ESS 524 Class #15

Highlights from last Wednesday– Erich
Today's highlights report on Wednesday – Shashank

Today

- Project updates
- Homework progress
- Calculating the flow field $\boldsymbol{u} = (u_1, u_2, u_3)$ in a viscous fluid
  - Momentum conservation – viscous stress τ_{ij}
  - strain rate $\dot{\epsilon}_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$
  - Constitutive relation – stress τ_{ij} and strain rate $\dot{\epsilon}_{ij}$
  - Navier-Stokes equation – velocity and pressure p

Reading

- Patankar – Chapter 6: Calculation of the Flow Field
- Versteeg and Malalasekara – Chapter 6: coupled p and v

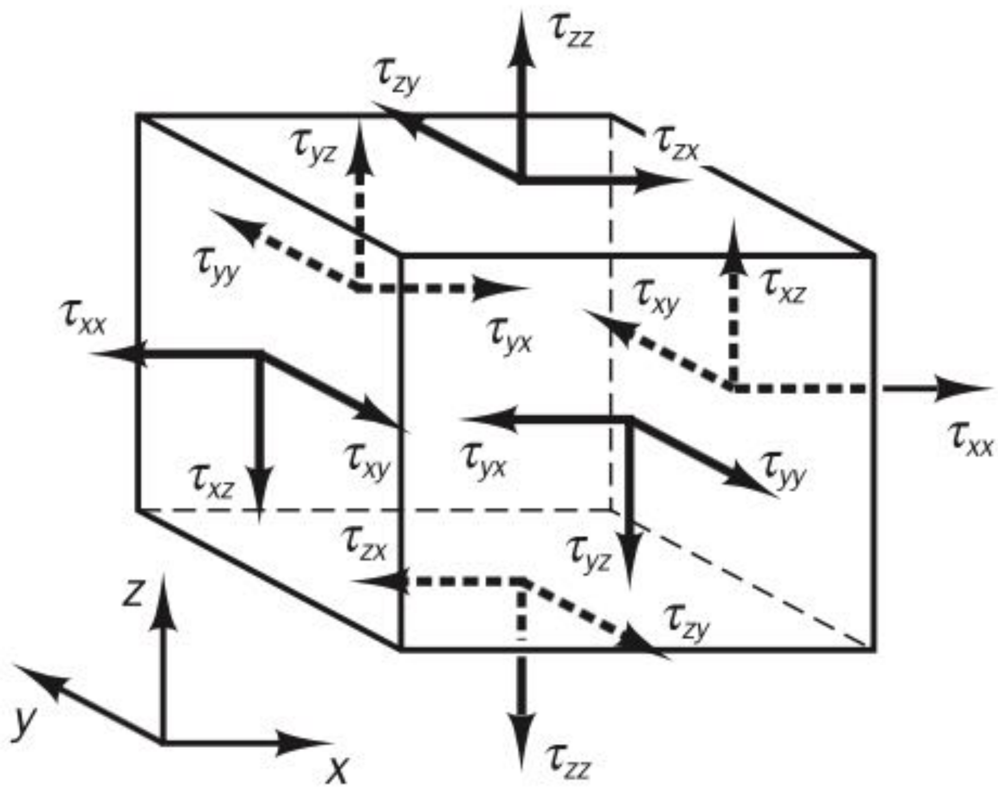
Next week

- Monday is Memorial Day
- Wednesday, Daniel Shapero will lead the first of 2 classes to introduce us to *Firedrake*, and publically available codes to solve advective-diffusive problems
- Do you have ready access to Python?

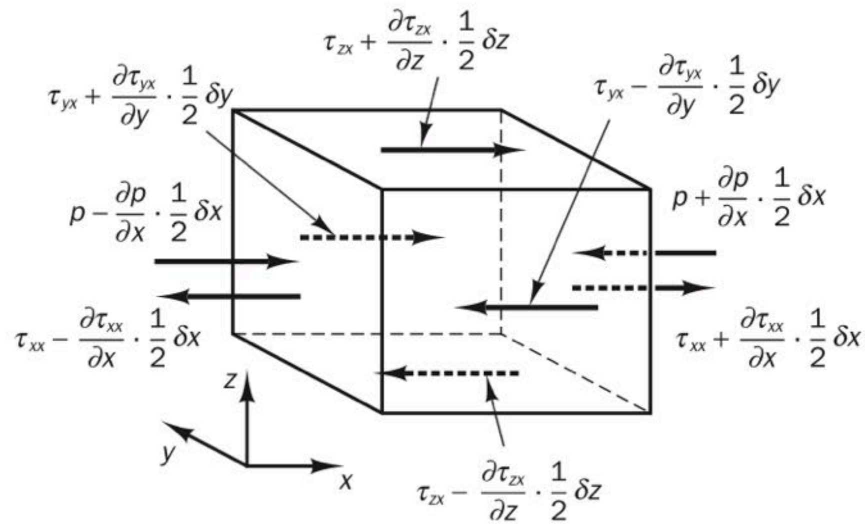
Viscous (deviatoric) stresses

Stress $\sigma_{ij} = \tau_{ij} - p$

Pressure $p = -(\sigma_{11}+\sigma_{22}+\sigma_{33})/3$



Force balance in *x*



On the pair of faces (E, W) we have

$$\left[\left(p - \frac{\partial p}{\partial x}\frac{1}{2}\delta x\right) - \left(\tau_{xx} - \frac{\partial \tau_{xx}}{\partial x}\frac{1}{2}\delta x\right)\right]\delta y \delta z + \left[-\left(p + \frac{\partial p}{\partial x}\frac{1}{2}\delta x\right)\right.$$

$$\left. + \left(\tau_{xx} + \frac{\partial \tau_{xx}}{\partial x}\frac{1}{2}\delta x\right)\right]\delta y \delta z = \left(-\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x}\right)\delta x \delta y \delta z \qquad (2.12a)$$

The net force in the x-direction on the pair of faces (N, S) is

$$-\left(\tau_{yx} - \frac{\partial \tau_{yx}}{\partial y}\frac{1}{2}\delta y\right)\delta x \delta z + \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}\frac{1}{2}\delta y\right)\delta x \delta z = \frac{\partial \tau_{yx}}{\partial y}\delta x \delta y \delta z \qquad (2.12b)$$

Finally the net force in the x-direction on faces T and B is given by

$$-\left(\tau_{zx} - \frac{\partial \tau_{zx}}{\partial z}\frac{1}{2}\delta z\right)\delta x \delta y + \left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}\frac{1}{2}\delta z\right)\delta x \delta y = \frac{\partial \tau_{zx}}{\partial z}\delta x \delta y \delta z \qquad (2.12c)$$

The total force per unit volume on the fluid due to these surface stresses is equal to the sum of (2.12a), (2.12b) and (2.12c) divided by the volume $\delta x \delta y \delta z$:

$$\frac{\partial(-p + \tau_{xx})}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \tag{2.13}$$

S_{Mj} = momentum source per unit volume in direction j

Now ***F = ma***

$$\rho \frac{Dv}{Dt} = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial(-p + \tau_{yy})}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + S_{My}$$

And in *y* and *z*:

$$\rho \frac{Du}{Dt} = \frac{\partial(-p + \tau_{xx})}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + S_{Mx}$$

$$\rho \frac{Dw}{Dt} = \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial(-p + \tau_{zz})}{\partial z} + S_{Mz}$$

Newtonian viscous fluids

Velocity gradients describe deformation rates

- (Dynamic) viscosity μ relates stresses to linear deformations
- (Bulk) viscosity λ relates stresses to volumetric deformations

Constitutive relations

$$\tau_{xx} = 2\mu\frac{\partial u}{\partial x} + \lambda \operatorname{div} \mathbf{u} \quad \tau_{yy} = 2\mu\frac{\partial v}{\partial y} + \lambda \operatorname{div} \mathbf{u} \quad \tau_{zz} = 2\mu\frac{\partial w}{\partial z} + \lambda \operatorname{div} \mathbf{u}$$

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \quad \tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \qquad (2.31)$$

Navier-Stokes velocity equations

Variables

- u, v, w, p, ρ

Parameters

- μ, λ

Source

- S_{Mj}

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left[2\mu \frac{\partial u}{\partial x} + \lambda \operatorname{div} \mathbf{u}\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right] + S_{Mx}$$

$$\rho \frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left[2\mu \frac{\partial v}{\partial y} + \lambda \operatorname{div} \mathbf{u}\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right] + S_{My}$$

$$\rho \frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right] + \frac{\partial}{\partial z}\left[2\mu \frac{\partial w}{\partial z} + \lambda \operatorname{div} \mathbf{u}\right] + S_{Mz}$$

Navier-Stokes equation for *u*

$$\rho\frac{Du}{Dt}=-\frac{\partial p}{\partial x}+\frac{\partial}{\partial x}\left[2\mu\frac{\partial u}{\partial x}+\lambda\,\mathrm{div}\,\mathbf{u}\right]+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right]+S_{Mx}$$

Variables

- u, v, w, p, ρ

Parameters

- μ, λ

Source

- S_{Mj}

u time marching equation also depends on derivatives of *v, w,* and *p*

The other 2 velocity equations are similarly coupled

We need another equation

Conservation of Mass relates velocity and density

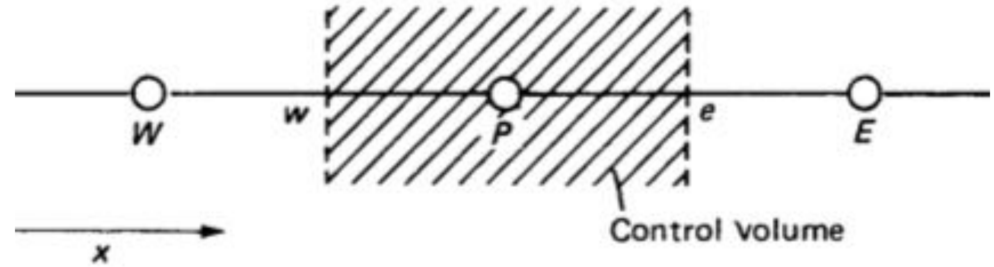
$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$

$$\boxed{\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{u}) = 0}$$

There isn't another obvious fundamental pressure equation.

- Iteratively adjust the pressure field until mass conservation is satisfied.

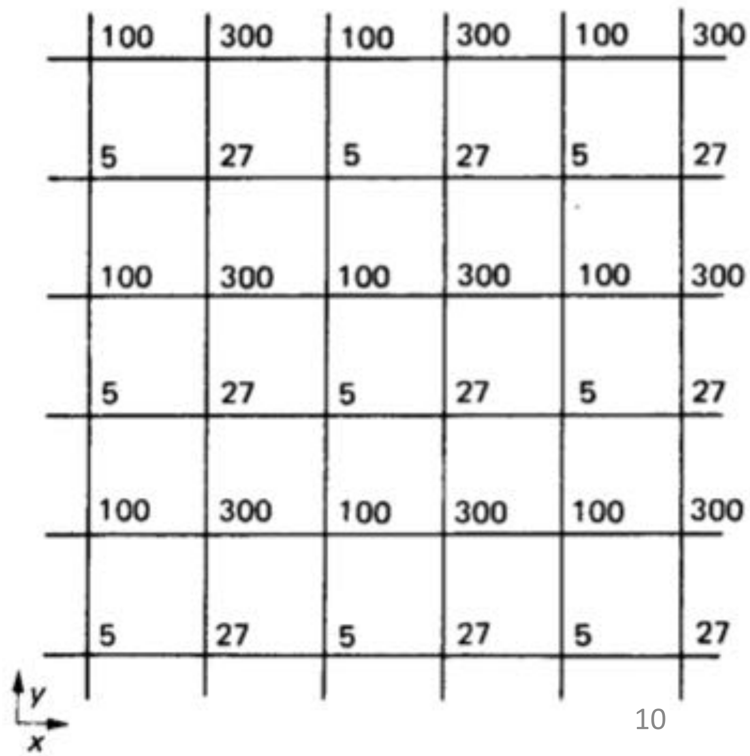
The pressure-gradient problem



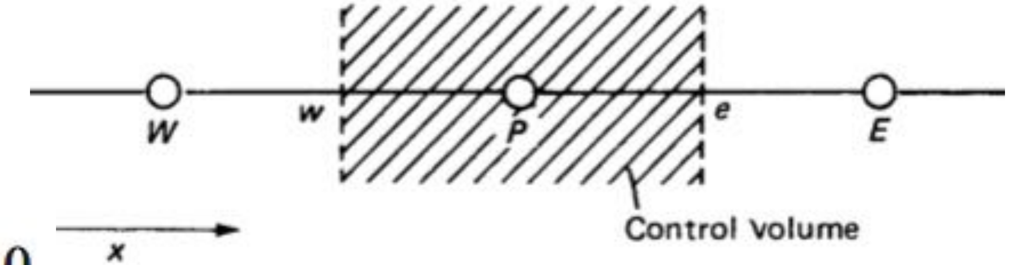
If pressure is found on the same nodes as *u*
Then when solving for *u* at node *P*

$$p_w - p_e = \frac{p_W + p_P}{2} - \frac{p_P + p_E}{2} = \frac{p_W - p_E}{2}$$

With this crazy pressure field, $p_w - p_e = 0$ everywhere, and discretization scheme thinks pressure is uniform.



The mass-conservation-equation problem



$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$

If *u* and *v* are found on the same nodes, *t*hen when at node *P* (e.g. in 2-D)

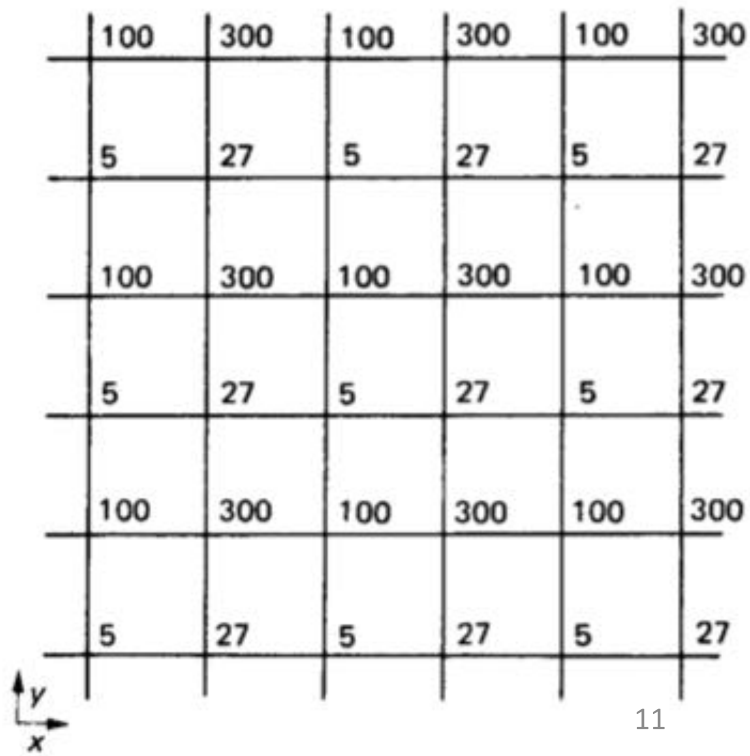
$$u_e - u_w = \frac{u_P + u_E}{2} - \frac{u_W + u_P}{2} =$$

$$= u_E - u_W$$

E.g. with this crazy velocity field,

- $u_w - u_e = 0$ everywhere,
- $v_n - v_s = 0$ everywhere,

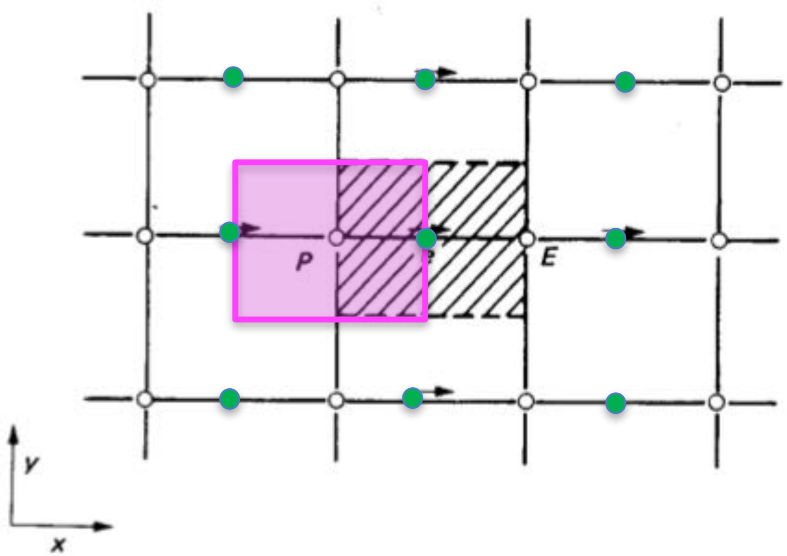
and discretization scheme thinks velocity is uniform.



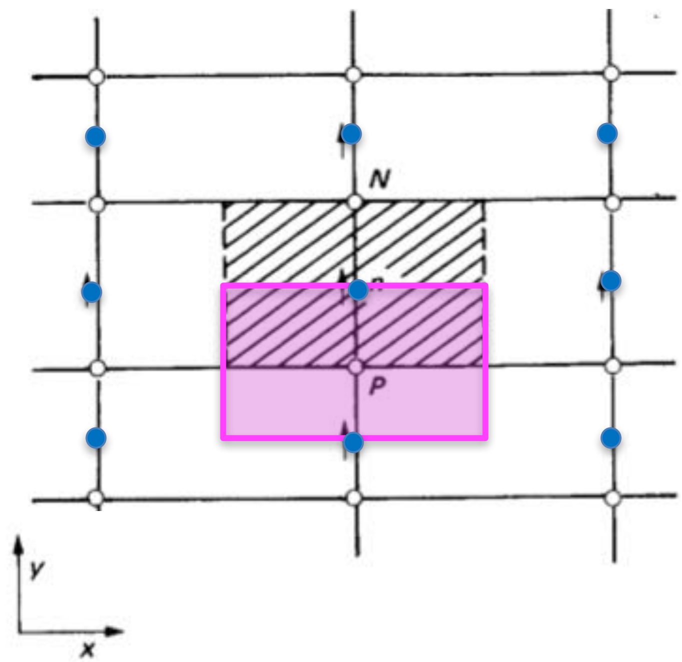
Staggered grids for mass conservation equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$

Evaluate density ρ on the initial mesh (magenta volumes) with volume centers at P, E, W, N, S.



Evaluate *u* on the east and west edges of the *P* volume, in hatched volumes.



Evaluate *v* on the north and south edges of the *P* volume, in the hatched volumes.

Residual in mass conservation

The SIMPLE algorithm