ESS 524 Class #16

Highlights from last Monday – Shashank
Today's highlights report next Wednesday – Erich

Today

- Homework progress
- Beyond the Cartesian mesh
- Structured vs unstructured grids
  - Boundary-fitted grids – see Price et al. (2007)
  - Fluxes across volume edges in non-cartesian grids

Reading

- Versteeg and Malalasekara – Chapter 11 Complex Geometries

Next week

- Monday is Memorial Day

- Wednesday, Daniel Shapero will lead the first of 2 classes to introduce us to *Firedrake*,
  https://www.firedrakeproject.org/
- solving advective-diffusive problems.
- Recall that the Divergence Theorem (converting volume integrals into surface integrals) underlies virtually all numerical methods to solve pde's ☺
  https://en.wikipedia.org/wiki/Divergence_theorem#Corollaries

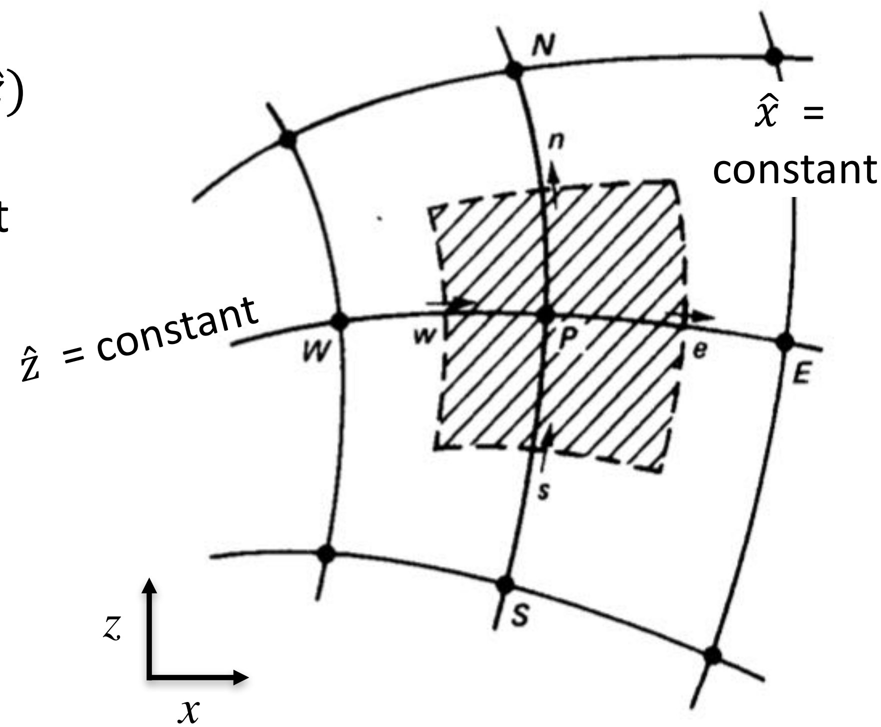
Do you have access to Python, including SymPy?

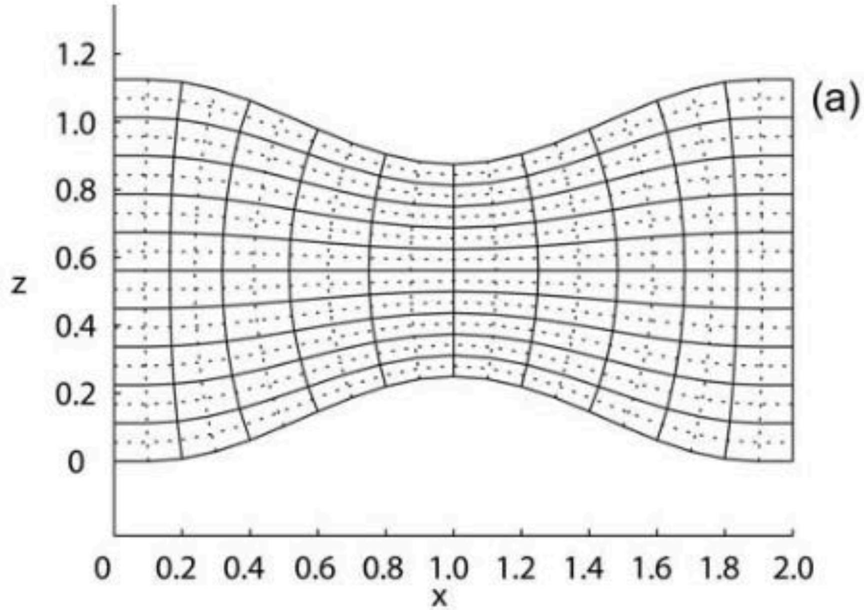
- (SymPy is included in Anaconda)
  https://www.anaconda.com/products/individual

Curvilinear orthogonal boundary-fitted mesh

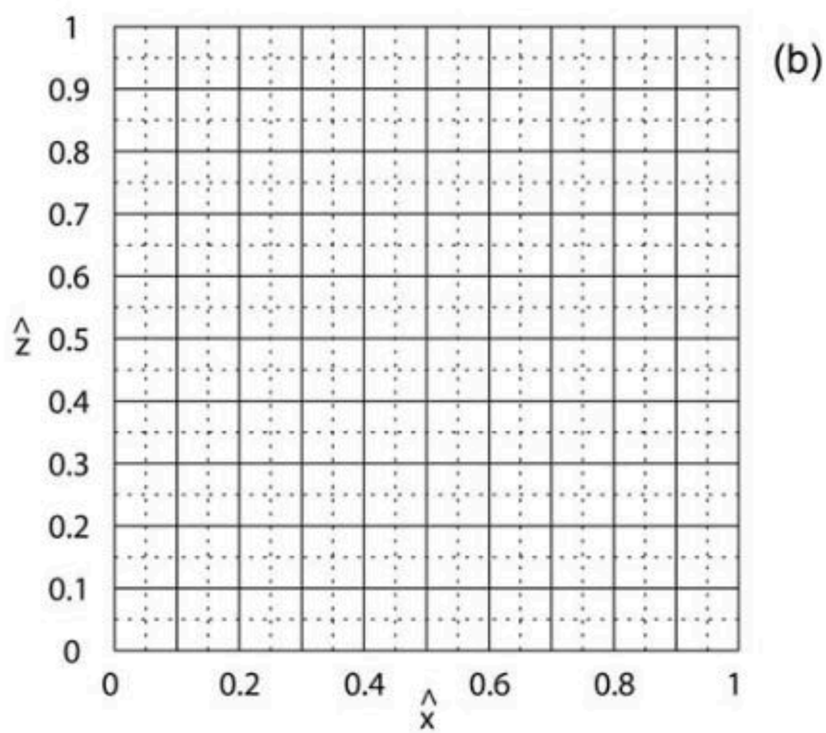
Lines of constant coordinate values ($\hat{x}$, $\hat{z}$) are curves in Cartesian (x, z) space

- $\hat{x}$ and $\hat{z}$ coordinate lines meet at right angles
- Finite-volume edges are curves
- Edges meet curves joining volume centers at right angles.





Map curvilinear geometry onto Cartesian geometry



Equations to be solved have the same form, apart from some geometric coefficients

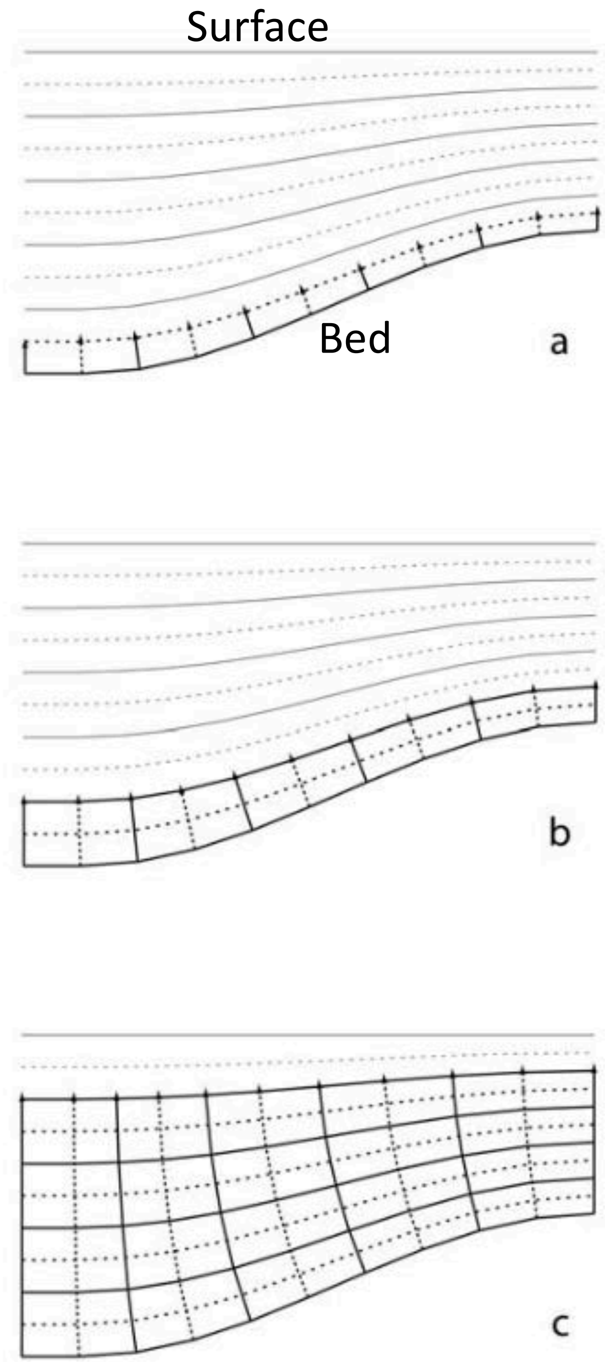
Creating orthogonal $(\hat{x}, \hat{z})$ coordinate system

Curves of constant $\hat{z}$

- Fractional distance between bed $B(x)$ and surface $S(x)$
- $\hat{z} = \frac{z - B(x)}{S(x) - B(x)}$

Curves of constant $\hat{x}$

- Constructed orthogonal to curves of constant $\hat{z}$



Metric tensor g_{ij}

$g_{ij} = \boldsymbol{e}_i \cdot \boldsymbol{e}_j$

If e_i and e_j are basis vectors in an orthonormal coordinate system, $g_{ij} = \delta_{ij}$

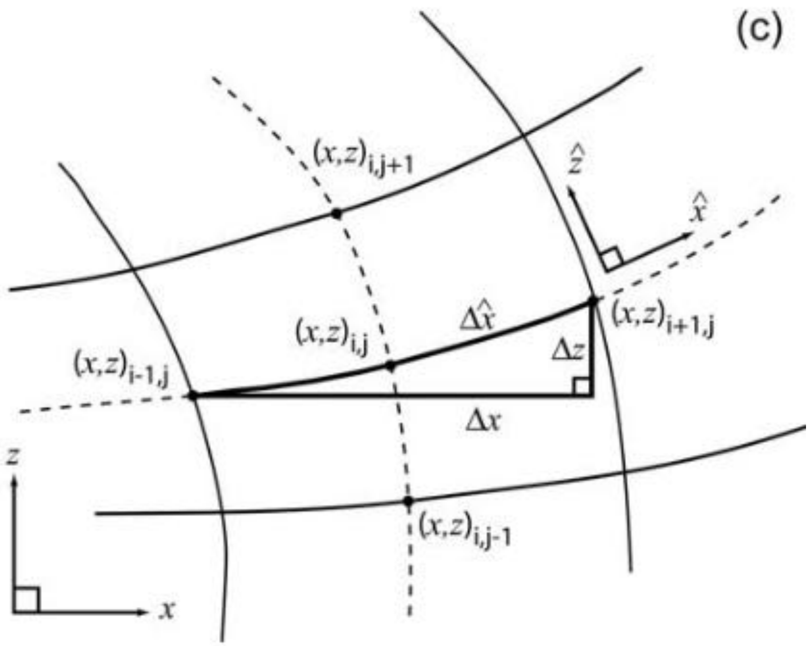
If e_i and e_j are basis vectors in 2 different coordinate systems, g_{ij} relates lengths in the 2 systems.

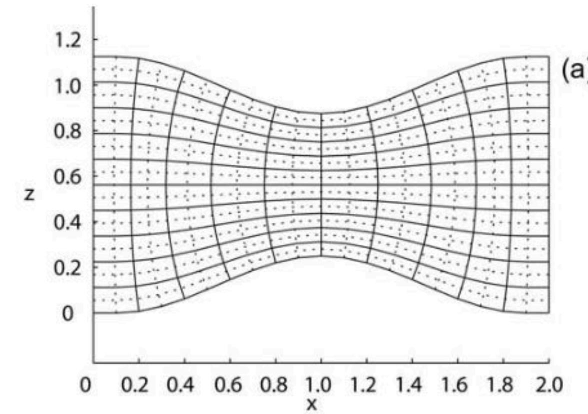
$g_{ij} = \boldsymbol{e}_i \cdot \hat{\boldsymbol{e}}_j$

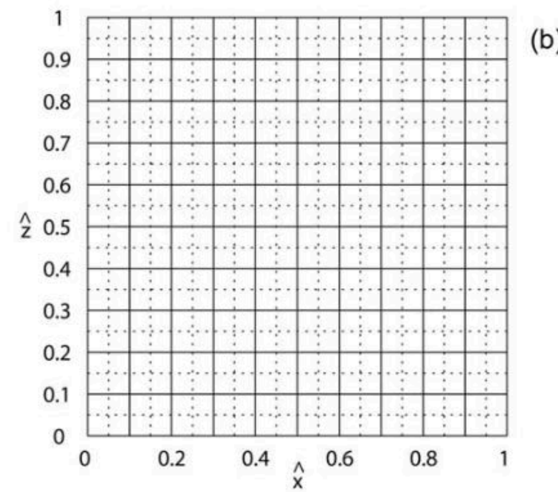
When $g_{ij} = \boldsymbol{e}_i \cdot \hat{\boldsymbol{e}}_j$

$$\frac{\partial x}{\partial \hat{z}} = -\frac{g_{12}}{g_{11}} \neq 0 \text{ necessarily}$$

Scale factors $h_{\hat{x}}$ and $h_{\hat{z}}$







$$\mathbf{r} = (x, z)$$

Coordinate unit vectors
in curvilinear coordinates

$$\hat{\mathbf{e}}_{\hat{x}} = \frac{\partial \mathbf{r}}{\partial \hat{x}} \left\| \frac{\partial \mathbf{r}}{\partial \hat{x}} \right\|^{-1} = \frac{\partial \mathbf{r}}{\partial \hat{x}} \frac{1}{h_{\hat{x}}}$$

$$h_{\hat{x}} = \left\| \frac{\partial \mathbf{r}}{\partial \hat{x}} \right\| = \left[\left(\frac{\partial x}{\partial \hat{x}} \right)^2 + \left(\frac{\partial z}{\partial \hat{x}} \right)^2 \right]^{\frac{1}{2}}$$

$$h_{\hat{z}} = \left\| \frac{\partial \mathbf{r}}{\partial \hat{z}} \right\| = \left[\left(\frac{\partial x}{\partial \hat{z}} \right)^2 + \left(\frac{\partial z}{\partial \hat{z}} \right)^2 \right]^{1/2}$$

Momentum equations

In Cartesian (x,z) coordinate system

- x-momentum equation

$$\frac{\partial(\rho u_i)}{\partial t}+u_j\frac{\partial(\rho u_i)}{\partial x_j}=\frac{\partial}{\partial x_j}\left(\eta\frac{\partial u_i}{\partial x_j}\right)-\frac{\partial P}{\partial x_i}+\rho g_i \qquad (i=x,y,z)$$

Accelerations are negligible in glacier flow, so

$$\frac{\partial}{\partial x_j}\left(\eta\frac{\partial u_i}{\partial x_j}\right)-\frac{\partial P}{\partial x_i}+\rho g_i=0 \qquad (i=x,y,z)$$

In curvilinear orthogonal $(\hat{x}, \hat{z})$ coordinate system (in 2-D)

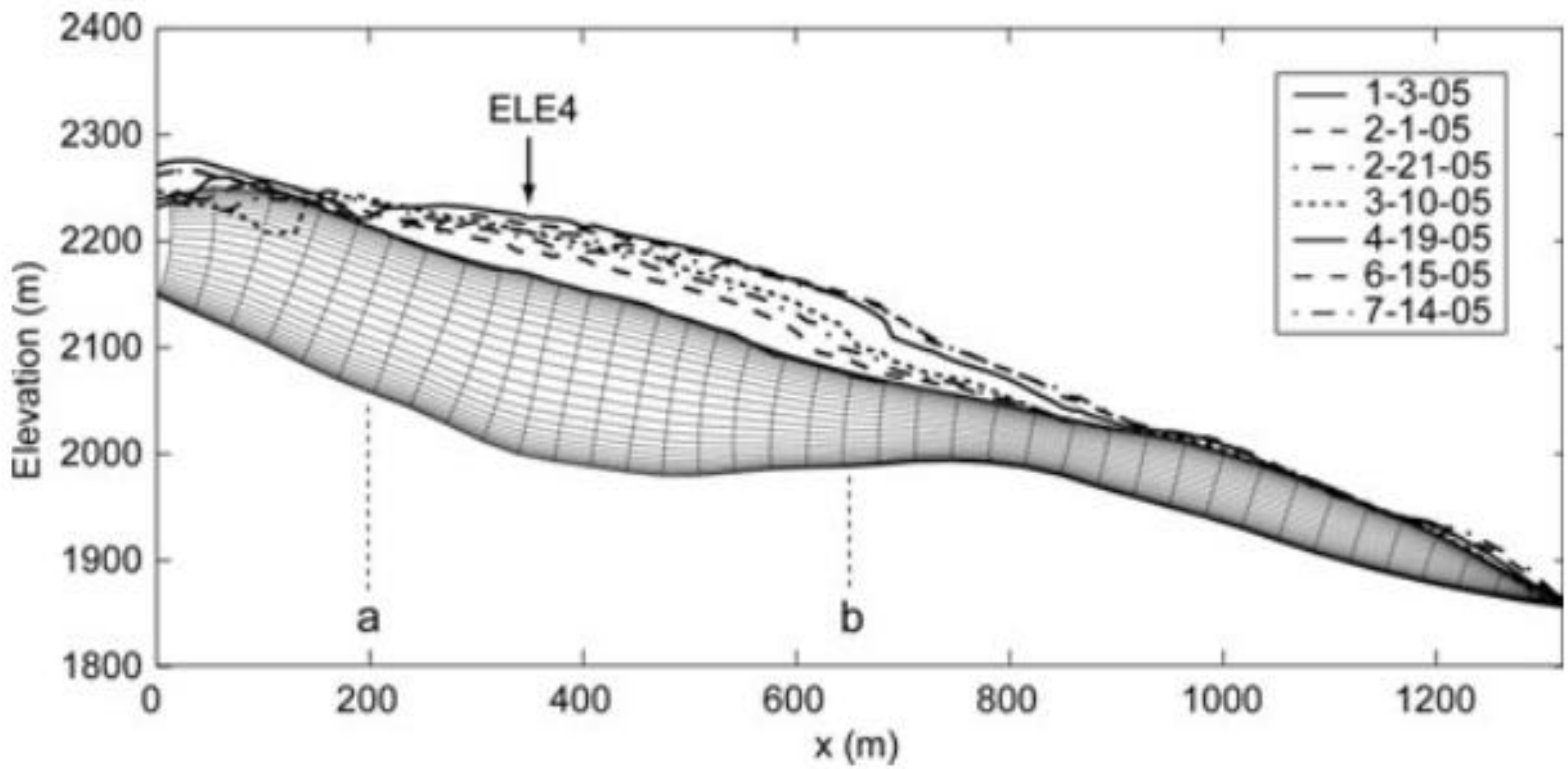
- x-momentum equation

$$\frac{1}{h_{\hat{x}}h_{\hat{z}}}\frac{\partial}{\partial\hat{x}}\left(\eta\frac{h_{\hat{z}}}{h_{\hat{x}}}\frac{\partial\tilde{u}}{\partial\hat{x}}\right)-\frac{1}{h_{\hat{x}}}\frac{\partial P}{\partial\hat{x}}+\rho g_{\hat{x}}=0,$$

New Crater Glacier on Mt St Helens

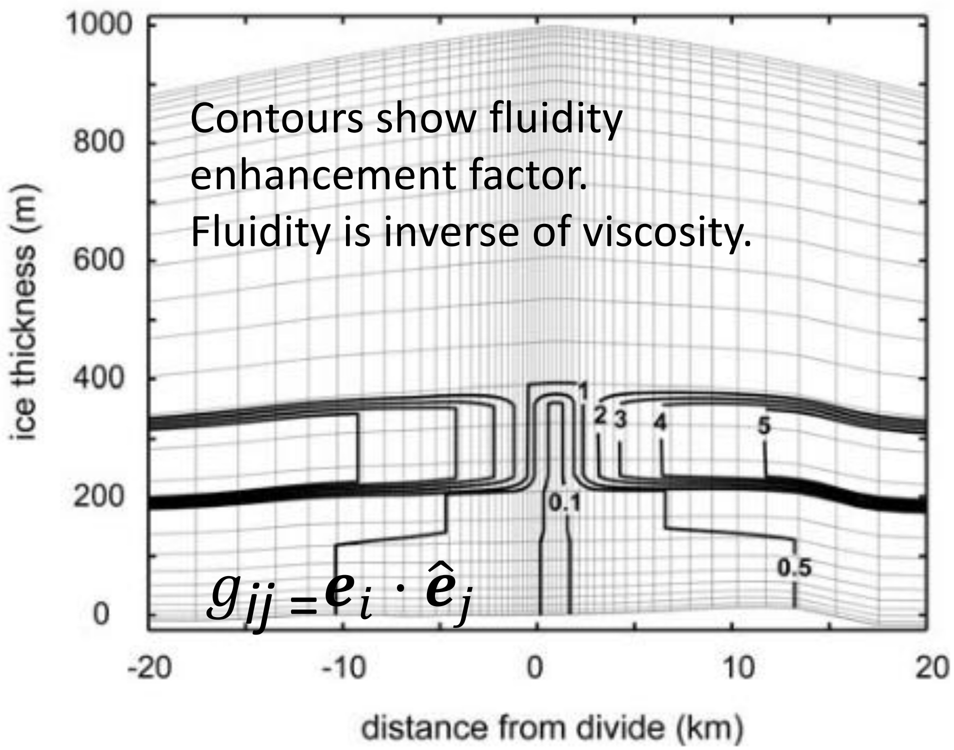
Monday (May 18) was 40th anniversary of the 1980 eruption.

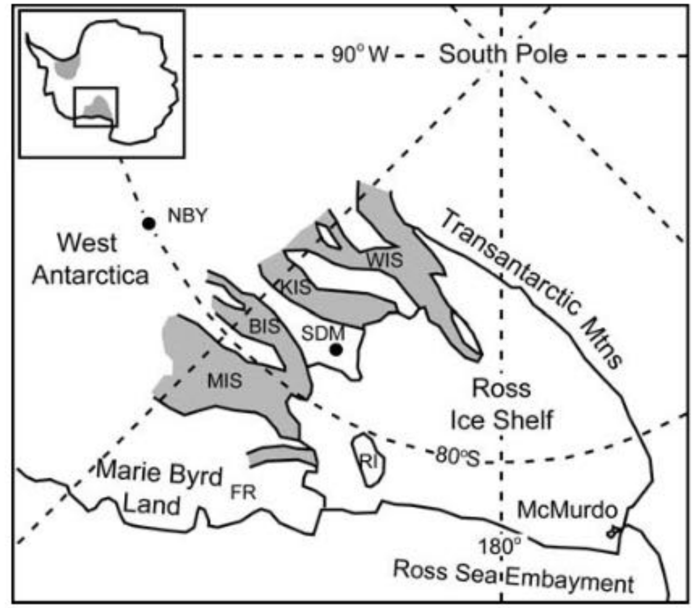
- Glacier began to grow in the caldera after the eruption.
- A bulge (kinematic wave) swept down the glacier in 2005.



Price, S.F. and J.S. Walder. (2007) Modeling the dynamic response of a crater glacier to lava-dome emplacement: Mount St Helens, Washington, USA. *Ann Glaciol*. 45, 21-28.

Deglacial thinning history of Siple Dome, West Antarctica





Antarctica resembles a pancake, so the transformation is barely noticeable.

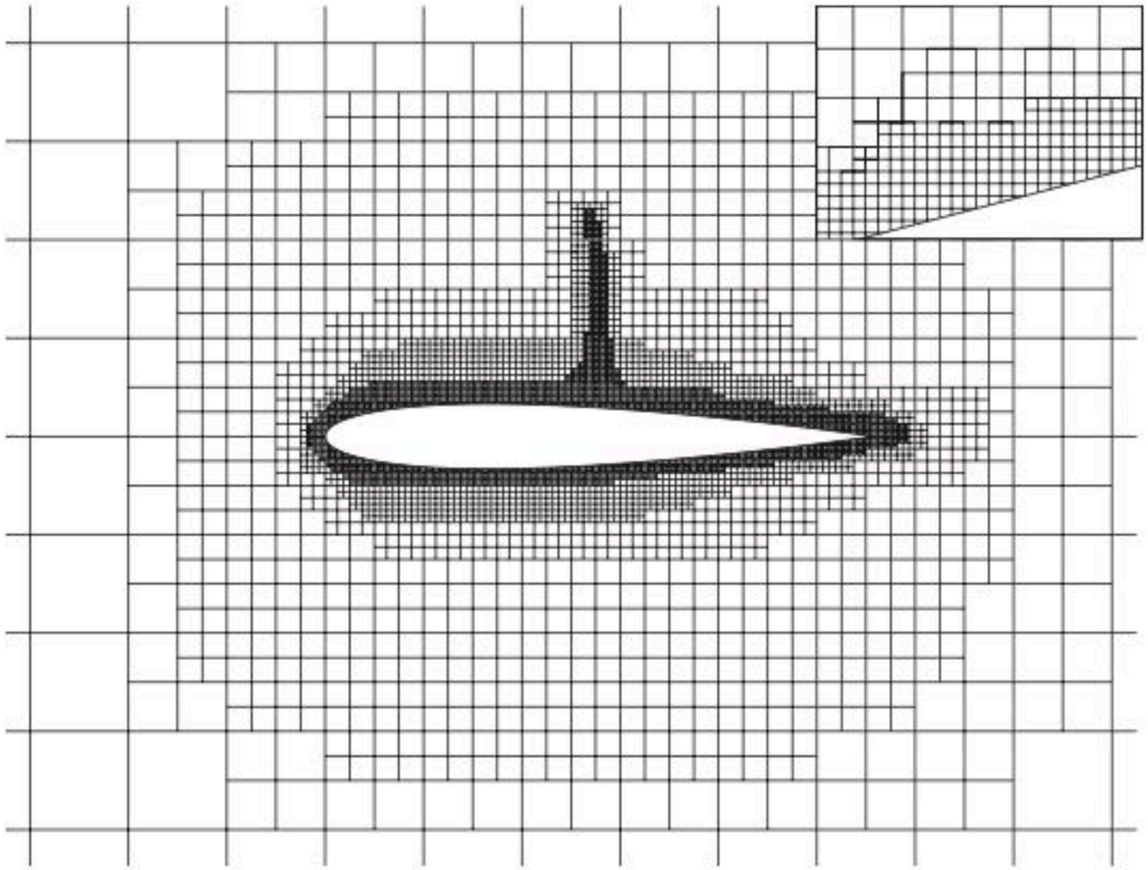
- Care is needed to treat transient boundary conditions.

Coupled thermomechanical FVM model used to match data sets

- internal stratigraphy (GPR) and borehole temperature profile.
- suggested SDM had thinned by ~300 m during deglaciation
- Between 15ka and 14ka

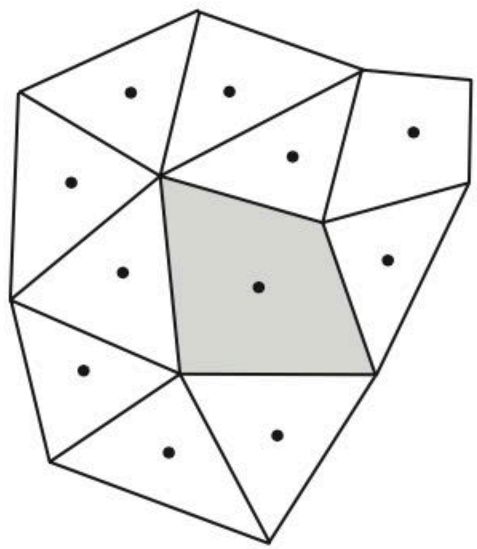
Price, S.F. et al. (2007) Evidence for late Pleistocene thinning of Siple Dome, West Antarctica. *JGR* 112, F03021, doi:10.1029/2006JF000725

Block-structured mesh

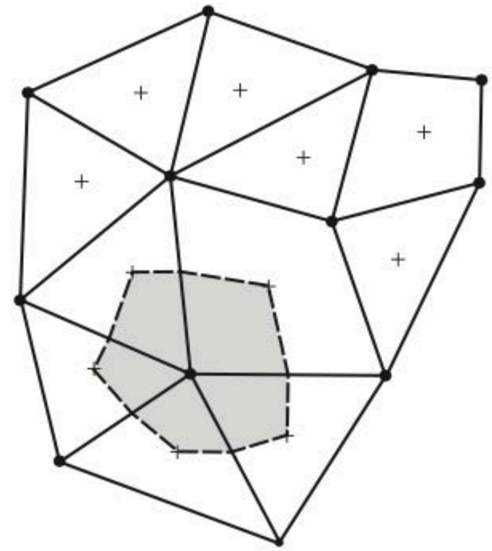


- This is like nesting a high-resolution weather model for WA state inside a coarser global model (a GCM),which provides boundary conditions.
- Care is need to get fluxes right on interfaces where mesh is sub-divided.

Unstructured grids



Cell-centered volumes



Vertex-based volumes

Select vertices

- nodes are centroids

Select nodes

- Volume edges connect centroids of figures formed by vertices

Deriving Finite Volume discretization

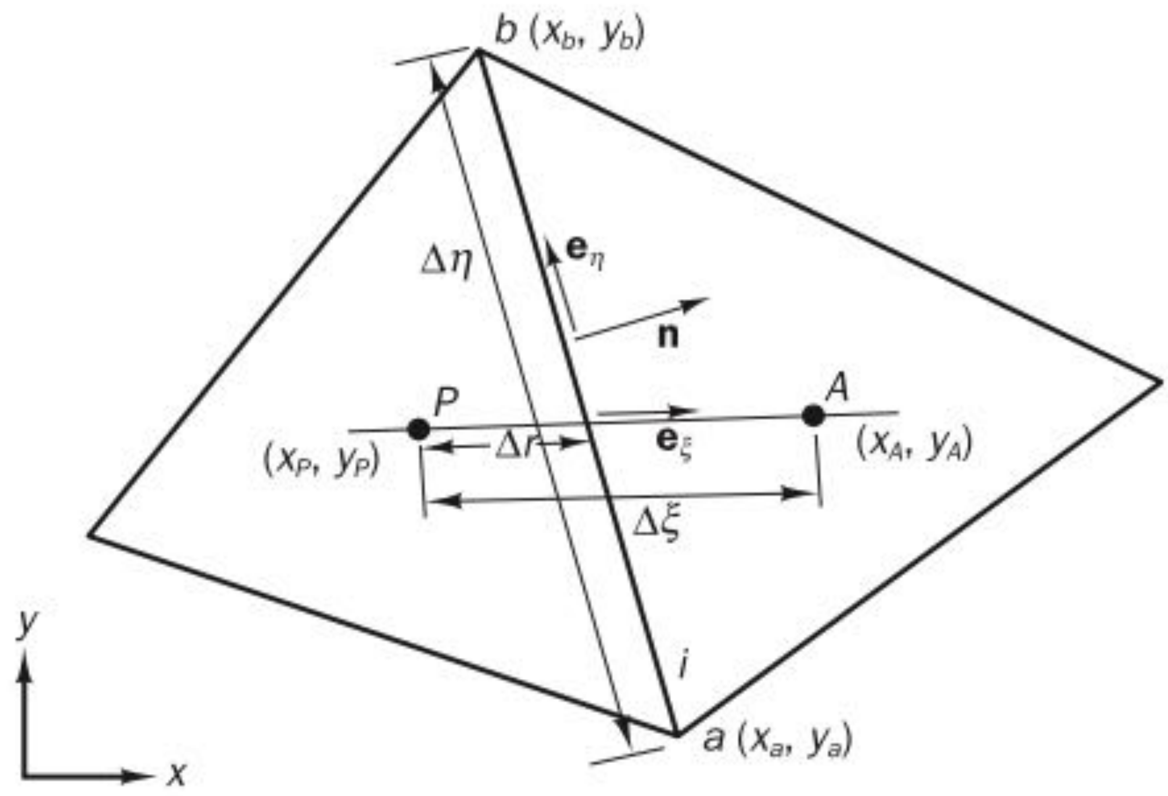
Integrate each term of general equation over Control Volume

$$\int_{CV} \frac{\partial}{\partial t}(\rho\phi)\mathrm{d}V + \int_{CV} \mathrm{div}(\rho\phi\mathbf{u})\mathrm{d}V = \int_{CV} \mathrm{div}(\Gamma\ \mathrm{grad}\ \phi)\mathrm{d}V + \int_{CV} S_\phi \mathrm{d}V$$

- When Cartesian volume edges aligned with coordinate directions, we previously just directly integrated over a volume and got boundary fluxes.
- Now we need Divergence Theorem to convert divergence inside a generic volume into flux across the surfaces.

$$\frac{\partial}{\partial t}\left(\int_{CV} \rho\phi \mathrm{d}V\right) + \int_{A} \mathbf{n} \cdot (\rho\phi\mathbf{u})\mathrm{d}A = \int_{A} \mathbf{n} \cdot (\Gamma\ \mathrm{grad}\ \phi)\mathrm{d}A + \int_{CV} S_\phi \mathrm{d}V$$

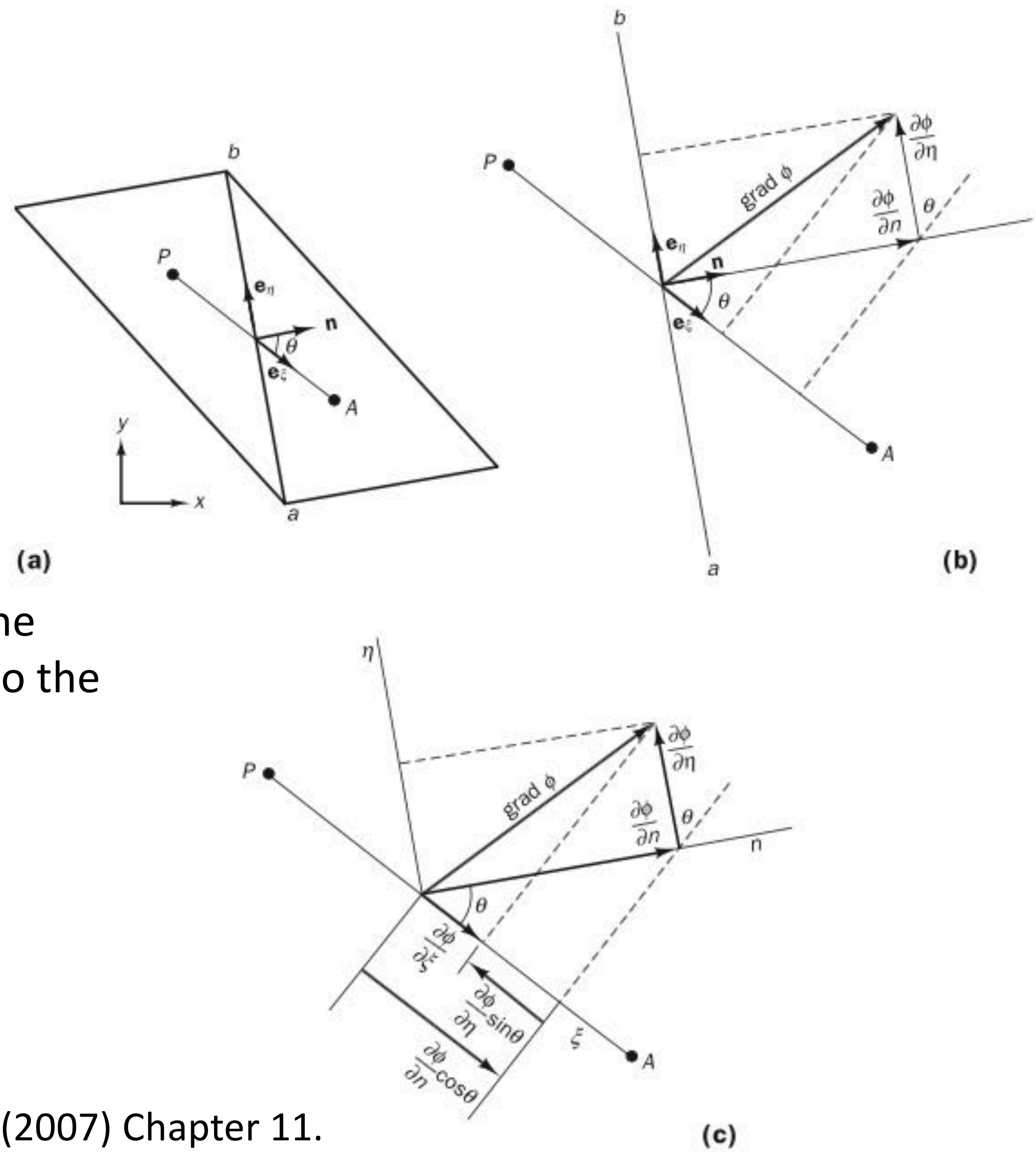
Fluxes at interfaces



We want to express fluxes in terms of values at *P* and *A*.

- But line *PA* joining nodes is not perpendicular to the interface *ab*

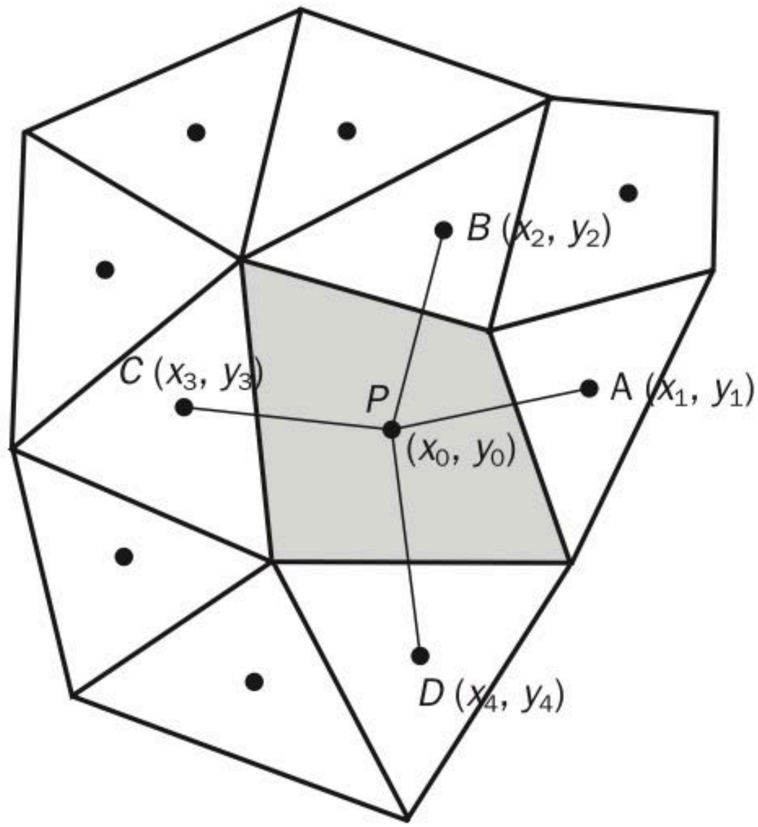
Cross-diffusion



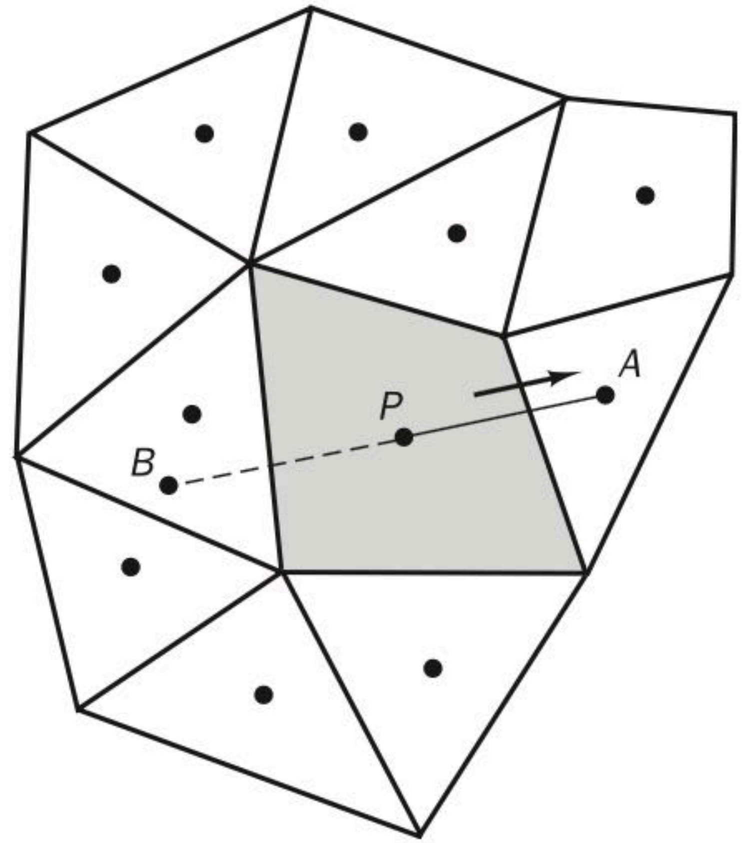
Gradient along *PA* is not the same as gradient normal to the interface *ab*

See: Versteeg & Malalsekera (2007) Chapter 11.

Upwinding



Control Volume at *P*, and its neighboring nodes



Need to compare $(\phi_P - \phi_B)$ to $(\phi_A - \phi_P)$

- But where is the upwind point *B*?

Higher-order schemes, collocated grids?