Beam deflections using discontinuity functions.

Consider the Heaviside or step function H(x) given by:

$H(x-a) = 0$ for $x \le a$ and $H(x-a)=1$ for $x > a$

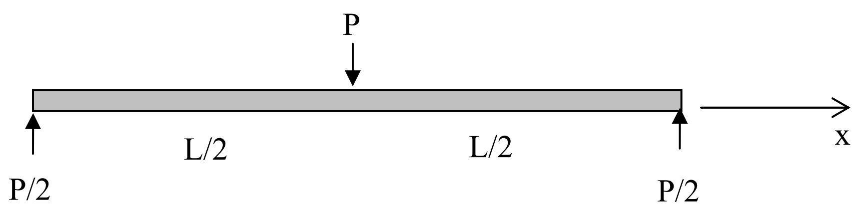
If we apply this to another function, it acts like a switch, causing that function to turn on at $x=a$. For example, $H(x-a)*(x-a)^n = 0$ for $x \le a$ and $= (x-a)^n$ for $x > a$

We can write this combined function as $f(x) = \langle x-a \rangle^n$

Note also that integration and differentiation work as usual:

$$\frac{d}{dx}\langle x-a\rangle^n = n\langle x-a\rangle^{n-1} \quad \text{and} \quad \int \langle x-a\rangle^n dx = \frac{\langle x-a\rangle^{n+1}}{n+1} + C$$

Now consider a simply supported beam of length L with a single central load P.



For $0 \le x \le L/2$ the bending moment is $M(x) = Px/2$

For $L/2 \le x \le L$ the bending moment is $M(x) = Px/2 - P(x - L/2)$

We can replace these two equations with one equation if we use discontinuity functions:

$$M(x) = \frac{P\langle x-0\rangle^1}{2} - P\langle x - L/2\rangle^1$$

Note that this equation has values corresponding to the two above equations in the corresponding ranges of x.

The differential equation for beam deflections is $EIv'' = -M(x)$, giving

$$EIv'' = -\frac{P\langle x-0\rangle^1}{2} + P\langle x-L/2\rangle^1 \text{ and integrating twice we get,}$$

$$EIv' = -\frac{P\langle x-0\rangle^2}{4} + \frac{P\langle x-L/2\rangle^2}{2} + C_1$$

$$EIv = -\frac{P\langle x-0\rangle^3}{12} + \frac{P\langle x-L/2\rangle^3}{6} + C_1 x + C_2$$

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Using the boundary conditions solves for C_1 and C_2:

$v(0) = 0$ gives $0 = 0 + 0 + 0 + C_2$ and thus $C_2 = 0$

$v(L) = 0$ gives $0 = -\frac{PL^3}{12} + \frac{PL^3}{48} + C_1 L$ and, thus $C_1 = \frac{PL^2}{16}$

The final deflection equation is then:

$$EIv = -\frac{P\langle x-0\rangle^3}{12} + \frac{P\langle x-L/2\rangle^3}{6} + \frac{PL^2 x}{16}$$

To find the maximum deflection use $EIv' = 0$ as usual but note that you must check each section of the beam separately to locate possible extrema in each section. This is because you can only solve an algebraic equation for the x-values of the extrema, not an equation in discontinuous form. Whenever you substitute actual values of x in the above equation it becomes an algebraic equation valid only for the range of x between two adjacent discontinuities.

Check values in the range $0 \le x \le L/2$: we have,

$EIv' = 0 = -\frac{P\langle x-0\rangle^2}{4} + \frac{P\langle x-L/2\rangle^2}{2} + \frac{PL^2}{16}$ and for x-values in this range, this becomes

$0 = Px^2/4 + 0 + PL^2/16$ giving x=L/2 as the location of the maximum deflection. Substituting this in the deflection equation gives:

$$EIv_{\max} = -\frac{PL^3}{96} + 0 + \frac{PL^3}{32} = \frac{PL^3}{48}$$

Discontinuity functions of bending moment due to various loads.

1. Point load P applied at x=a: $M(x) = P\langle x-a\rangle^1$

2. Uniform load of w (lb/ft or N/m) applied over whole length of beam:

$$M(x) = \frac{w\langle x-0\rangle^2}{2}$$

3. Uniform load of w (lb/ft or N/m) starting at x = a and continuing to end of beam:

$$M(x) = \frac{w\langle x-a\rangle^2}{2}$$

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4. Uniform load of w (lb/ft or N/m) starting at x = a and continuing only to x = b :

$$M(x) = \frac{w\langle x-a \rangle^2}{2} - \frac{w\langle x-b \rangle^2}{2}$$

5. Point couple C applied at x = a :

$$M(x) = C\langle x-a \rangle^0$$