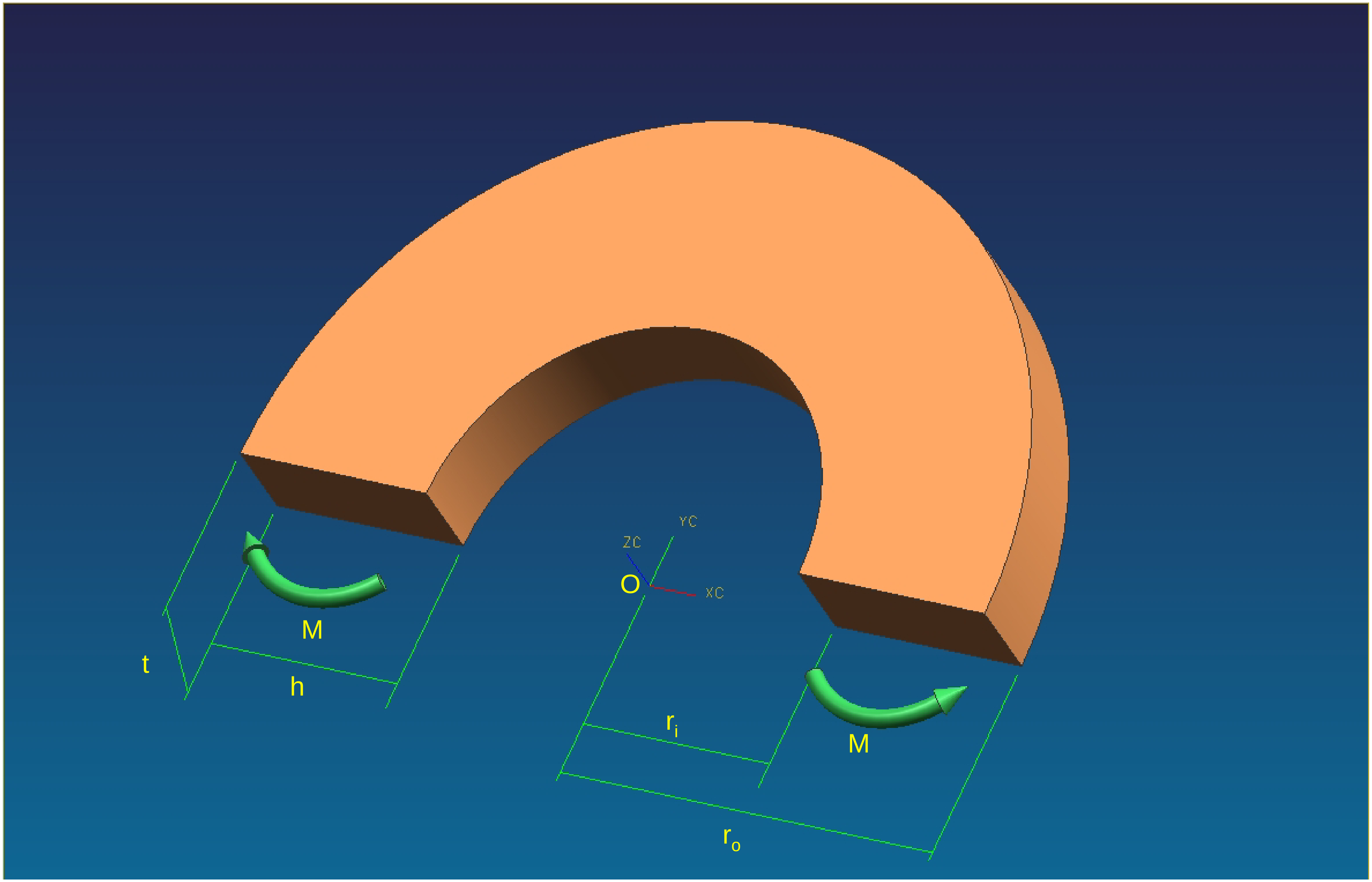
Curved Beams

Derivation of stress equations


M
t
h
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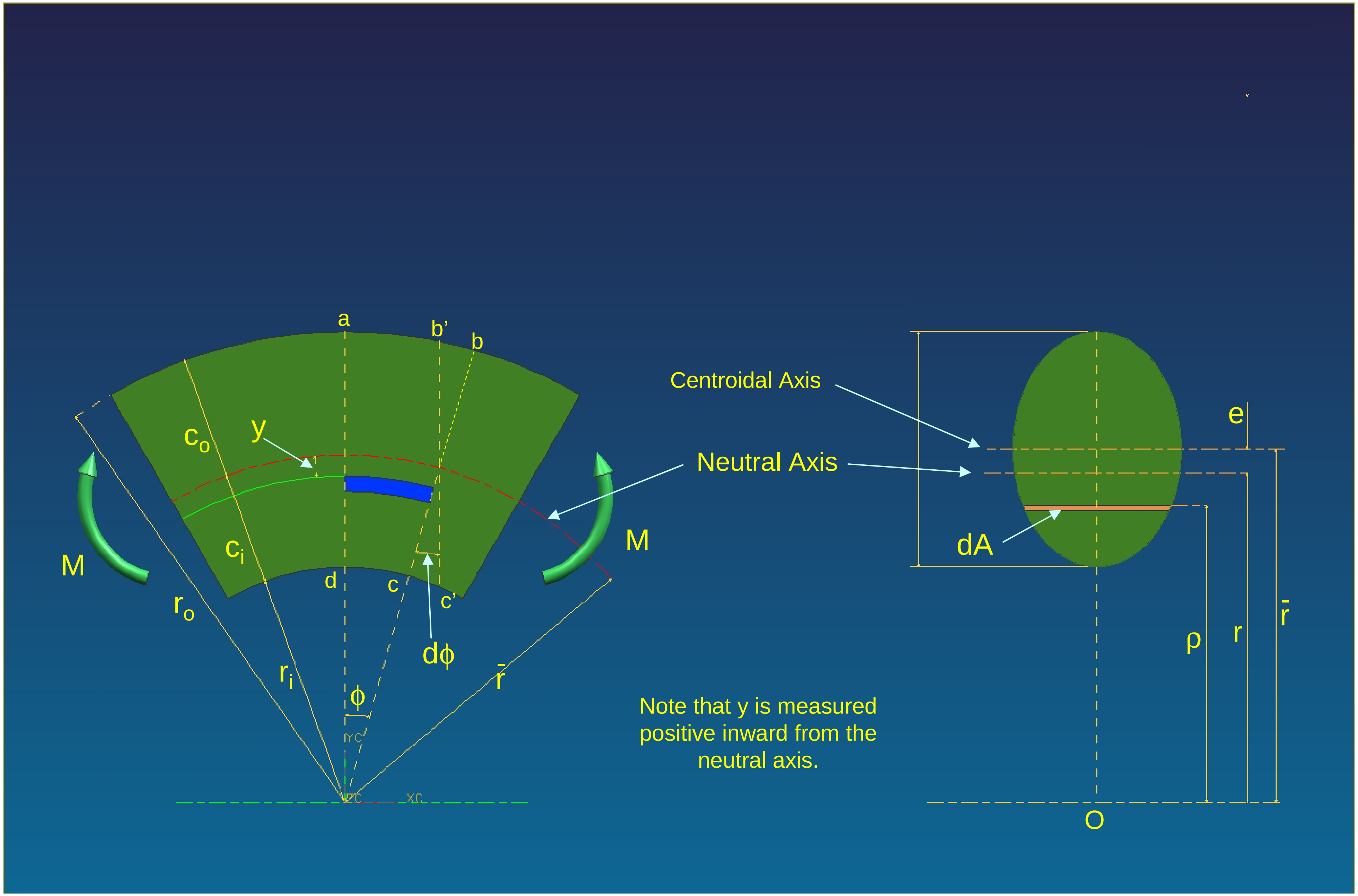

a
b'
b
Centroidal Axis
Neutral Axis
c_o
y
c_i
M
M
d
c
c'
r_o
r_i
dϕ
ϕ
$\bar{r}$
e
dA
ρ
r
$\bar{r}$
O
Note that y is measured positive inward from the neutral axis.

CURVED MEMBERS IN FLEXURE

The distribution of stress in a curved flexural member is determined by using the following assumptions.

1 The cross section has an axis of symmetry in a plane along the length of the beam.
2 Plane cross sections remain plane after bending.
3 The modulus of elasticity is the same in tension as in compression.

It will be found that the neutral axis and the centroidal axis of a curved beam, unlike a straight beam, are not coincident and also that the stress does not vary linearly from the neutral axis. The notation shown in the above figures is defined as follows:

r_o = radius of outer fiber
r_i = radius of inner fiber
h = depth of section
c_o = distance from neutral axis to outer fiber
c_i = distance from neutral axis to inner fiber
r = radius of neutral axis
$\bar{r}$ = radius of centroidal axis
e = distance from centroidal axis to neutral axis

To begin, we define the element *abcd* by the angle ϕ. A bending moment *M* causes section *bc* to rotate through $d\phi$ to *b'c'.* The strain on any fiber at distance ρ from the center *0* is

$$\varepsilon = \frac{\delta l}{l} = \frac{(r-\rho)\,d\phi}{\rho\phi}$$

The normal stress corresponding to this strain is

$$\sigma = \varepsilon E = \frac{E\left(r-\rho\right)d\phi}{\rho\phi} \tag{1}$$

Since there are no axial external forces acting on the beam, the sum of the normal forces acting on the section must be zero. Therefore

$$\int \sigma dA = E\frac{d\phi}{\phi}\int \frac{\left(r-\rho\right)dA}{\rho} = 0 \tag{2}$$

Now arrange Eq. (2) in the form

$$E\frac{d\phi}{\phi}\left(r\int \frac{dA}{\rho} - \int dA\right) = 0 \tag{3}$$

and solve the expression in parentheses. This gives

$$r\int \frac{dA}{\rho} - A = 0 \qquad \text{or} \qquad r = \frac{A}{\int \frac{dA}{\rho}} \tag{4}$$

This important equation is used to find the location of the neutral axis with respect to the center of curvature *0* of the cross section. The equation indicates that the neutral and the centroidal axes are not coincident.

Our next problem is to determine the stress distribution. We do this by balancing the external applied moment against the internal resisting moment. Thus, from Eq. (2),

$$\int (r-\rho)(\sigma dA) = E\frac{d\phi}{\phi}\int \frac{(r-\rho)^2 dA}{\rho} = M \tag{5}$$

Since $(r-\rho)^2 = r^2 - 2\rho r + \rho^2$, Eq. (5) can be written in the form

$$M = E\frac{d\phi}{\phi}\left(r^2 \int \frac{dA}{\rho} - r\int dA - r\int dA + \int \rho dA \right) \tag{6}$$

Note that *r* is a constant; then compare the first two terms in parentheses with Eq. (4). These terms vanish, and we have left

$$M = E\frac{d\phi}{\phi}\left(-r\int dA + \int \rho dA\right)$$

The first integral in this expression is the area *A,* and the second is the product *rA.* Therefore

$$M = E\frac{d\phi}{\phi}(\bar{r} - r)A = E\frac{d\phi}{\phi}eA$$

Now, using Eq. (1) once more, and rearranging, we finally obtain $\sigma = \dfrac{My}{Ae(r-y)}$

This equation shows that the stress distribution is hyperbolic. The algebraic *maximum* stresses occur at the inner and outer fibers and are

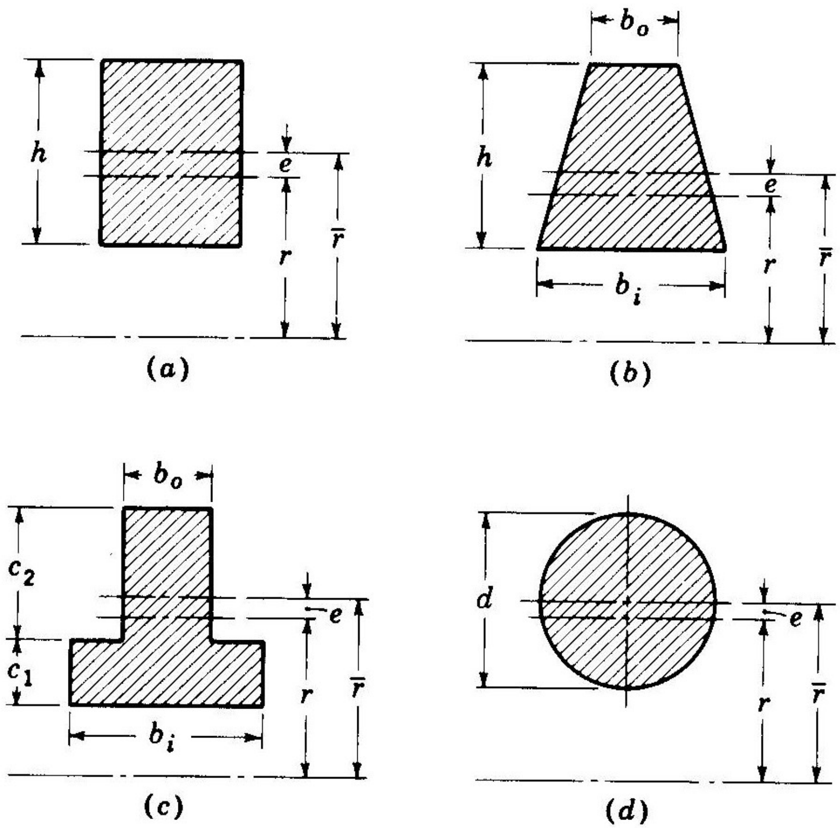
$$\sigma_i = \frac{Mc_i}{Aer_i} \qquad\qquad \sigma_o = \frac{Mc_o}{Aer_o} \qquad\qquad (7)$$

The sign convention used is that M is positive if it acts to straighten on the beam. The distance y is positive inwards to the center of curvature and is measured from the neutral axis. It follows that c_i is positive and c_o is negative.

These equations are valid for pure bending. In the usual and more general case such as a crane hook, the U frame of a press, or the frame of a clamp, the bending moment is due to forces acting to one side of the cross section under consideration. In this case the bending moment is computed about the *centroidal axis, not* the neutral axis. Also, an additional axial tensile (P/A) or compressive ($-P/A$) stress must be added to the bending stress given by Eq. (7) to obtain the resultant stress acting on the section.

Formulas for Some Common Sections

Sections most frequently encountered in the stress analysis of curved beams are shown below.



For the rectangular section shown in (a), the formulae are

$$\bar{r} = r_i + \frac{h}{2} \qquad and \qquad r = \frac{h}{\ln(r_o / r_i)}$$

For the trapezoidal section in (b), the formulae are

$$\bar{r} = r_i + \frac{h}{3}\frac{b_i + 2b_o}{b_i + b_o}$$

$$r = \frac{A}{b_o - b_i + [(b_i r_o - b_o r_i)/h]\ln(r_o / r_i)}$$

For the T section in we have

$$\bar{r} = r_i + \frac{b_i c_1^2 + 2b_o c_1 c_2 + b_o c_2^2}{2(b_o c_2 + b_i c_1)}$$

$$r = \frac{b_i c_1 + b_o c_2}{b_i \ln[(r_i + c_1)/r_i] + b_o \ln[r_o /(r_i + c_1)]}$$

The equations for the solid round section of Fig. *(d)* are

$$\bar{r} = r_i + \frac{d}{2}$$

$$r = \frac{d^2}{4\left(2\bar{r} - \sqrt{4\bar{r}^2 - d^2}\right)}$$