UNIVERSITY of WASHINGTON
COLLEGE of ENGINEERING

Mechanical Engineering

Summary of Classical Lamination Theory (CLT) Calculations

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Numerical examples illustrating discussion in:

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Section 6.8.1: A CLT Analysis When Loads Are Known

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Numerical examples illustrating discussion in:

Section 6.8.1: A CLT Analysis When Loads Are Known

Section 6.8.2: A CLT Analysis When Midplane Strains and Curvatures are Known

(Sections 6.8.1 and 6.8.2 are nearly identical…)

Summary of Classical Lamination Theory (CLT) Calculations

Numerical examples illustrating discussion in:

Section 6.8.1: A CLT Analysis When Loads Are Known

Section 6.8.2: A CLT Analysis When Midplane Strains and Curvatures are Known

(Sections 6.8.1 and 6.8.2 are nearly identical…)

Section 6.8.1: A CLT Analysis When Loads Are Known

1. Define the problem:
   a) Specify number of different materials used
   b) Specify properties for each material

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Example: Suppose two materials are used…graphite-epoxy and glass epoxy. From Table 3.1, typical properties are

Mat'l Name	Mat'l Number	E_{11} (psi)	E_{22} (psi)	ν_{12}	G_{12} (psi)	α_{11} (in/in-°F)	α_{22} (in/in-°F)	Ply thick (in)
Gr/Ep	1	25×10^6	1.5×10^6	0.30	1.9×10^6	-0.5×10^{-6}	15×10^{-6}	0.005
Gl/Ep	2	8.0×10^6	2.3×10^6	0.28	1.1×10^6	3.7×10^{-6}	14×10^{-6}	0.005

Section 6.8.1: A CLT Analysis When Loads Are Known

1. Define the problem:
   a) Specify number of different materials used
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Example:

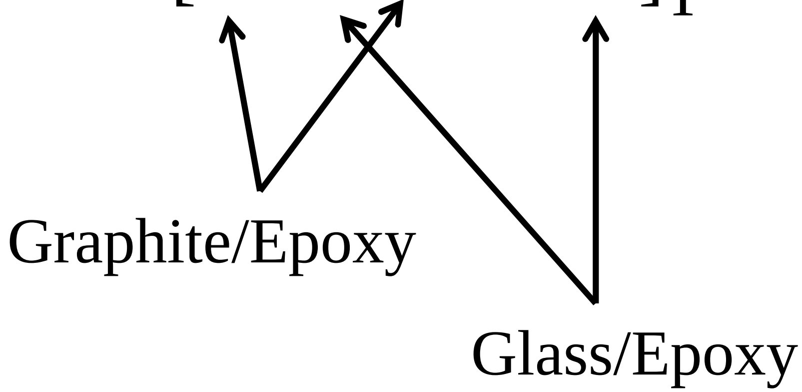
$[0/30/90/-30]_T$ ⟵ A 4-ply laminate. This description is adequate if same material is used for all plies.

Section 6.8.1: A CLT Analysis When Loads Are Known

1. Define the problem:
    a) Specify number of different materials used
    b) Specify properties for each material
    c) Specify laminate description

Example:

$[0/30/90/-30]_T$ ← A 4-ply laminate. This description is adequate if same material is used for all plies. For illustrative purposes assume Gr/Ep used for 0º and 90º plies and Gl/Ep used for ±30º plies



Section 6.8.1: A CLT Analysis When Loads Are Known

1. Define the problem:
   a) Specify number of different materials used
   b) Specify properties for each material
   c) Specify laminate description

Example:

$$\text{Total laminate thickness} = 4(0.005in) = 0.020in$$

$$z_0 = -t/2 = -(.020in)/2 = -0.010in$$

$$z_1 = z_0 + t_1 = -0.01 + 0.005m = -0.005in$$

$$z_2 = z_1 + t_2 = -0.005 + 0.005in = 0.000in$$

$$z_3 = z_2 + t_3 = -0.000 + 0.005in = 0.005in$$

$$z_4 = z_3 + t_4 = 0.005 + 0.005in = 0.010in$$

Section 6.8.1: A CLT Analysis When Loads Are Known

1. Define the problem:
   a) Specify number of different materials used
   b) Specify properties for each material
   c) Specify laminate description
   d) Specify mechanical and thermal loads

Section 6.8.1: A CLT Analysis When Loads Are Known

1. Define the problem:
   a) Specify number of different materials used
   b) Specify properties for each material
   c) Specify laminate description
   d) Specify mechanical and thermal loads

Example:

$$N_{xx} = 520 \text{ lbf/in} \qquad M_{xx} = -4.0 \text{ lbf} - \text{in/in}$$

$$N_{yy} = 377 \text{ lbf/in} \qquad M_{yy} = 0.22 \text{ lbf} - \text{in/in}$$

$$N_{xy} = 64.4 \text{ lbf/in} \qquad M_{xy} = -0.0854 \text{ lbf} - \text{in/in}$$

$$T_{cure} = 350°F \quad T_{service} = 75°F \quad \Rightarrow \quad \Delta T = -275°F$$

Section 6.8.1: A CLT Analysis When Loads Are Known

2. Calculate the [*ABD*] matrix:

Section 6.8.1: A CLT Analysis When Loads Are Known

2. Calculate the [*ABD*] matrix:
   a) Calculate the [*Q*] matrix for each material

Section 6.8.1: A CLT Analysis When Loads Are Known

2. Calculate the [ABD] matrix:
    a) Calculate the [Q] matrix for each material

$$[Q]=\begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix}=\begin{bmatrix} \left(\dfrac{E_{11}^2}{E_{11}-\nu_{12}^2 E_{22}}\right) & \left(\dfrac{\nu_{12}E_{11}E_{22}}{E_{11}-\nu_{12}^2 E_{22}}\right) & (0) \\ \left(\dfrac{\nu_{12}E_{11}E_{22}}{E_{11}-\nu_{12}^2 E_{22}}\right) & \left(\dfrac{E_{11}E_{22}}{E_{11}-\nu_{12}^2 E_{22}}\right) & (0) \\ (0) & (0) & (G_{12}) \end{bmatrix}$$

Section 6.8.1: A CLT Analysis When Loads Are Known

2. Calculate the [*ABD*] matrix:
   a) Calculate the [*Q*] matrix for each material

Example:

$$[Q]_{Gr/Ep} = \begin{bmatrix} 25.14x10^6 & 0.452x10^6 & 0 \\ 0.452x10^6 & 1.508x10^6 & 0 \\ 0 & 0 & 1.90x10^6 \end{bmatrix} psi$$

$$[Q]_{Gl/Ep} = \begin{bmatrix} 8.184x10^6 & 0.659x10^6 & 0 \\ 0.659x10^6 & 2.353x10^6 & 0 \\ 0 & 0 & 1.10x10^6 \end{bmatrix} psi$$

Section 6.8.1: A CLT Analysis When Loads Are Known

2. Calculate the [*ABD*] matrix:
   a) Calculate the [*Q*] matrix for each material
   b) Calculate the $[\bar{Q}]$ matrix for each ply

Section 6.8.1: A CLT Analysis When Loads Are Known

2. Calculate the [*ABD*] matrix:

a) Calculate the [*Q*] matrix for each material

b) Calculate the $[\overline{Q}]$ matrix for each ply

$$[\overline{Q}] = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix}$$

Section 6.8.1: A CLT Analysis When Loads Are Known

2. Calculate the [*ABD*] matrix:

   a) Calculate the [*Q*] matrix for each material

   b) Calculate the $[\overline{Q}]$ matrix for each ply

$$\overline{Q}_{11} = Q_{11}\cos^4\theta + 2(Q_{12} + 2Q_{66})\cos^2\theta\sin^2\theta + Q_{22}\sin^4\theta$$

$$\overline{Q}_{12} = \overline{Q}_{21} = Q_{12}(\cos^4\theta + \sin^4\theta) + (Q_{11} + Q_{22} - 4Q_{66})\cos^2\theta\sin^2\theta$$

$$\overline{Q}_{16} = \overline{Q}_{61} = (Q_{11} - Q_{12} - 2Q_{66})\cos^3\theta\sin\theta - (Q_{22} - Q_{12} - 2Q_{66})\cos\theta\sin^3\theta$$

$$\overline{Q}_{22} = Q_{11}\sin^4\theta + 2(Q_{12} + 2Q_{66})\cos^2\theta\sin^2\theta + Q_{22}\cos^4\theta$$

$$\overline{Q}_{26} = \overline{Q}_{62} = (Q_{11} - Q_{12} - 2Q_{66})\cos\theta\sin^3\theta - (Q_{22} - Q_{12} - 2Q_{66})\cos^3\theta\sin\theta$$

$$\overline{Q}_{66} = (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})\cos^2\theta\sin^2\theta + Q_{66}(\cos^4\theta + \sin^4\theta)$$

Section 6.8.1: A CLT Analysis When Loads Are Known

2. Calculate the [*ABD*] matrix:

a) Calculate the [*Q*] matrix for each material

b) Calculate the $[\overline{Q}]$ matrix for each ply

Example: For ply 1,

$$[\overline{Q}]^{0^\circ ply}_{Gr/Ep} = \begin{bmatrix} 25.14x10^6 & 0.452x10^6 & 0 \\ 0.452x10^6 & 1.508x10^6 & 0 \\ 0 & 0 & 1.90x10^6 \end{bmatrix} psi$$

Section 6.8.1: A CLT Analysis When Loads Are Known

2. Calculate the [*ABD*] matrix:
   a) Calculate the [*Q*] matrix for each material
   b) Calculate the $[\overline{Q}]$ matrix for each ply

Example: For ply 2,

$$[\overline{Q}]^{30^\circ ply}_{Gl/Ep} = \begin{bmatrix} 5.823x10^6 & 1.563x10^6 & 1.784x10^6 \\ 1.563x10^6 & 2.907x10^6 & 0.741x10^6 \\ 1.784x10^6 & 0.741x10^6 & 2.00x10^6 \end{bmatrix} psi$$

Section 6.8.1: A CLT Analysis When Loads Are Known

2. Calculate the [*ABD*] matrix:
   a) Calculate the [*Q*] matrix for each material
   b) Calculate the $[\bar{Q}]$ matrix for each ply

Example: For ply 3,

$$[\bar{Q}]^{90^\circ ply}_{Gr/Ep} = \begin{bmatrix} 1.508x10^6 & 0.452x10^6 & 0 \\ 0.452x10^6 & 25.14x10^6 & 0 \\ 0 & 0 & 1.90x10^6 \end{bmatrix} psi$$

Section 6.8.1: A CLT Analysis When Loads Are Known

2. Calculate the [*ABD*] matrix:

 a) Calculate the [*Q*] matrix for each material

 b) Calculate the $[\overline{Q}]$ matrix for each ply

Example: For ply 4,

$$[\overline{Q}]_{Gl/Ep}^{-30^\circ\,ply} = \begin{bmatrix} 5.823x10^6 & 1.563x10^6 & -1.784x10^6 \\ 1.563x10^6 & 2.907x10^6 & -0.741x10^6 \\ -1.784x10^6 & -0.741x10^6 & 2.00x10^6 \end{bmatrix} psi$$

Section 6.8.1: A CLT Analysis When Loads Are Known

2. Calculate the [*ABD*] matrix:
   a) Calculate the [*Q*] matrix for each material
   b) Calculate the $[\bar{Q}]$ matrix for each ply
   c) Calculate the $[A_{ij}]$, $[B_{ij}]$, and $[D_{ij}]$ matrices

Section 6.8.1: A CLT Analysis When Loads Are Known

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   a) Calculate the [*Q*] matrix for each material

   b) Calculate the $[\overline{Q}]$ matrix for each ply

   c) Calculate the $[A_{ij}]$, $[B_{ij}]$, and $[D_{ij}]$ matrices

Example:

$$A_{ij} = \sum_{k=1}^{n} \left\{\overline{Q}_{ij}\right\}_k (z_k - z_{k-1}) = \begin{bmatrix} 191.4x10^3 & 20.15x10^3 & 0 \\ 20.15x10^3 & 162.3x10^3 & 0 \\ 0 & 0 & 39.04x10^3 \end{bmatrix} \frac{lbf}{in}$$

Section 6.8.1: A CLT Analysis When Loads Are Known

2. Calculate the [*ABD*] matrix:
   a) Calculate the [*Q*] matrix for each material
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   c) Calculate the $[A_{ij}]$, $[B_{ij}]$, and $[D_{ij}]$ matrices

Example:

$$B_{ij} = \frac{1}{2}\sum_{k=1}^{n}\left\{\overline{Q}_{ij}\right\}_k (z_k^2 - z_{k-1}^2) = \begin{bmatrix} -778 & 27.8 & -89.2 \\ 27.8 & 330 & -37.0 \\ -89.2 & -37.0 & 2.59 \end{bmatrix} lbf$$

Section 6.8.1: A CLT Analysis When Loads Are Known

2. Calculate the [*ABD*] matrix:

   a) Calculate the [*Q*] matrix for each material

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   c) Calculate the $[A_{ij}]$, $[B_{ij}]$, and $[D_{ij}]$ matrices

Example:

$$D_{ij} = \frac{1}{3}\sum_{k=1}^{n}\left\{\overline{Q}_{ij}\right\}_k (z_k^3 - z_{k-1}^3) = \begin{bmatrix} 9.34 & 0.672 & -0.446 \\ 0.672 & 2.46 & -0.185 \\ -0.446 & -0.185 & 1.30 \end{bmatrix} lbf-in$$

Section 6.8.1: A CLT Analysis When Loads Are Known

2. Calculate the [*ABD*] matrix:
   a) Calculate the [*Q*] matrix for each material
   b) Calculate the $[\bar{Q}]$ matrix for each ply
   c) Calculate the $[A_{ij}]$, $[B_{ij}]$, and $[D_{ij}]$ matrices
   d) Assemble the [*ABD*] matrix

Section 6.8.1: A CLT Analysis When Loads Are Known

2. Calculate the [*ABD*] matrix:

   a) Calculate the [*Q*] matrix for each material

   b) Calculate the $[\bar{Q}]$ matrix for each ply

   c) Calculate the $[A_{ij}]$, $[B_{ij}]$, and $[D_{ij}]$ matrices

   d) Assemble the [*ABD*] matrix

$$[ABD]=\begin{bmatrix} A_{ij} & B_{ij} \\ B_{ij} & D_{ij} \end{bmatrix}=\begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix}$$

Section 6.8.1: A CLT Analysis When Loads Are Known

2. Calculate the [*ABD*] matrix:
   - a) Calculate the [*Q*] matrix for each material
   - b) Calculate the $[\bar{Q}]$ matrix for each ply
   - c) Calculate the [A_{ij}], [B_{ij}], and [D_{ij}] matrices
   - d) Assemble the [*ABD*] matrix

Example:

$$[ABD] = \begin{bmatrix} 191x10^3 & 20.1x10^3 & 0 & -778 & 27.8 & -89.2 \\ 20.1x10^3 & 162x10^3 & 0 & 27.8 & 330 & -37.0 \\ 0 & 0 & 39.0x10^3 & -89.2 & -37.0 & 2.59 \\ -778 & -27.8 & -89.2 & 9.34 & 0.672 & -0.446 \\ -27.8 & 330 & -37.0 & 0.672 & 2.46 & -0.185 \\ -89.2 & -37.0 & 2.59 & -0.446 & -0.185 & 1.30 \end{bmatrix}$$

Section 6.8.1: A CLT Analysis When Loads Are Known

3. Calculate the $[abd] = [ABD]^{-1}$ matrix:

Section 6.8.1: A CLT Analysis When Loads Are Known

3. Calculate the $[abd] = [ABD]^{-1}$ matrix:

$$[abd] = [ABD]^{-1} = \begin{bmatrix} a_{11} & a_{12} & a_{16} & b_{11} & b_{12} & b_{16} \\ a_{12} & a_{22} & a_{26} & b_{21} & b_{22} & b_{26} \\ a_{16} & a_{26} & a_{66} & b_{61} & b_{62} & b_{66} \\ b_{11} & b_{21} & b_{61} & d_{11} & d_{12} & d_{16} \\ b_{12} & b_{22} & b_{62} & d_{12} & d_{22} & d_{26} \\ b_{16} & b_{26} & b_{66} & d_{16} & d_{26} & d_{66} \end{bmatrix}$$

Section 6.8.1: A CLT Analysis When Loads Are Known

3. Calculate the $[abd] = [ABD]^{-1}$ matrix:

Example:

$$[abd] = \begin{bmatrix} 9.09x10^{-6} & -0.811x10^{-6} & 0 & 0.827x10^{-3} & -0.129x10^{-3} & 0.862x10^{-3} \\ -0.811x10^{-6} & 8.64x10^{-6} & 0 & -0.196x10^{-4} & -1.16x10^{-3} & 0.203x10^{-4} \\ 0 & 0 & 26.9x10^{-6} & 0.381x10^{-3} & 0.454x10^{-3} & 0.226x10^{-3} \\ 0.827x10^{-3} & -0.196x10^{-4} & 0.381x10^{-3} & 0.188 & -0.044 & 0.114 \\ -0.129x10^{-3} & -1.16x10^{-3} & 0.454x10^{-3} & -0.044 & 0.586 & 0.025 \\ 0.862x10^{-3} & 0.203x10^{-4} & 0.226x10^{-3} & 0.114 & 0.025 & 0.870 \end{bmatrix}$$

Section 6.8.1: A CLT Analysis When Loads Are Known

4. Calculate thermal stress and moment resultants:

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   a) Calculate effective thermal expansion coefficients for each ply

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   a) Calculate effective thermal expansion coefficients for each ply

$$\alpha_{xx} = \alpha_{11}\cos^2(\theta) + \alpha_{22}\sin^2(\theta)$$

$$\alpha_{yy} = \alpha_{11}\sin^2(\theta) + \alpha_{22}\cos^2(\theta)$$

$$\alpha_{xy} = 2\cos(\theta)\sin(\theta)(\alpha_{11} - \alpha_{22})$$

Section 6.8.1: A CLT Analysis When Loads Are Known

4. Calculate thermal stress and moment resultants:
   a) Calculate effective thermal expansion coefficients for each ply

Example:

Ply Number	Mat'l Number	Fiber angle (deg)	α_{xx} (in/in- ºF)	α_{yy} (in/in- ºF)	α_{xy} (in/in- ºF)
1	1	0	$-0.5x10^{-6}$	$15\ x10^{-6}$	0
2	2	30	$6.28\ x10^{-6}$	$11.4\ x\ 10^{-6}$	$-8.92\ x\ 10^{-6}$
3	1	90	$15\ x10^{-6}$	$-0.5x10^{-6}$	0
4	2	-30	$6.28\ x10^{-6}$	$11.4\ x\ 10^{-6}$	$8.92\ x\ 10^{-6}$

Section 6.8.1: A CLT Analysis When Loads Are Known

4. Calculate thermal stress and moment resultants:
   a) Calculate effective thermal expansion coefficients for each ply
   b) Calculate thermal stress & moment resultants

$$N_{xx}^{T} \equiv \Delta T \sum_{k=1}^{n} \left\{ \left[\overline{Q}_{11}\alpha_{xx} + \overline{Q}_{12}\alpha_{yy} + \overline{Q}_{16}\alpha_{xy} \right]_k \left[z_k - z_{k-1} \right] \right\}$$

$$N_{yy}^{T} \equiv \Delta T \sum_{k=1}^{n} \left\{ \left[\overline{Q}_{12}\alpha_{xx} + \overline{Q}_{22}\alpha_{yy} + \overline{Q}_{26}\alpha_{xy} \right]_k \left[z_k - z_{k-1} \right] \right\}$$

$$N_{xy}^{T} \equiv \Delta T \sum_{k=1}^{n} \left\{ \left[\overline{Q}_{16}\alpha_{xx} + \overline{Q}_{26}\alpha_{yy} + \overline{Q}_{66}\alpha_{xy} \right]_k \left[z_k - z_{k-1} \right] \right\}$$

Section 6.8.1: A CLT Analysis When Loads Are Known

4. Calculate thermal stress and moment resultants:
   a) Calculate effective thermal expansion coefficients for each ply
   b) Calculate thermal stress & moment resultants

$$M_{xx}^{T} \equiv \frac{\Delta T}{2} \sum_{k=1}^{n} \left\{ \left[\overline{Q}_{11}\alpha_{xx} + \overline{Q}_{12}\alpha_{yy} + \overline{Q}_{16}\alpha_{xy} \right]_k \left[z_k^2 - z_{k-1}^2 \right] \right\}$$

$$M_{yy}^{T} \equiv \frac{\Delta T}{2} \sum_{k=1}^{n} \left\{ \left[\overline{Q}_{12}\alpha_{xx} + \overline{Q}_{22}\alpha_{yy} + \overline{Q}_{26}\alpha_{xy} \right]_k \left[z_k^2 - z_{k-1}^2 \right] \right\}$$

$$M_{xy}^{T} \equiv \frac{\Delta T}{2} \sum_{k=1}^{n} \left\{ \left[\overline{Q}_{16}\alpha_{xx} + \overline{Q}_{26}\alpha_{yy} + \overline{Q}_{66}\alpha_{xy} \right]_k \left[z_k^2 - z_{k-1}^2 \right] \right\}$$

Section 6.8.1: A CLT Analysis When Loads Are Known

4. Calculate thermal stress and moment resultants:
   a) Calculate effective thermal expansion coefficients for each ply
   b) Calculate thermal stress & moment resultants

Example:

$$N_{xx}^{T} = -129 lbf/in \qquad M_{xx}^{T} = -0.401\, lbf-in/in$$

$$N_{yy}^{T} = -123 lbf/in \qquad M_{yy}^{T} = 0.0005\, lbf-in/in$$

$$N_{xy}^{T} = 0 \qquad M_{xy}^{T} = -0.0246\, lbf-in/in$$

Section 6.8.1: A CLT Analysis When Loads Are Known

5. Calculate midplane strains and curvatures

Section 6.8.1: A CLT Analysis When Loads Are Known

5. Calculate midplane strains and curvatures

$$\begin{Bmatrix} \varepsilon_{xx}^{o} \\ \varepsilon_{yy}^{o} \\ \gamma_{xy}^{o} \\ \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{16} & b_{11} & b_{12} & b_{16} \\ a_{12} & a_{22} & a_{26} & b_{21} & b_{22} & b_{26} \\ a_{16} & a_{26} & a_{66} & b_{61} & b_{62} & b_{66} \\ b_{11} & b_{21} & b_{61} & d_{11} & d_{12} & d_{16} \\ b_{12} & b_{22} & b_{62} & d_{12} & d_{22} & d_{26} \\ b_{16} & b_{26} & b_{66} & d_{16} & d_{26} & d_{66} \end{bmatrix} \begin{Bmatrix} N_{xx} + N_{xx}^{T} \\ N_{yy} + N_{yy}^{T} \\ N_{xy} + N_{xy}^{T} \\ M_{xx} + M_{xx}^{T} \\ M_{yy} + M_{yy}^{T} \\ M_{xy} + M_{xy}^{T} \end{Bmatrix}$$

Section 6.8.1: A CLT Analysis When Loads Are Known

5. Calculate midplane strains and curvatures

Example:

$$\begin{Bmatrix} \varepsilon_{xx}^{o} \\ \varepsilon_{yy}^{o} \\ \gamma_{xy}^{o} \\ \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -1300x10^{-6}\ in/in \\ 900x10^{-6}\ \mathrm{rad} \\ -0.50\ in^{-1} \\ 0.40\ in^{-1} \\ -0.20\ in^{-1} \end{Bmatrix}$$

Section 6.8.1: A CLT Analysis When Loads Are Known

6. For each ply:

a) Calculate ply strains in *x*-*y* coordinate system (usually at ply interfaces)

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon^{o}_{xx} \\ \varepsilon^{o}_{yy} \\ \gamma^{o}_{xy} \end{Bmatrix} + z \begin{Bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{Bmatrix}$$

Section 6.8.1: A CLT Analysis When Loads Are Known

6. For each ply:
   a) Calculate ply strains in *x*-*y* coordinate system (usually at ply interfaces)

Example:

LAMINATE PLY STRAINS, x-y COORDINATE SYSTEM:

PLY NO	Z-COORD	EPSxx	EPSyy	GAMxy
-------	-0.100E-01	0.005000	-0.005300	0.002900
1				
-------	-0.500E-02	0.002500	-0.003300	0.001900
2				
-------	0.000E+00	0.000000	-0.001300	0.000900
3				
-------	0.500E-02	-0.002500	0.000700	-0.000100
4				
-------	0.100E-01	-0.005000	0.002700	-0.001100

Section 6.8.1: A CLT Analysis When Loads Are Known

6. For each ply:

a) …usually also transform strains to 1-2 coordinate system

$$\left.\left\{\begin{matrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \gamma_{12}/2\end{matrix}\right\}\right|_z = [T]\big|_z \left.\left\{\begin{matrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}/2\end{matrix}\right\}\right|_z$$

Section 6.8.1: A CLT Analysis When Loads Are Known

6. For each ply:
   a) …usually also transform strains to 1-2 coordinate system

<u>Example</u>:

LAMINATE PLY STRAINS, 1-2 COORDINATE SYSTEM:

PLY NO	Z-COORD	EPS11	EPS22	GAM12
1	-0.10000E-01	0.005000	-0.005300	0.002900
	-0.50000E-02	0.002500	-0.003300	0.001900
2	-0.50000E-02	0.001873	-0.002673	-0.004073
	0.00000E+00	0.000065	-0.001365	-0.000676
3	0.00000E+00	-0.001300	0.000000	-0.000900
	0.50000E-02	0.000700	-0.002500	0.000100
4	0.50000E-02	-0.001657	-0.000143	-0.002821
	0.10000E-01	-0.002599	0.000299	-0.007218

Section 6.8.1: A CLT Analysis When Loads Are Known

6. For each ply:
   b) Calculate ply stresses in *x*-*y* coordinate system

Section 6.8.1: A CLT Analysis When Loads Are Known

6. For each ply:

b) Calculate ply stresses in *x-y* coordinate system

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} - \Delta T \alpha_{xx} \\ \varepsilon_{yy} - \Delta T \alpha_{yy} \\ \gamma_{xy} - \Delta T \alpha_{xy} \end{Bmatrix} \quad \text{(for each ply)}$$

Section 6.8.1: A CLT Analysis When Loads Are Known

6. For each ply:

b) Calculate ply stresses in *x-y* coordinate system

Example:

LAMINATE PLY STRESSES, x-y COORDINATE SYSTEM:

PLY NO	Z-COORD	SIGxx	SIGyy	TAUxy
1	-0.10000E-01	0.12169E+06	0.42794E+03	0.55100E+04
	-0.50000E-02	0.59756E+05	0.23131E+04	0.36100E+04
2	-0.50000E-02	0.23372E+05	0.57335E+04	0.63146E+04
	0.00000E+00	0.10155E+05	0.69006E+04	0.13317E+04
3	0.00000E+00	0.55707E+04	-0.34266E+05	0.17100E+04
	0.50000E-02	0.27052E+04	0.14874E+05	-0.19000E+03
4	0.50000E-02	-0.27044E+04	0.82159E+04	0.32505E+04
	0.10000E-01	-0.12352E+05	0.10865E+05	0.42260E+04

Section 6.8.1: A CLT Analysis When Loads Are Known

6. For each ply:

b) usually also transform stresses to *1-2* coord system

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{Bmatrix} = [T] \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix}$$

Section 6.8.1: A CLT Analysis When Loads Are Known

6. For each ply:

b) usually also transform stresses to *1-2* coord system

Example:

LAMINATE PLY STRESSES, 1-2 COORDINATE SYSTEM:

PLY NO	Z-COORD	SIG11	SIG22	TAU12
1	-0.10000E-01	0.12169E+06	0.42794E+03	0.55100E+04
	-0.50000E-02	0.59756E+05	0.23131E+04	0.36100E+04
2	-0.50000E-02	0.24431E+05	0.46744E+04	-0.44802E+04
	0.00000E+00	0.10495E+05	0.65610E+04	-0.74342E+03
3	0.00000E+00	-0.34266E+05	0.55707E+04	-0.17100E+04
	0.50000E-02	0.14874E+05	0.27052E+04	0.19000E+03
4	0.50000E-02	-0.27893E+04	0.83009E+04	-0.31034E+04
	0.10000E-01	-0.10208E+05	0.87202E+04	-0.79402E+04

Summary of Classical Lamination Theory (CLT) Calculations

Numerical examples illustrating discussion in:

Section 6.8.1: A CLT Analysis When Loads Are Known

Section 6.8.2: A CLT Analysis When Midplane Strains and Curvatures are Known

(Sections 6.8.1 and 6.8.2 are nearly identical…)

Section 6.8.2: A CLT Analysis When Midplane Strains and Curvatures Are Known

1. Define the problem:
   a) Specify number of different materials used
   b) Specify properties for each material
   c) Specify laminate description
   d) Specify midplane strains/curv & therm loads

Section 6.8.2: A CLT Analysis When Midplane Strains and Curvatures Are Known

1. Define the problem:
   a) Specify number of different materials used
   b) Specify properties for each material
   c) Specify laminate description
   d) Specify midplane strains/curv & therm loads

Example:

$$\varepsilon_{xx}^{o} = 0 \qquad \kappa_{xx} = -0.50\,in^{-1}$$

$$\varepsilon_{yy}^{o} = -1300\ \mu in/in \qquad \kappa_{yy} = 0.40\,in^{-1}$$

$$\gamma_{xy}^{o} = 900\ \mu\text{rad} \qquad \kappa_{xy} = -0.20\,in^{-1}$$

$$T_{cure} = 350^{\circ}F \qquad T_{service} = 75^{\circ}F \quad \Rightarrow \quad \Delta T = -275^{\circ}F$$

Section 6.8.2: A CLT Analysis When Midplane Strains and Curvatures Are Known

2. Calculate the [*ABD*] matrix
   a) Calculate the $[\underline{Q}]$ matrix for each material
   b) Calculate the $[Q]$ matrix for each ply
   c) Calculate the $[A_{ij}]$, $[B_{ij}]$, and $[D_{ij}]$ matrices
   d) Assemble the [*ABD*] matrix

(in this case do not need the [*abd*] matrix)

Section 6.8.2: A CLT Analysis When Midplane Strains and Curvatures Are Known

3. Calculate thermal stress and moment resultants:
   a) Calculate effective thermal expansion coefficients for each ply
   b) Complete calculation of thermal stress and moment resultants

Section 6.8.2: A CLT Analysis When Midplane Strains and Curvatures Are Known

3. Calculate thermal stress and moment resultants:
   a) Calculate effective thermal expansion coefficients for each ply
   b) Complete calculation of thermal stress and moment resultants

$$N_{xx}^{T} = -129 lbf/in \qquad M_{xx}^{T} = -0.401\, lbf-in/in$$

$$N_{yy}^{T} = -123 lbf/in \qquad M_{yy}^{T} = 0.0005\, lbf-in/in$$

$$N_{xy}^{T} = 0 \qquad M_{xy}^{T} = -0.0246\, lbf-in/in$$

Section 6.8.2: A CLT Analysis When Midplane Strains and Curvatures Are Known

5. Calculate stress and moment resultants

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \\ M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{o} \\ \varepsilon_{yy}^{o} \\ \gamma_{xy}^{o} \\ \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{Bmatrix} - \begin{Bmatrix} N_{xx}^{T} \\ N_{yy}^{T} \\ N_{xy}^{T} \\ M_{xx}^{T} \\ M_{yy}^{T} \\ M_{xy}^{T} \end{Bmatrix}$$

Section 6.8.2: A CLT Analysis When Midplane Strains and Curvatures Are Known

5. Calculate stress and moment resultants

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \\ M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{o} \\ \varepsilon_{yy}^{o} \\ \gamma_{xy}^{o} \\ \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{Bmatrix} - \begin{Bmatrix} N_{xx}^{T} \\ N_{yy}^{T} \\ N_{xy}^{T} \\ M_{xx}^{T} \\ M_{yy}^{T} \\ M_{xy}^{T} \end{Bmatrix}$$

(this is the only difference in calculation….)

Section 6.8.2: A CLT Analysis When Midplane Strains and Curvatures Are Known

5. Calculate stress and moment resultants

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \\ M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} 191E3 & 20.1E3 & 0 & -778 & 27.8 & -89.2 \\ 20.1E3 & 162E3 & 0 & 27.8 & 330 & -37.0 \\ 0 & 0 & 39.0E3 & -89.2 & -37.0 & 2.59 \\ -778 & -27.8 & -89.2 & 9.34 & 0.672 & -0.446 \\ -27.8 & 330 & -37.0 & 0.672 & 2.46 & -0.185 \\ -89.2 & -37.0 & 2.59 & -0.446 & -0.185 & 1.30 \end{bmatrix} \begin{Bmatrix} 0 \\ -1300\mu \\ 900\mu \\ -0.50 \\ 0.40 \\ -0.20 \end{Bmatrix} - \begin{Bmatrix} -129 \\ -123 \\ 0 \\ -0.401 \\ 0.0005 \\ -0.0246 \end{Bmatrix}$$

Section 6.8.2: A CLT Analysis When Midplane Strains and Curvatures Are Known

5. Calculate stress and moment resultants

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \\ M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \begin{Bmatrix} 520\ \text{lbf/in} \\ 377\ \text{lbf/in} \\ 64.4\ \text{lbf/in} \\ -4.0\ \text{lbf-in/in} \\ 0.22\ \text{lbf-in/in} \\ -0.085\ \text{lbf-in/in} \end{Bmatrix}$$

Section 6.8.2: A CLT Analysis When Midplane Strains and Curvatures Are Known

6. For each ply:

   a) Calculate ply strains in *x-y* and *1-2* coordinate systems (at ply interfaces)

   b) Calculate ply stresses in *x-y* and *1-2* coordinate systems (at ply interfaces)

Analysis complete!