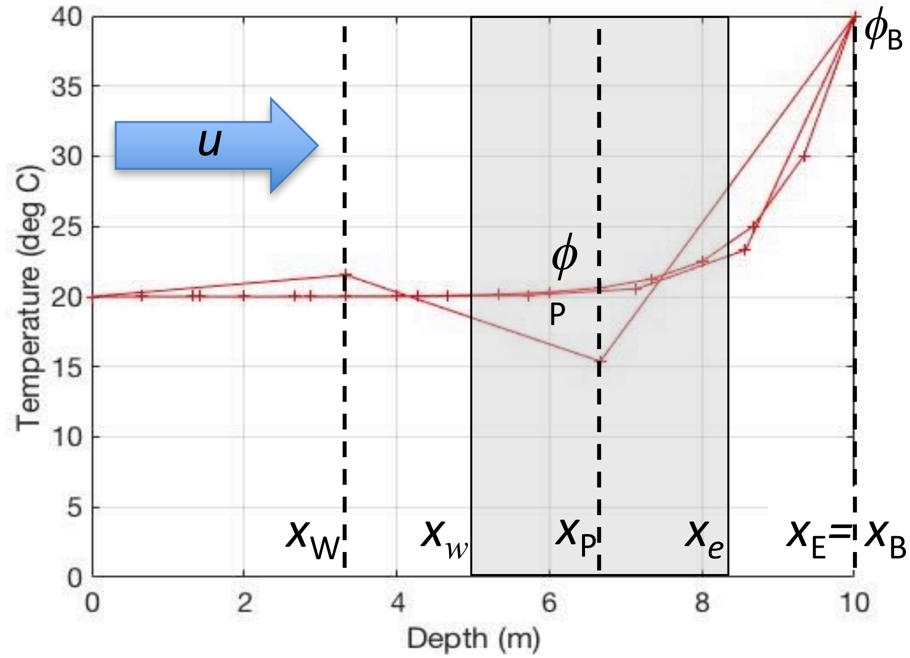
$$\Delta x = \frac{\Gamma}{u}$$



If advection carries value ϕ_P from x_P to x_B much faster than diffusion from x_B can inform node at x_P that a fixed value $\phi = \phi_B$ is ahead at the boundary, then the numerical scheme should not incorporate ϕ_B into the equation for ϕ_P by giving equal weights to both nodes.

- Node spacing Δx must be reduced until $\Delta x \sim \Gamma/u$ or less.

Advecting a signal



If node spacing $\Delta x > \Gamma/u$,

- x_P is too far from x_B to “feel” the warm boundary condition ϕ_B.
- But central-difference scheme with large Δx forces volume edge at x_e to “feel” the boundary value ϕ_B anyway.
- Correct slope at x_e should be vey small.
- Erroneously large positive slope at x_e delivers too much heat into the finite volume from x_B.
- This excess diffusive heat gain must be offset by advective heat loss in steady state.
- Since u is uniform, $\phi_e - \phi_w > 0$ produces net advective export $u(\phi_e - \phi_w)$
- ϕ_e is constrained to fall on the line joining ϕ_P and ϕ_E, so ϕ_w is forced leave the correct solution to produce correct advective export (high or low depends on u and Γ).
- ϕ_P can be forced to fall below the correct solution in order to make ϕ_w satisfy the pde.
- In example shown, additional heat conducted across x_w must be compensated by reduced advective flux, i.e. reduced ϕ_w.
- Adjustments to compensate taper off upstream.
- Cause is inabilit y to represent curvature near boundary.