10. COMPRESSION AND BUCKLING

Whenever a structural member is designed, it is necessary that it satisfies specific strength, deflection and stability requirements. Typically strength (or in some cases fracture toughness) is used to determine failure, while assuming that the member will always be in static equilibrium. However, when certain structural members are subjected to compressive loads, they may either fail due to the compressive stress exceeding the yield strength (i.e., yielding) or they may fail due to lateral deflection (i.e., buckling).

The maximum axial load that a structural component (a.k.a., column) can support when it is on the verge of buckling is called the critical load, P_{cr}. Any additional load greater than P will cause the column to buckle and therefore deflect laterally. Buckling is a geometric instability and is related to material stiffness, column length, and column cross sectional dimensions. Strength does not play a role in buckling but does play a role in yielding.

Recall a slender column under load with a restoring force related to lateral deflection as shown in Fig. 10.1. Two moments are generated about the pinned base, O. The first moment is the restoring moment:

$$FL = K(\Delta x)L = K(L\theta)L = KL^2\theta \qquad (10.1)$$

where F is the restoring force, L is the length of the column, K is the spring constant, and θ is the angle. The second moment is the overturning moment:

$$P(\Delta x) = P(L\theta)L \qquad (10.2)$$

where P is the axial force.

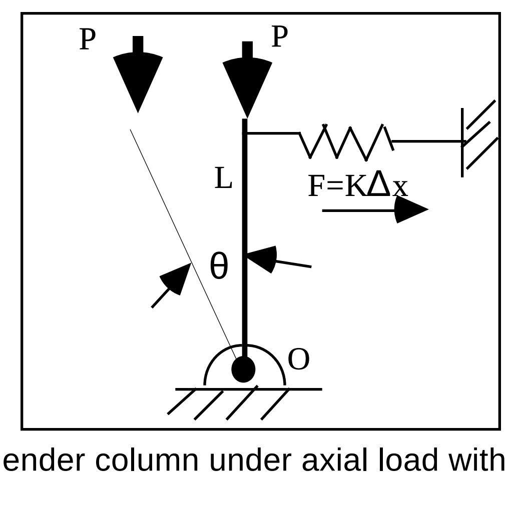

Figure 10.1 Model of a slender column under axial load with a restoring force/moment

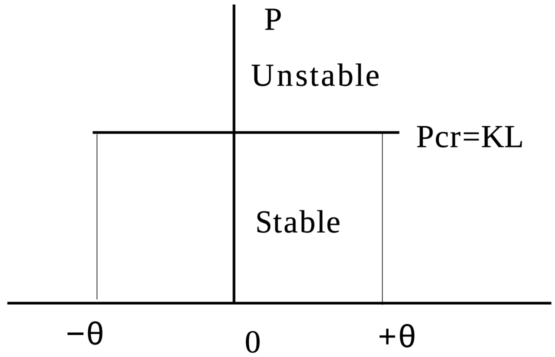

Figure 10.2 Illustration of stable and unstable conditions along with critical load

The following conditions apply as illustrated in Fig. 10.2:

If $PL\theta < K\theta L^2$ then P < KL and a stable condition results
If $PL\theta > K\theta L^2$ then P > KL and an unstable condition results (10.3)

The critical load separating stability and instability is

$$P_{cr} = KL \tag{10.4}$$

This situation can also be thought of as a ball on a surface as shown in Fig. 10.3.

The ideal column (pinned at both ends as shown in Fig. 10.4) can be modeled mathematically using the equation of the elastic curve. Note that assumptions for the ideal column include: i) column is initially perfectly straight, ii) load is applied through the center of the cross section, iii) material is homogeneous and linear elastic, and iv) column buckles and bends in a single plane. The elastic curve equation is:

$$EIv'' = -M \tag{10.5}$$

where E is the elastic modulus, I is the moment of inertia, $v''=\dfrac{d^2v}{dx^2}$, and M is the internal moment introduced by defection, v, and the applied load, P, such that M=Pv.

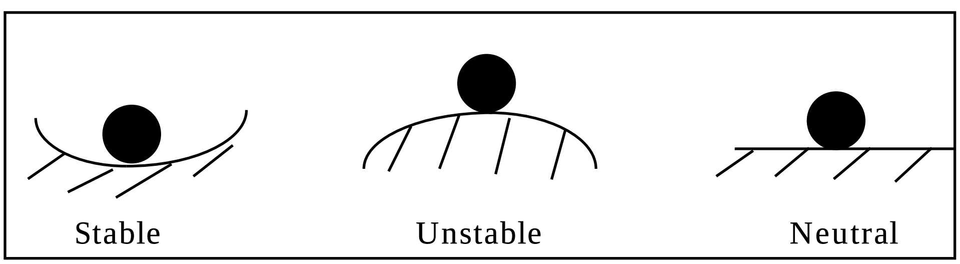

Figure 10.3 Illustration of stable, unstable and neutral conditions

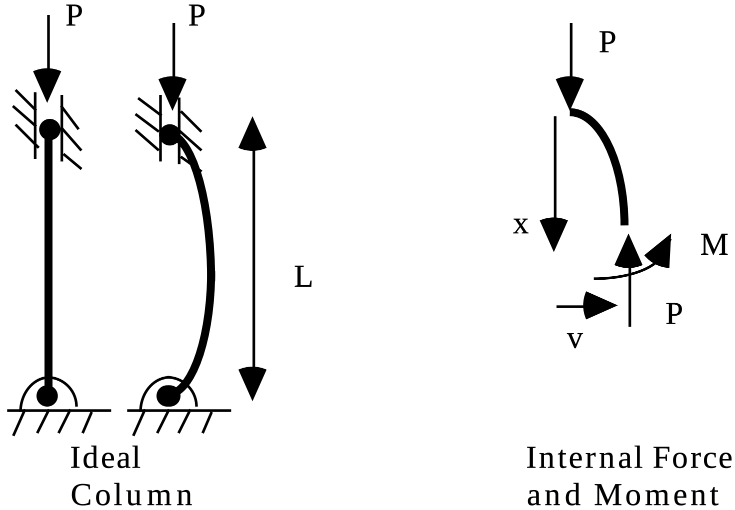

Figure 10.4 Ideal column and internal force/moment

Substituting this relation for M into Eq. 10.5 gives

$$EIv'' = -Pv \qquad v'' + \frac{P}{EI}v = 0 \tag{10.6}$$

Equation 10.6 is a homogeneous, second-order, linear differential equation with constant coefficients. The general solution for this equation is:

$$v = C_1 \sin\left(\sqrt{\frac{P}{EI}}x\right) + C_2 \cos\left(\sqrt{\frac{P}{EI}}x\right) \tag{10.6}$$

The constants of integration are found from the boundary conditions at the ends of the column. Since v=0 at x=0, then C_2=0. In addition, at x=L, v=0. In this case the trivial solution is that C_1=0 which means that the column is always straight. The other possibly is that $\sin\left(\sqrt{\frac{P}{EI}}L\right)=0$ which is satisfied if $\sqrt{\frac{P}{EI}}L = n\pi$. This solution can be rearranged such that

$$P = \frac{n^2\pi^2 EI}{L^2} \tag{10.7}$$

The smallest value of P (in other words, the value of P reached first when loading from P=0 to some critical load) is obtained when n=1, so that the critical load for the column (known as the Euler load) is:

$$P_{cr} = \frac{\pi^2 EI}{L^2} \tag{10.8}$$

Note that this critical load is independent of the strength of the material, but rather depends on the column dimensions (actual length, L, and smallest moment of inertia, I) and material stiffness, E. Note also that load-carrying ability increases as the moment of inertia increases, as elastic modulus increases but at length decreases. Thus, short, fat columns made of stiff material will have very low tendency to buckle. A key to understanding buckling is to create a design whereby the required (i.e., applied) design load is always much less than the theoretical buckling load. The reasons for this are that the theoretical buckling load is determined based on an ideal column with many inherent assumptions. Thus, a large safety margin must be placed between the design load and the calculated critical buckling load.

Equation 10.8 is the solution for an ideal column with pinned/pinned end conditions. If other boundary conditions are applied then the following critical load equations result for the indicated end conditions.

$$
\begin{aligned}
P_{cr} &= \frac{\pi^2 EI}{L^2} && \text{Pinned} - \text{Pinned} \\
P_{cr} &= \frac{\pi^2 EI}{4L^2} && \text{Free} - \text{Fixed} \\
P_{cr} &= \frac{4\pi^2 EI}{L^2} && \text{Fixed} - \text{Fixed} \\
P_{cr} &= \frac{2.046\pi^2 EI}{L^2} && \text{Pinned} - \text{Fixed}
\end{aligned}
\tag{10.9}
$$

The equations in Eq. 10.9 are a bit cumbersome because four different relations need to be remembered. So instead an effective length is defined such that $L_e = KL$. Now a single equation can be used to replace the four equations of Eq. 10.9 and effective length constants, K, for each end condition can be used to define the effective length. The effective length constants are as shown in Fig. 10.5 and the single equation is:

$$P_{cr} = \frac{\pi^2 EI}{L_e^2} = \frac{\pi^2 EI}{(KL)^2} \tag{10.10}$$

For purposes of design, it is more useful to express the critical loading condition in terms of a stress such that:

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{\pi^2 EI}{AL_e^2} \qquad \sigma_{cr} = \frac{\pi^2 E}{(L_e/k)^2} \tag{10.11}$$

where $k=\sqrt{I/A}$ is the smallest radius of gyration determined from the least moment of inertia, I, for the cross section. The term, L_e/k is known as the slenderness ratio and contains information about the length and the cross section.

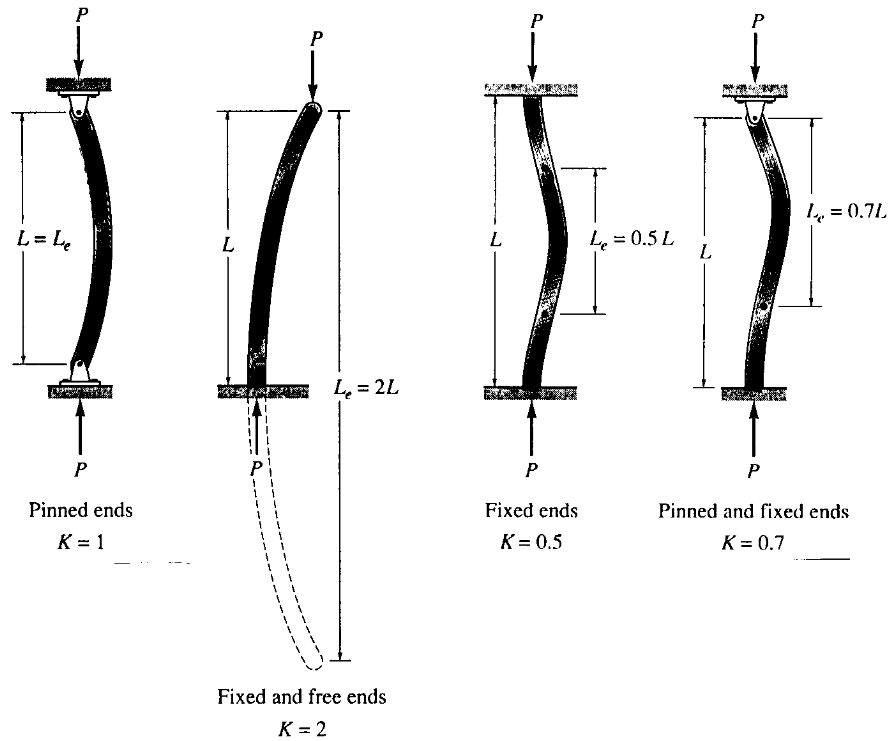

Figure 10.5 Effective length constants for various end conditions

It is interesting to investigate the case where the applied stress, σ, is equal to the critical buckling stress, σ_{cr}, and the generalized yield strength, σ_o. At this point there is a transition between yield and buckling:

$$\sigma = \sigma_o = \sigma_{cr} \qquad \sigma_o = \frac{\pi^2 E}{(L_e / k)^2} \tag{10.12}$$

If Eq. 10.12 is solved for L_e / k, the resulting relation marks the combination of length and cross section at which the compressive behaviour transitions from yielding to buckling. This relation is known as the minimum slenderness ratio:

$$\left.\frac{L_e}{k}\right|_{min} = \sqrt{\frac{\pi^2 E}{\sigma_o}} \tag{10.13}$$

This transition can be illustrated by plotting the relation between stress, σ, and slenderness ratio, L_e / k, as shown in Fig. 10.6. Note that in reality there is no sharply divided transition between yielding and buckling. Instead the σ vs. L_e / k curve can be divided into three regions as shown in Fig. 10.7. Region I is the short-column region in which general yielding occurs when $\sigma = \sigma_o$. Region II is an intermediate-column region in which i) the column may yield or may buckle and ii) empirical relations are used to approximate the resulting curve. Region III is for long columns and buckling will occur.

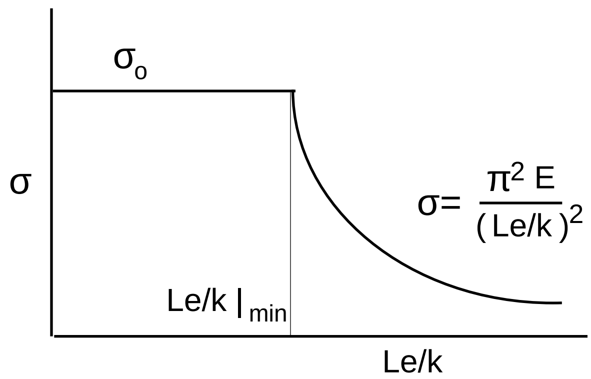

Figure 10.6 Stress vs. slenderness ratio relation

Regions I and III are straight forward to determine (either apply $\sigma = \sigma_o$. in Region I or $\sigma_{cr} = \dfrac{\pi^2 E}{(L_e/k)^2}$ in Region III). Region II is material/ geometry/loading dependent and empirical relations are often used. One approach is to fit a parabola to the σ vs. L_e/k curve from $\sigma = \sigma_o$. to $\sigma = \sigma_o/2$ such that.

$$\sigma = \sigma_o \left[1 - \frac{(L_e/k)^2}{\left(2 L_e/k|_{min}\right)^2} \right] \quad \text{for} \quad 0 \le \frac{L_e}{k} \le \left.\frac{L_e}{k}\right|_{min}$$

and (10.14)

$$\sigma = \frac{\pi^2 E}{(L_e/k)^2} \qquad \text{for} \quad \frac{L_e}{k} > \left.\frac{L_e}{k}\right|_{min}$$

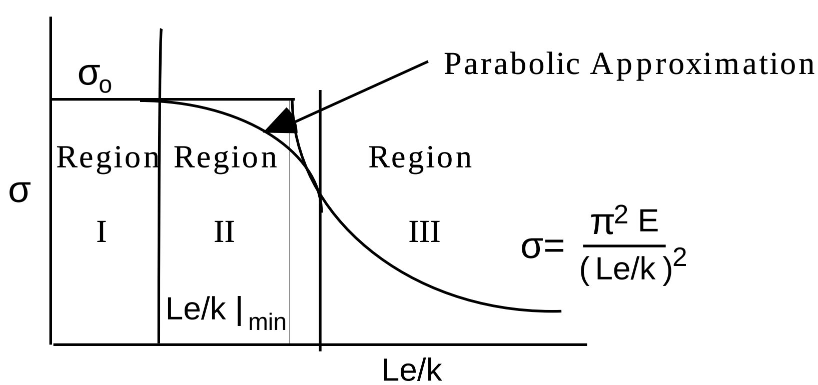

Figure 10.7 Stress vs. slenderness ratio relation with parabolic fit in Region II

In summary, note that buckling loads and stresses are sensitive to the following

1) stiffness of material
2) length of column
3) cross section dimensions
4) cross sectional shape
5) end conditions
6) initial eccentricity
7) eccentric loading
8) end conditions

Thus, the moral of the story is to always keep design loads well below the calculated critical buckling loads.