2

Review of Force, Stress, and Strain Tensors

2.1 The Force Vector

Forces can be grouped into two broad categories: *surface* forces and *body* forces. Surface forces are those that act over a surface (as the name implies), and result from direct physical contact between two bodies. In contrast, body forces are those that act at a distance, and do not result from direct physical contact of one body with another. The force of gravity is the most common type of body force. In this chapter we are primarily concerned with surface forces, the effects of body forces (such as the weight of a structure) will be ignored.

A force is a three-dimensional (3D) vector. A force is defined by a *magnitude* and a *line of action*. In SI units, the magnitude of a force is expressed in *Newtons*, abbreviated *N*, whereas in English units the magnitude of a force is expressed in *pounds-force*, abbreviated lbf. A force vector $\bar{F}$ acting at a point P and referenced to a right-handed x–y–z coordinate system is shown in Figure 2.1. Components of $\bar{F}$ acting parallel to the x–y–z coordinate axes, F_x, F_y, and F_z, respectively, are also shown in the figure. The algebraic sign of each force component is defined in accordance with the algebraically positive direction of the corresponding coordinate axis. All force components shown in Figure 2.1 are algebraically positive, as each component "points" in the corresponding positive coordinate direction.

The reader is likely to have encountered several different ways of expressing force vectors in a mathematical sense. Three methods will be described here. The first is called *vector notation*, and involves the use of unit vectors. Unit vectors parallel to the x-, y-, and z-coordinate axes are typically labeled $\hat{i}$, $\hat{j}$, and $\hat{k}$, respectively, and by definition have a magnitude equal to unity. A force vector $\bar{F}$ is written in vector notation as follows:

$$\bar{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k} \tag{2.1}$$

The magnitude of the force is given by

$$\left|\bar{F}\right| = \sqrt{F_x^2 + F_y^2 + F_z^2} \tag{2.2}$$

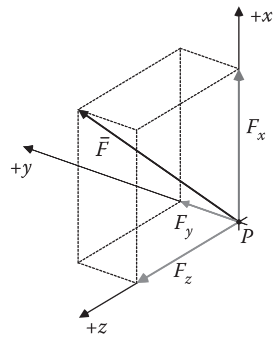

FIGURE 2.1
A force vector $\bar{F}$ acting at point P. Force components F_x, F_y, and F_z acting parallel to the x–y–z coordinate axes, respectively, are also shown.

A second method of defining a force vector is through the use of *indicial notation*. In this case, a subscript is used to denote individual components of the vectoral quantity:

$$\bar{F} = (F_x, F_y, F_z)$$

The subscript denotes the coordinate direction of each force component. One of the advantages of indicial notation is that it allows a shorthand notation to be used, as follows:

$$\bar{F} = F_i, \quad \text{where } i = x, y, \text{ or } z \tag{2.3}$$

Note that a *range* has been explicitly specified for the subscript "i" in Equation 2.3. That is, it is explicitly stated that the subscript i may take on values of x, y, or z. Usually, however, the range of a subscript(s) is not stated explicitly but rather is implied. For example, Equation 2.3 is normally written simply as

$$\bar{F} = F_i$$

where it is understood that the subscript i takes on values of x, y, and z.

The third approach is called *matrix notation*. In this case, individual components of the force vector are listed within braces in the form of a column array:

$$\bar{F} = \begin{Bmatrix} F_x \\ F_y \\ F_z \end{Bmatrix} \tag{2.4}$$

Indicial notation is sometimes combined with matrix notation as follows:

$$\bar{F} = \{F_i\} \tag{2.5}$$

2.2 Transformation of a Force Vector

One of the most common requirements in the study of mechanics is the need to describe a vector in more than one coordinate system. For example, suppose all components of a force vector F_i are known in one coordinate system (the x–y–z coordinate system, say) and it is desired to express this force vector in a second coordinate system (the x'–y'–z' coordinate system, say). To describe the force vector in the new coordinate system, we must calculate the components of the force parallel to the x', y', and z' axes—that is, we must calculate $F_{x'}$, $F_{y'}$, and $F_{z'}$. The process of relating force components in one coordinate system to those in another coordinate system is called *transformation* of the force vector. This terminology is perhaps unfortunate, in the sense that the force vector itself is not "transformed" but rather our *description* of the force vector transforms as we change from one coordinate system to another.

It can be shown [1,2] that the force components in the x'–y'–z' coordinate system ($F_{x'}$, $F_{y'}$, and $F_{z'}$) are related to the components in x–y–z coordinate system (F_x, F_y, F_z) according to:

$$\begin{aligned} F_{x'} &= c_{x'x}F_x + c_{x'y}F_y + c_{x'z}F_z \\ F_{y'} &= c_{y'x}F_x + c_{y'y}F_y + c_{y'z}F_z \\ F_{z'} &= c_{z'x}F_x + c_{z'y}F_y + c_{z'z}F_z \end{aligned} \tag{2.6a}$$

The terms $c_{i'j}$ that appear in Equation 2.6a are called *direction cosines*, and are to equal the cosine of the angle between the axes of the new and original coordinate systems. An angle of rotation is defined *from* the original x–y–z coordinate system *to* the new x'–y'–z' coordinate system. The algebraic sign of the angle of rotation is defined in accordance with the right-hand rule.

Equation 2.6a can be succinctly written using the summation convention as follows:

$$F_{i'} = c_{i'j}F_j \tag{2.6b}$$

Alternatively, these three equations can be written using matrix notation as

$$\begin{Bmatrix} F_{x'} \\ F_{y'} \\ F_{z'} \end{Bmatrix} = \begin{bmatrix} c_{x'x} & c_{x'y} & c_{x'z} \\ c_{y'x} & c_{y'y} & c_{y'z} \\ c_{z'x} & c_{z'y} & c_{z'z} \end{bmatrix} \begin{Bmatrix} F_x \\ F_y \\ F_z \end{Bmatrix} \tag{2.6c}$$

Note that although values of individual force components vary as we change from one coordinate system to another, the magnitude of the force vector (given by Equation 2.2) does not. The magnitude is independent of the coordinate system used, and is called an *invariant* of the force tensor.

Direction cosines relate unit vectors in the "new" and "old" coordinate systems. For example, a unit vector directed along the x'-axis (i.e., unit vector $\hat{i}'$) is related to unit vectors in the x–y–z coordinate system as follows:

$$\hat{i}' = c_{x'x}\,\hat{i} + c_{x'y}\,\hat{j} + c_{x'z}\,\hat{k} \tag{2.7}$$

As $\hat{i}'$ is a unit vector, then in accordance with Equation 2.2:

$$(c_{x'x})^2 + (c_{x'y})^2 + (c_{x'z})^2 = 1 \tag{2.8}$$

To this point we have referred to a force as a *vector.* A force vector can also be called a force *tensor.* The term "tensor" refers to any quantity that transforms in a physically meaningful way from one Cartesian coordinate system to another. The *rank* of a tensor equals the number of subscripts that must be used to describe the tensor. A force can be described using a single subscript, F_i, and therefore a force is said to be a *tensor of rank one*, or equivalently, *a first-order tensor*. Equation 2.6 is called the *transformation law for a first-order tensor*.

It is likely that the reader is already familiar with two other tensors: the *stress tensor*, σ_{ij}, and the strain tensor, ε_{ij}. The stress and strain tensors will be reviewed later in this chapter, but at this point it can be noted that *two* subscripts are used to describe stress and strain tensors. Hence, stress and strain tensors are said to be *tensors of rank two*, or equivalently, *second-order tensors*.

Example Problem 2.1

Given: All components of a force vector $\bar{F}$ are known in a given x–y–z coordinate system. It is desired to express this force in a new x''–y''–z'' coordinate system, where the x''–y''–z'' system is generated from the original x–y–z system by the following two rotations (see Figure 2.2):

- A rotation of θ-degrees about the original z-axis (which defines an intermediate x'–y'–z' coordinate system), followed by
- A rotation of β-degrees about the x'-axis (which defines the final x''–y''–z'' coordinate system)

PROBLEM

a. Determine the direction cosines $c_{i''j}$ relating the original x–y–z coordinate system to the new x''–y''–z'' coordinate system.
b. Obtain a general expression for the force vector $\bar{F}$ in the x''–y''–z'' coordinate system.

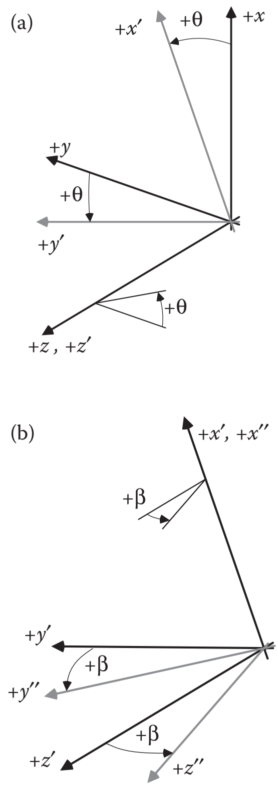

FIGURE 2.2
Generation of the x''–y''–z'' coordinate system from the x–y–z coordinate system. (a) Rotation of θ-degrees about the original z-axis (which defines an intermediate x'–y'–z' coordinate system); (b) rotation of β-degrees about the x'-axis (which defines the final x''–y''–z'' coordinate system).

c. Calculate numerical values of the force vector $\bar{F}$ in the x''–y''–z'' coordinate system if $\theta = -20°$, $\beta = 60°$, and $F_x = 1000\ N$, $F_y = 200\ N$, $F_z = 600\ N$.

SOLUTION

a. One way to determine direction cosines $c_{i''j}$ is to rotate unit vectors. In this approach unit vectors are first rotated from the original x–y–z coordinate system to the intermediate x'–y'–z' coordinate system, and then from the x'–y'–z' system to the final x''–y''–z'' coordinate system.

Define a unit vector $\bar{I}$ that is aligned with the x-axis:

$$\bar{I} \equiv (1)\hat{i}$$

That is, vector $\bar{I}$ is a vector for which $I_x = 1$, $I_y = 0$, and $I_z = 0$. The vector $\bar{I}$ can be rotated to the intermediate x'–y'–z' coordinate system using Equation 2.6:

$$I_{x'} = c_{x'x}I_x + c_{x'y}I_y + c_{x'z}I_z$$

$$I_{y'} = c_{y'x}I_x + c_{y'y}I_y + c_{y'z}I_z$$

$$I_{z'} = c_{z'x}I_x + c_{z'y}I_y + c_{z'z}I_z$$

The direction cosines associated with a transformation from the x–y–z coordinate system to the intermediate x'–y'–z' coordinate system can be determined by inspection (see Figure 2.2a), and are given by

$c_{x'x}$ = cosine(angle between x'- and x-axes) = cos θ

$c_{x'y}$ = cosine(angle between x'- and y-axes) = cos(90° − θ) = sin θ

$c_{x'z}$ = cosine(angle between x'- and z-axes) = cos(90°) = 0

$c_{y'x}$ = cosine(angle between the y'- and x-axes) = cos(90° + θ) = –sin θ

$c_{y'y}$ = cosine(angle between the y'- and y-axes) = cos θ

$c_{y'z}$ = cosine(angle between the y'- and z-axes) = cos(90°) = 0

$c_{z'x}$ = cosine(angle between the z'- and x-axes) = cos(90°) = 0

$c_{z'y}$ = cosine(angle between the z'- and y-axes) = cos(90°) = 0

$c_{z'z}$ = cosine(angle between the z'- and z-axes) = cos(0°) = 1

Using these direction cosines:

$$I_{x'} = c_{x'x}I_x + c_{x'y}I_y + c_{x'z}I_z = (\cos\ \theta)(1) + (\sin\theta)(0) + (0)(0) = \cos\ \theta$$

$$I_{y'} = c_{y'x}I_x + c_{y'y}I_y + c_{y'z}I_z = (-\sin\theta)(1) + (\cos\theta)(0) + (0)(0) = -\sin\theta$$

$$I_{z'} = c_{z'x}I_x + c_{z'y}I_y + c_{z'z}I_z = (0)(1) + (0)(0) + (1)(0) = 0$$

Therefore, in the x'–y'–z' coordinate system, the vector $\bar{I}$ is written:

$$\bar{I} = (\cos\theta)\,\hat{i}' + (-\sin\theta)\,\hat{j}'$$

Now define two additional unit vectors, one aligned with the original y-axis (vector $\bar{J}$) and one aligned with the original z-axis (vector $\bar{K}$); that is, let $\bar{J} = (1)\,\hat{j}$ and $\bar{K} = (1)\,\hat{k}$. Transforming these vectors to the x'–y'–z' coordinate system, again using the direction cosines listed above, results in

$$\bar{J} = (\sin\theta)\,\hat{i}' + (\cos\theta)\,\hat{j}'$$

$$\bar{K} = (1)\,\hat{k}'$$

We now rotate vectors $\bar{I}$, $\bar{J}$ and $\bar{K}$ from the *intermediate* x'–y'–z' coordinate system to the *final* x''–y''–z'' coordinate system. The direction cosines associated with a transformation from the x'–y'–z' coordinate system to the final x''–y''–z'' coordinate system are easily determined by inspection (see Figure 2.2b) and are given by

$$\begin{aligned}
c_{x''x'} &= 1 \quad c_{x''y'} = 0 \qquad\quad c_{x''z'} = 0\\
c_{y''x'} &= 0 \quad c_{y''y'} = \cos\beta \quad\ c_{y''z'} = \sin\beta\\
c_{z''x'} &= 0 \quad c_{z''y'} = -\sin\beta \quad c_{z''z'} = \cos\beta
\end{aligned}$$

These direction cosines together with Equation 2.6 can be used to rotate the vector $\bar{I}$ from the intermediate x'–y'–z' coordinate system to the final x''–y''–z'' coordinate system:

$$\begin{aligned}
I_{x''} &= c_{x''x'}I_{x'} + c_{x''y'}I_{y'} + c_{x''z'}I_{z'} = (1)(\cos\theta) + (0)(-\sin\theta) + (0)(0)\\
I_{x''} &= \cos\theta
\end{aligned}$$

$$\begin{aligned}
I_{y''} &= c_{y''x'}I_{x'} + c_{y''y'}I_{y'} + c_{y''z'}I_{z'} = (0)(\cos\theta) + (\cos\beta)(-\sin\theta) + (\sin\beta)(0)\\
I_{y''} &= -\cos\beta\ \sin\theta
\end{aligned}$$

$$\begin{aligned}
I_{z''} &= c_{z''x'}I_{x'} + c_{z''y'}I_{y'} + c_{z''z'}I_{z'} = (0)(\cos\theta) + (-\sin\beta)(-\sin\theta) + (\cos\beta)(0)\\
I_{z''} &= \sin\beta\ \sin\theta
\end{aligned}$$

Therefore, in the final x''–y''–z'' coordinate system, the vector $\bar{I}$ is written:

$$\bar{I} = (\cos\theta)\,\hat{i}'' + (-\cos\beta\sin\theta)\,\hat{j}'' + (\sin\beta\ \sin\theta)\,\hat{k}'' \tag{a}$$

Recall that in the original x–y–z coordinate system $\bar{I}$ is simply a unit vector aligned with the original x-axis: $\bar{I} \equiv (1)\,\hat{i}$. Therefore, result (a) defines the direction cosines associated with the angle between the original x-axis and the final x''-, y''-, and z''-axes. That is

$$\begin{aligned}
c_{x''x} &= \cos\theta\\
c_{y''x} &= -\cos\beta\,\sin\theta\\
c_{z''x} &= \sin\beta\,\sin\theta
\end{aligned}$$

A similar procedure is used to rotate the unit vectors $\bar{J}$ and $\bar{K}$ from the intermediate x'–y'–z' coordinate system to the final x''–y''–z'' coordinate system. These rotations result in

$$\bar{J} = (\sin\theta)\hat{i}'' + (\cos\beta\cos\theta)\hat{j}'' + (-\sin\beta\cos\theta)\hat{k}'' \quad \text{(b)}$$

$$\bar{K} = (0)\hat{i}'' + (\sin\beta)\hat{j}'' + (\cos\beta)\hat{k}'' \quad \text{(c)}$$

As vector $\bar{J}$ is a unit vector aligned with the original y-axis, $\bar{J} = (1)\hat{j}$, result (b) defines the direction cosines associated with the angle between the original y-axis and the final x''-, y''-, and z''-axes:

$$c_{x''y} = \sin\theta$$

$$c_{y''y} = \cos\beta\cos\theta$$

$$c_{z''y} = -\sin\beta\cos\theta$$

Finally, result (c) defines the direction cosines associated with the angle between the original z-axis and the final x''-, y''-, and z''-axes:

$$c_{x''z} = 0$$

$$c_{y''z} = \sin\beta$$

$$c_{z''z} = \cos\beta$$

Assembling the preceding results, the set of direction cosines relating the original x–y–z coordinate system to the final x''–y''–z'' coordinate system can be written:

$$\begin{bmatrix} c_{x''x} & c_{x''y} & c_{x''z} \\ c_{y''x} & c_{y''y} & c_{y''z} \\ c_{z''x} & c_{z''y} & c_{z''z} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\cos\beta\sin\theta & \cos\beta\cos\theta & \sin\beta \\ \sin\beta\sin\theta & -\sin\beta\cos\theta & \cos\beta \end{bmatrix}$$

b. As direction cosines have been determined, transformation of force vector $\bar{F}$ can be accomplished using any version of Equation 2.6. For example, using matrix notation, Equation 2.6c:

$$\begin{Bmatrix} F_{x''} \\ F_{y''} \\ F_{z''} \end{Bmatrix} = \begin{bmatrix} c_{x''x} & c_{x''y} & c_{x''z} \\ c_{y''x} & c_{y''y} & c_{y''z} \\ c_{z''x} & c_{z''y} & c_{z''z} \end{bmatrix} \begin{Bmatrix} F_x \\ F_y \\ F_z \end{Bmatrix}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\cos\beta\sin\theta & \cos\beta\cos\theta & \sin\beta \\ \sin\beta\sin\theta & -\sin\beta\cos\theta & \cos\beta \end{bmatrix} \begin{Bmatrix} F_x \\ F_y \\ F_z \end{Bmatrix}$$

$$\begin{Bmatrix} F_{x''} \\ F_{y''} \\ F_{z''} \end{Bmatrix} = \begin{Bmatrix} (\cos\theta)F_x + (\sin\theta)F_y \\ (-\cos\beta\sin\theta)F_x + (\cos\beta\cos\theta)F_y + (\sin\beta)F_z \\ (\sin\beta\sin\theta)F_x + (-\sin\beta\cos\theta)F_y + (\cos\beta)F_z \end{Bmatrix}$$

c. Using the specified numerical values and the results of part (b):

$$\begin{Bmatrix} F_{x''} \\ F_{y''} \\ F_{z''} \end{Bmatrix} = \begin{Bmatrix} [\cos(-20°)](1000\ N) + [\sin(-20°)]200\ N \\ [-\cos(60°)(\sin -20°)](1000\ N) + [\cos(60°)\cos(-20°)](200\ N) \\ + [\sin(60°)](600\ N) \\ [\sin(60°)\sin(-20°)](1000\ N) + [-\sin(60°)\cos(-20°)](200\ N) \\ + (\cos 60°)(600\ N) \end{Bmatrix}$$

$$\begin{Bmatrix} F_{x''} \\ F_{y''} \\ F_{z''} \end{Bmatrix} = \begin{Bmatrix} 873.1\ N \\ 784.6\ N \\ -159.0\ N \end{Bmatrix}$$

Using vector notation, $\bar{F}$ can now be expressed in the two different coordinate systems as

$$\bar{F} = (1000\ N)\hat{i} + (200\ N)\hat{j} + (600\ N)\hat{k}$$

or, equivalently

$$\bar{F} = (873.1\ N)\hat{i}'' + (784.6\ N)\hat{j}'' + (-159.0\ N)\hat{k}''$$

Where $\hat{i}$, $\hat{j}$, $\hat{k}$ and $\hat{i}''$, $\hat{j}''$, $\hat{k}''$ are unit vectors in the x–y–z and x''–y''–z'' coordinate systems, respectively. Force vector $\bar{F}$ drawn in the x–y–z and x''–y''–z'' coordinate systems is shown in Figure 2.3a and b, respectively. The two descriptions of $\bar{F}$ are entirely equivalent. A convenient way of (partially) verifying this equivalence is to calculate the magnitude of the original and transformed force vectors. As the magnitude is an invariant, it is independent of the coordinate system used to describe the force vector. Using Equation 2.2, the magnitude of the force vector in the x–y–z coordinate system is

$$|\bar{F}| = \sqrt{F_x^2 + F_y^2 + F_z^2} = \sqrt{(1000\,N)^2 + (200\,N)^2 + (600\,N)^2} = 1183\ N$$

The magnitude of the force vector in the x''–y''–z'' coordinate system is

$$|\bar{F}| = \sqrt{(F_{x''})^2 + (F_{y''})^2 + (F_{z''})^2} = \sqrt{(871.3\ N)^2 + (784.6\ N)^2 + (-159.0\ N)^2}$$

$$= 1183\ N \quad \text{(agrees)}$$

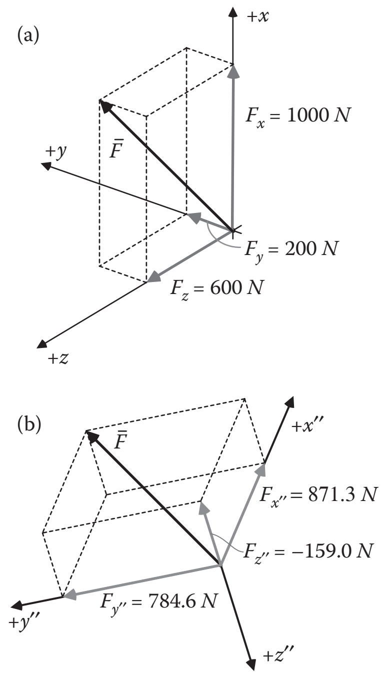

FIGURE 2.3
Force vector $\bar{F}$ drawn in two different coordinate systems. (a) Force vector $\bar{F}$ in the original *x–y–z* coordinate system; (b) force vector $\bar{F}$ in a new *x″–y″–z″* coordinate system.

2.3 Normal Forces, Shear Forces, and Free-Body Diagrams

Force $\bar{F}$ acting at an angle to a planar surface is shown in Figure 2.4. As force is a vector, it can always be decomposed into two force components, a *normal* force component and a *shear* force component. The line-of-action of the normal force component is orthogonal to the surface, whereas the line-of-action of the shear force component is tangent to the surface.

Internal forces induced within a solid body by externally applied forces can be investigated with the aid of *free-body diagrams*. A simple example is shown in Figure 2.5, which shows a straight circular rod with constant diameter subjected to two external forces of equal magnitude (R) but opposite direction. The internal force ($\bar{F}_I$, say) induced at any cross-section of the rod can be investigated by making an imaginary cut along the plane of interest. Suppose an imaginary cut is made along plane *a–a–a–a*, which is perpendicular to the axis of the rod. The resulting free-body diagram for the lower half of the rod is shown in Figure 2.5a, where a *x–y–z* coordinate system has been assigned such that the *x*-axis is parallel to the rod axis, as shown. On the basis of this free-body diagram, it is concluded that an internal force $\bar{F}_I = (R)\hat{i} + (0)\hat{j} + (0)\hat{k}$ is induced at cross-section *a–a*. That

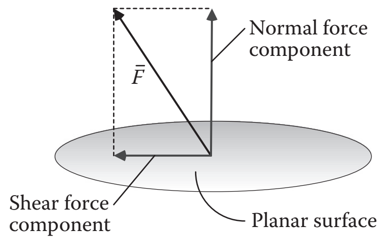

FIGURE 2.4
A force $\bar{F}$ acting at an angle to a planar surface.

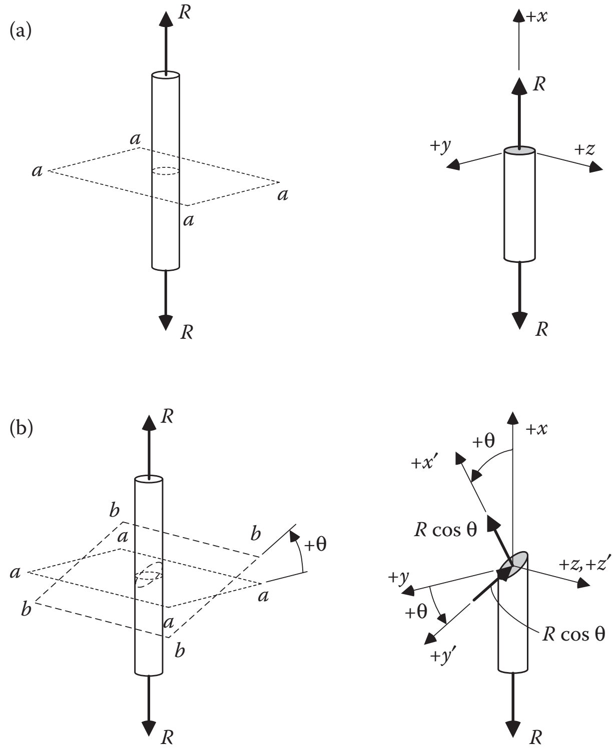

FIGURE 2.5
The use of free-body diagrams to determine internal forces acting on planes *a–a–a–a* and *b–b–b–b*. (a) Free-body diagram based on plane *a–a–a–a*, perpendicular to rod axis; (b) free-body diagram based on plane *b–b–b–b*, inclined at angle +θ to the rod axis.

is, only a normal force of magnitude R is induced at cross-section *a–a–a–a*, which has been defined to be perpendicular to the axis of the rod.

In contrast, the imaginary cut need not be made perpendicular to the axis of the rod. Suppose the imaginary cut is made along plane *b–b–b–b*, which is inclined at an angle of +θ with respect to the axis of the rod. The resulting free-body diagram for the lower half of the rod is shown in Figure 2.5b. A new x'–y'–z' coordinate system has been assigned so that the x'-axis is perpendicular to plane b–b–b–b and the z'-axis is coincident with the z-axis, that is, the x'–y'–z' coordinate system is generated from the x–y–z coordinate system by a rotation of +θ about the original z-axis. The internal force $\bar{F}_I$ can be expressed with respect to the x'–y'–z' coordinate system by transforming $\bar{F}_I$ from the x–y–z coordinate system to the x'–y'–z' coordinate system.

This coordinate transformation is a special case of the transformation considered in Example Problem 2.1. The direction cosines now become (with $\beta = 0°$):

$$\begin{array}{lll} c_{x'x} = \cos\theta & c_{x'y} = \sin\theta & c_{x'z} = 0 \\ c_{y'x} = -\sin\theta & c_{y'y} = \cos\theta & c_{y'z} = 0 \\ c_{z'x} = 0 & c_{z'y} = 0 & c_{z'z} = 1 \end{array}$$

Applying Equation 2.6, we have

$$\begin{Bmatrix} F_{x'} \\ F_{y'} \\ F_{z'} \end{Bmatrix} = \begin{bmatrix} c_{x'x} & c_{x'y} & c_{x'z} \\ c_{y'x} & c_{y'y} & c_{y'z} \\ c_{z'x} & c_{z'y} & c_{z'z} \end{bmatrix} \begin{Bmatrix} F_x \\ F_y \\ F_z \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} R \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} (\cos\theta)R \\ (-\sin\theta)R \\ 0 \end{Bmatrix}$$

In the x'–y'–z' coordinate system the internal force is $\bar{F}_I = (R\cos\theta)\hat{i}' - (R\sin\theta)\hat{j}' + (0)\hat{k}'$. Hence, by defining a coordinate system which is inclined to the axis of the rod, we conclude that *both* a normal force (R cos θ) *and* a shear force (–R sin θ) are induced in the rod.

Although the preceding discussion may seem simplistic, it has been included to demonstrate the following:

A specific coordinate system must be specified before a force vector can be defined in a mathematical sense. In general, the coordinate system is defined by the imaginary cut(s) used to form the free-body diagram.

All components of a force must be specified to fully define the force vector. Further, the individual components of a force change as the vector is transformed from one coordinate system to another.

These two observations are valid for all tensors, not just for force vectors. In particular, these observations hold in the case of stress and strain tensors, which will be reviewed in the following sections.

2.4 Definition of Stress

There are two fundamental types of stress: *normal* stress and *shear* stress. Both types of stress are defined as a force divided by the area over which it acts.

A general 3D solid body subjected to a system of external forces is shown in Figure 2.6a. It is assumed that the body is in static equilibrium, that is, it is assumed that the sum of all external forces is zero, $\Sigma \overline{F}_i = 0$. These external forces induce internal forces acting within the body. In general, the internal forces will vary in both magnitude and direction throughout the body. An illustration of the variation of internal forces along a single line within an internal plane is shown in Figure 2.6b. A small area (ΔA) isolated from this plane is shown in Figure 2.6c. Area ΔA is assumed to be "infinitesimally small." That

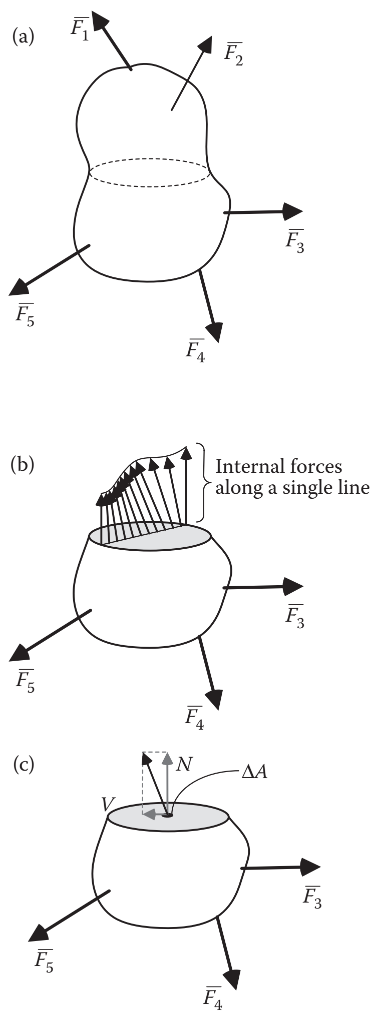

FIGURE 2.6
A solid 3-D body in equilibrium. (a) A solid 3-D body subject to external forces $\overline{F}_1 \rightarrow \overline{F}_5$; (b) variation of internal forces along an internal line; (c) internal force acting over infinitesimal area ΔA.

is, the area ΔA is small enough such that the internal forces acting over ΔA can be assumed to be of constant magnitude and direction. Therefore, the internal forces acting over ΔA can be represented by a force vector which can be decomposed into a normal force, N, and a shear force, V, as shown in Figure 2.6c.

Normal stress (usually denoted "σ"), and shear stress (usually denoted "τ") are defined as the force per unit area acting perpendicular and tangent to the area ΔA, respectively. That is,

$$\sigma \equiv \lim_{\Delta A \to 0} \frac{N}{\Delta A} \quad \tau \equiv \lim_{\Delta A \to 0} \frac{V}{\Delta A} \tag{2.9}$$

Note that by definition the area ΔA shrinks to zero: $\Delta A \to 0$. Stresses σ and τ are therefore said to exist "at a point." As internal forces generally vary from point-to-point (as shown in Figure 2.6), stresses also vary from point-to-point.

Stress has units of force per unit area. In SI units stress is reported in terms of "Pascals" (abbreviated "Pa"), where 1 Pa = 1 N/m^2. In English units stress is reported in terms of pounds-force per square inch (abbreviated "psi"), that is, 1 psi = 1 lbf/in.^{2} Conversion factors between the two systems of measurement are 1 psi = 6895 Pa, or equivalently, 1 Pa = 0.1450×10^{-3} psi. Common abbreviations used throughout this chapter are as follows:

1×10^3 Pa = 1 kilo-Pascals = 1 kPa $\quad$ 1×10^3 psi = 1 kilo-psi = 1 ksi

1×10^6 Pa = 1 Mega-Pascals = 1 MPa $\quad$ 1×10^6 psi = 1 mega-psi = 1 Msi

1×10^9 Pa = 1 Giga-Pascals = 1 GPa.

2.5 The Stress Tensor

A general 3D solid body subjected to a system of external forces is shown in Figure 2.7a. It is assumed that the body is in static equilibrium and that body forces are negligible, that is, is it assumed that the sum of all external forces is zero, $\Sigma \bar{F}_i = 0$. A free-body diagram of an infinitesimally small cube removed from the body is shown in Figure 2.7b. The cube is referenced to a x–y–z coordinate system, and the cube edges are aligned with these axes. The lengths of the cube edges are denoted dx, dy, and dz. Although (in general) internal forces are induced over all six faces of the cube, for clarity the forces acting on only three faces have been shown.

The force acting over each cube face can be decomposed into a normal force component and two shear force components, as illustrated in Figure 2.7c. Although each force component could be identified with a single subscript (as force is a first-order tensor), for convenience two subscripts have been used. The first subscript identifies the face over which the force is distributed, whereas the second subscript identifies the direction in which the

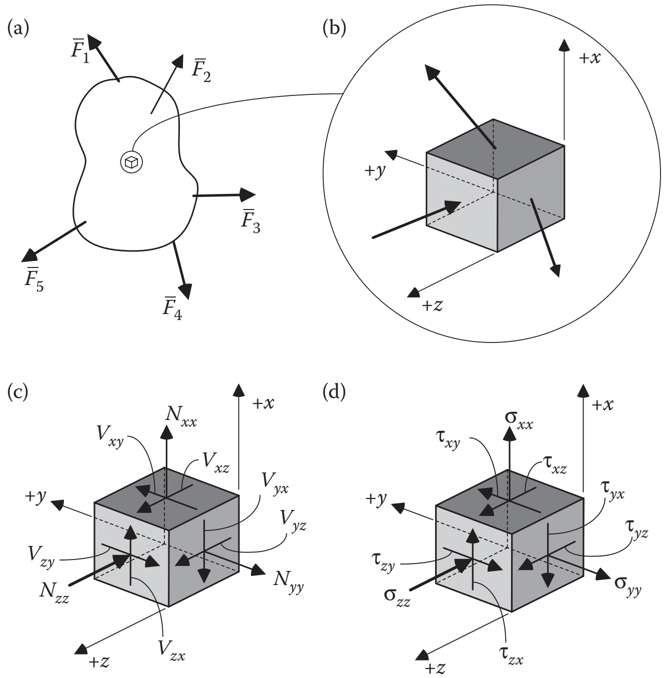

FIGURE 2.7
Free-body diagrams used to define stress induced in a solid body. (a) 3-D solid body in equilibrium; (b) infinitesimal cube removed from the solid body (internal forces acting on three faces shown); (c) normal force and two shear forces act over each face of the cube; (d) normal stress and two shear stresses act over each face of the cube.

force is oriented. For example, N_{xx} refers to a normal force component which is distributed over the *x*-face and which is acting parallel to the *x*-direction. Similarly, V_{zy} refers to a shear force distributed over the *z*-face which is acting parallel to the *y*-direction.

Three stress components can now be defined for each cube face, in accordance with Equation 2.9. For example, for the three faces of the infinitesimal element shown in Figure 2.7:

Stresses acting on the –x-face:

$$\sigma_{xx} = \lim_{dy,dz\to 0}\left(\frac{N_{xx}}{dy\ dz}\right) \quad \tau_{xy} = \lim_{dy,dz\to 0}\left(\frac{V_{xy}}{dy\,dz}\right) \quad \tau_{xz} = \lim_{dy,dz\to 0}\left(\frac{V_{xz}}{dy\,dz}\right)$$

Stresses acting on the –y-face:

$$\sigma_{yy} = \lim_{dx,dz\to 0}\left(\frac{N_{yy}}{dx\ dz}\right) \quad \tau_{yx} = \lim_{dx,dz\to 0}\left(\frac{V_{yx}}{dx\,dz}\right) \quad \tau_{yz} = \lim_{dx,dz\to 0}\left(\frac{V_{yz}}{dx\,dz}\right)$$

Stresses acting on the +z-face:

$$\sigma_{zz} = \lim_{dx,dy\to 0}\left(\frac{N_{zz}}{dx\,dy}\right) \quad \tau_{zx} = \lim_{dx,dy\to 0}\left(\frac{V_{zx}}{dx\,dy}\right) \quad \tau_{zy} = \lim_{dx,dy\to 0}\left(\frac{V_{zy}}{dx\,dy}\right)$$

As three force components (and therefore three stress components) exist on each of the six faces of the cube, it would initially appear that there are 18 independent force (stress) components. However, it is easily shown that for static equilibrium to be maintained (assuming body forces are negligible):

- Normal forces acting on opposite faces of the infinitesimal element must be of equal magnitude and opposite direction.
- Shear forces acting within a plane of the element must be orientated either "tip-to-tip" (e.g., forces V_{xz} and V_{zx} in Figure 2.7c) or "tail-to-tail" (e.g., forces V_{xy} and V_{yx}), and be of equal magnitude. That is $|V_{xy}| = |V_{yx}|$, $|V_{xz}| = |V_{zx}|$, $|V_{yz}| = |V_{zy}|$.

These restrictions reduce the number of independent force (stress) components from 18 to 6, as follows:

Independent Force Component(s)	Corresponding Stress Component(s)
N_{xx}	σ_{xx}
N_{yy}	σ_{yy}
N_{zz}	σ_{zz}
V_{xy} $(=V_{yx})$	$\tau_{xy}(=\tau_{yx})$
V_{xz} $(=V_{zx})$	$\tau_{xz}(=\tau_{zx})$
V_{yz} $(=V_{zy})$	$\tau_{yz}(=\tau_{zy})$

We must next define the algebraic sign convention we will use to describe individual stress components. The components acting on three faces of an infinitesimal element are shown in Figure 2.8. We first associate an algebraic sign with each face of the infinitesimal element. A cube face is *positive* if the outward unit normal of the face (i.e., the unit normal pointing away from the interior of the element) points in a positive coordinate direction; otherwise, the face is negative. For example, faces (*ABCD*) and (*ADEG*) in Figure 2.8 are a positive faces, whereas face (*DCFE*) is a negative face.

Having identified the positive and negative faces of the element, a stress component is *positive* if

- The stress component acts on a positive face and points in a positive coordinate direction, or if
- The stress component acts on a negative face and points in a negative coordinate direction.

If neither of these conditions are met, then the stress component is negative. This convention can be used to confirm that all stress components

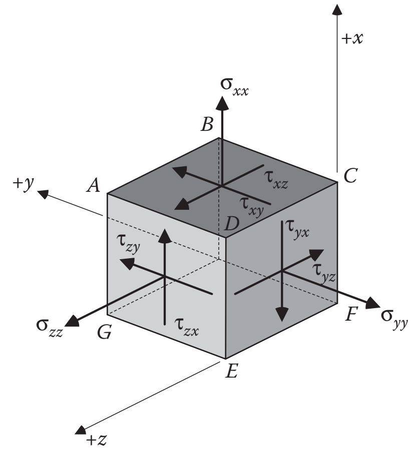

FIGURE 2.8
An infinitesimal stress element (all stress components shown in a positive sense).

shown in Figure 2.8 are algebraically positive. For example, to determine the algebraic sign of the normal stress σ_{xx} which acts on face *ABCD* in Figure 2.8, note that (a) face *ABCD* is positive, and (b) the normal stress σ_{xx} which acts on this face points in the positive *x*-direction. Therefore, σ_{xx} is positive. As a second example, the shear stress τ_{yz} which acts on cube face *DCFE* is positive because (a) face *DCFE* is a negative face, and (b) τ_{yz} points in the negative *z*-direction.

The preceding discussion shows that the *state of stress* at a point is defined by six components of stress: three normal stress components and three shear stress components. The state of stress is written using matrix notation as follows:

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix} \tag{2.10}$$

To express the state of stress using indicial notation we must first make the following change in notation:

$$\tau_{xy} \rightarrow \sigma_{xy}$$

$$\tau_{xz} \rightarrow \sigma_{xz}$$

$$\tau_{yx} \rightarrow \sigma_{yx}$$

$$\tau_{yz} \rightarrow \sigma_{yz}$$

$$\tau_{zx} \rightarrow \sigma_{zx}$$

$$\tau_{zy} \rightarrow \sigma_{zy}$$

With this change the matrix on the left side of the equality sign in Equation 2.10 becomes:

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \rightarrow \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$

Which can be succinctly written using indicial notation as

$$\sigma_{ij}, \quad i,j = x,y, \text{ or } z \tag{2.11}$$

In Section 2.1 it was noted that a force vector is a *first-order tensor,* as only one subscript is required to describe a force tensor, F_i. From Equation 2.11 is it clear that stress is a *second-order tensor* (or equivalently, a *tensor of rank two*), as two subscripts are required to describe a state of stress.

Example Problem 2.2

Given: The stress element referenced to a *x–y–z* coordinate system and subject to the stress components shown in Figure 2.9.
Determine: Label all stress components, including algebraic sign.

SOLUTION

The magnitude and algebraic sign of each stress component is determined using the sign convention defined above. The procedure will

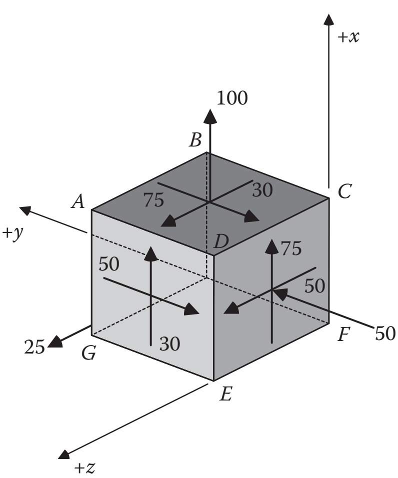

FIGURE 2.9
Stress components acting on an infinitesimal element (all stresses in MPa).

be illustrated using the stress components acting on face *DCFE*. First, note that face *DCFE* is a negative face, as an outward unit normal for this face points in the negative *y*-direction. The normal stress which acts on face *DCFE* has a magnitude of 50 MPa, and points in the positive *y*-direction. Hence, this stress component is negative and is labeled $\sigma_{yy} = -50$ MPa. One of the shear stress components acting on face *DCFE* has a magnitude of 75 MPa, and points in the positive *x*-direction. Hence, this stress component is also negative and is labeled $\tau_{yx} = -75$ MPa (or equivalently, $\tau_{xy} = -75$ MPa). Finally, the second shear force component acting on face *DCFE* has a magnitude of 50 MPa, and points in the positive *z*-direction. Hence, this component is labeled $\tau_{yz} = -50$ MPa (or equivalently, $\tau_{zy} = -50$ MPa).

Following this process for all faces of the element, the state of stress represented by the element shown in Figure 2.9 can be written:

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} 100\,\text{MPa} & -75\,\text{MPa} & 30\,\text{MPa} \\ -75\,\text{MPa} & -50\,\text{MPa} & -50\,\text{MPa} \\ 30\,\text{MPa} & -50\,\text{MPa} & 25\,\text{MPa} \end{bmatrix}$$

2.6 Transformation of the Stress Tensor

In Section 2.5 the stress tensor was defined using a free-body diagram of an infinitesimal element removed from a 3D body in static equilibrium. This concept is again illustrated in Figure 2.10a, which shows the stress element referenced to an *x–y–z* coordinate system.

Now, the infinitesimal element need not be removed in the orientation shown in Figure 2.10a. An infinitesimal element removed from precisely the same point within the body but at a different orientation is shown in Figure 2.10b. This stress element is referenced to a new *x′–y′–z′* coordinate system. The state of stress at the point of interest is dictated by the external loads applied to the body, and is independent of the coordinate system used to describe it. Hence, the stress tensor referenced to the *x′–y′–z′* coordinate system is equivalent to the stress tensor referenced to the *x–y–z* coordinate system, although the direction and magnitude of individual stress components will differ.

The process of relating stress components in one coordinate system to those in another is called *transformation* of the stress tensor. This terminology is perhaps unfortunate, in the sense that the state of stress itself is not "transformed" but rather our *description* of the state of stress transforms as we change from one coordinate system to another.

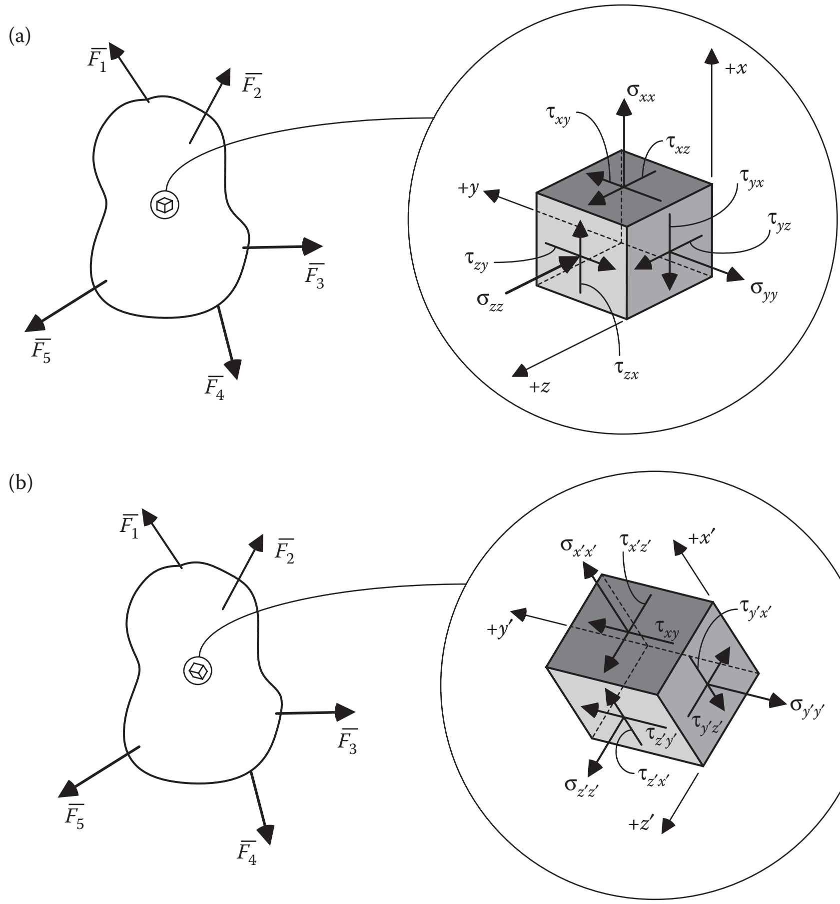

FIGURE 2.10
Infinitesimal elements removed from the same point within a 3D solid but in two different orientations. (a) Infinitesimal element referenced to the x–y–z coordinate system; (b) infinitesimal element referenced to the x'–y'–z' coordinate system.

It can be shown [1,2] that the stress components in the new x'–y'–z' coordinate system ($\sigma_{i'j'}$) are related to the components in the original x–y–z coordinate system (σ_{ij}) according to

$$\sigma_{i'j'} = c_{i'k}\, c_{j'l}\, \sigma_{kl} \quad \text{where} \quad i, j, k, l = x, y, z \tag{2.12a}$$

Or, equivalently (using matrix notation):

$$[\sigma_{i'j'}] = [c_{i'j}][\sigma_{ij}][c_{i'j}]^T$$

where $[c_{i'j}]^T$ is the transpose of the direction cosine array. Writing in full matrix form:

$$\begin{bmatrix} \sigma_{x'x'} & \sigma_{x'y'} & \sigma_{x'z'} \\ \sigma_{y'x'} & \sigma_{y'y'} & \sigma_{y'z'} \\ \sigma_{z'x'} & \sigma_{z'y'} & \sigma_{z'z'} \end{bmatrix} = \begin{bmatrix} c_{x'x} & c_{x'y} & c_{x'z} \\ c_{y'x} & c_{y'y} & c_{y'z} \\ c_{z'x} & c_{z'y} & c_{z'z} \end{bmatrix} \times \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \begin{bmatrix} c_{x'x} & c_{y'x} & c_{z'x} \\ c_{x'y} & c_{y'y} & c_{z'y} \\ c_{x'z} & c_{y'z} & c_{z'z} \end{bmatrix} \tag{2.12b}$$

As discussed in Section 2.2, the terms $c_{i'j}$ which appear in Equation 2.12a,b are *direction cosines* and equal the cosine of the angle between the axes of the x–y–z and x'–y'–z' coordinate systems. Recall that the algebraic sign of an angle of rotation is defined in accordance with the right-hand rule, and that angles are defined *from* the x–y–z coordinate system *to* x'–y'–z' coordinate system. Equation 2.12a,b is called the *transformation law for a second-order tensor*.

If an analysis is being performed with the aid of a digital computer, which nowadays is almost always the case, then matrix notation (Equation 2.12b) will most likely be used to transform a stress tensor from one coordinate system to another. Conversely, if a stress transformation is to be accomplished using hand calculations, then indicial notation (Equation 2.12a) may be the preferred choice. To apply Equation 2.12a, the stress component of interest is specified by selecting the appropriate values for subscripts i' and j', and then the terms on the right side of the equality are summed over the entire range of the remaining two subscripts, k and l. For example, suppose we wish write the relationship between $\sigma_{x'z'}$ and the stress components in the x–y–z coordinate system in expanded form. We first specify that $i' = x'$ and $j' = z'$, and Equation 2.12a becomes:

$$\sigma_{x'z'} = c_{x'k}\, c_{z'l}\, \sigma_{kl} \quad \text{where} \quad k, l = x, y, z$$

We then sum all terms on the right side of the equality formed by cycling through the entire range of k and l. In expanded form, we have

$$\begin{aligned} \sigma_{x'z'} = {} & c_{x'x}\, c_{z'x}\, \sigma_{xx} + c_{x'x}\, c_{z'y}\, \sigma_{xy} + c_{x'x}\, c_{z'z}\, \sigma_{xz} \\ & + c_{x'y}\, c_{z'x}\, \sigma_{yx} + c_{x'y}\, c_{z'y}\, \sigma_{yy} + c_{x'y}\, c_{z'z}\, \sigma_{yz} \\ & + c_{x'z}\, c_{z'x}\, \sigma_{zx} + c_{x'z}\, c_{z'y}\, \sigma_{zy} + c_{x'z}\, c_{z'z}\, \sigma_{zz} \end{aligned} \tag{2.13}$$

Equations 2.12a and 2.12b show that the value of any individual stress component $\sigma_{i'j'}$ varies as the stress tensor is transformed from one coordinate system to another. However, it can be shown [1,2] that there are features of the *total* stress tensor that do not vary when the tensor is transformed from one coordinate system to another. These features are called the *stress invariants*. For a second-order tensor three independent stress invariants exist, and are defined as follows:

$$\text{First stress invariant} = \Theta = \sigma_{ii} \tag{2.14a}$$

$$\text{Second stress invariant} = \Phi = \frac{1}{2}(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ij}) \tag{2.14b}$$

$$\text{Third stress invariant} = \Psi = \frac{1}{6}(\sigma_{ii}\sigma_{jj}\sigma_{kk} - 3\sigma_{ii}\sigma_{jk}\sigma_{jk} + 2\sigma_{ij}\sigma_{jk}\sigma_{ki}) \tag{2.14c}$$

Alternatively, by expanding these equations over the range $i,j,k = x,y,z$ and simplifying, the stress invariants can be written:

$$\text{First stress invariant} = \Theta = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} \tag{2.15a}$$

$$\text{Second stress invariant} = \Phi = \sigma_{xx}\sigma_{yy} + \sigma_{xx}\sigma_{zz} + \sigma_{yy}\sigma_{zz} - (\sigma_{xy}^2 + \sigma_{xz}^2 + \sigma_{yz}^2) \tag{2.15b}$$

$$\text{Third stress invariant} = \Psi = \sigma_{xx}\sigma_{yy}\sigma_{zz} - \sigma_{xx}\sigma_{yz}^2 - \sigma_{yy}\sigma_{xz}^2 - \sigma_{zz}\sigma_{xy}^2 + 2\sigma_{xy}\sigma_{xz}\sigma_{yz} \tag{2.15c}$$

The three stress invariants are conceptually similar to the magnitude of a force tensor. That is, the value of the three stress invariants is independent of the coordinate used to describe the stress tensor, just as the magnitude of a force vector is independent of the coordinate system used to describe the force. This invariance will be illustrated in the following Example Problem.

Example Problem 2.3

Given: A state of stress referenced to a x–y–z coordinate is known to be

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} 50 & -10 & 15 \\ -10 & 25 & 30 \\ 15 & 30 & -5 \end{bmatrix} (\text{ksi})$$

It is desired to express this state of stress in an x''–y''–z'' coordinate system, generated by the following two sequential rotations:

i. Rotation of $\theta = 20°$ about the original z-axis (which defines an intermediate x'–y'–z' coordinate system), followed by
ii. Rotation of $\beta = 35°$ about the x'-axis (which defines the final x''–y''–z'' coordinate system)

PROBLEM

a. Rotate the stress tensor to the x''–y''–z'' coordinate system.
b. Calculate the first, second, and third invariants of the stress tensor using (i) elements of the stress tensor referenced to the x–y–z coordinate system, σ_{ij}, and (ii) elements of the stress tensor referenced to the x''–y''–z'' coordinate system, $\sigma_{i''j''}$.

SOLUTION

a. General expressions for direction cosines relating the x–y–z and x''–y''–z'' coordinate systems were determined as a part of Example Problem 2.1. The direction cosines were found to be

$$\begin{aligned}
c_{x''x} &= \cos\theta \\
c_{x''y} &= \sin\theta \\
c_{x''z} &= 0 \\
c_{y''x} &= -\cos\beta\,\sin\theta \\
c_{y''y} &= \cos\beta\,\cos\theta \\
c_{y''z} &= \sin\beta \\
c_{z''x} &= \sin\beta\,\sin\theta \\
c_{z''y} &= -\sin\beta\,\cos\theta \\
c_{z''z} &= \cos\beta
\end{aligned}$$

As in this problem $\theta = 20°$ and $\beta = 35°$, the numerical values of the direction cosines are

$$\begin{aligned}
c_{x''x} &= \cos(20°) = 0.9397 \\
c_{x''y} &= \sin(20°) = 0.3420 \\
c_{x''z} &= 0 \\
c_{y''x} &= -\cos(35°)\sin(20°) = -0.2802 \\
c_{y''y} &= \cos(35°)\cos(20°) = 0.7698 \\
c_{y''z} &= \sin(35°) = 0.5736 \\
c_{z''x} &= \sin(35°)\sin(20°) = 0.1962 \\
c_{z''y} &= -\sin(35°)\cos(20°) = -0.5390 \\
c_{z''z} &= \cos(35°) = 0.8192
\end{aligned}$$

Each component of the transformed stress tensor is now found through application of either Equation 2.12a or 2.12b. For example, if indicial notation is used stress component $\sigma_{x''z''}$ can be found using Equation 2.13:

$$\begin{aligned}\sigma_{x''z''} &= c_{x''x}\, c_{z''x}\, \sigma_{xx} + c_{x''x}\, c_{z''y}\, \sigma_{xy} + c_{x''x}\, c_{z''z}\, \sigma_{xz} \\ &\quad + c_{x''y}\, c_{z''x}\, \sigma_{yx} + c_{x''y}\, c_{z''y}\, \sigma_{yy} + c_{x''y}\, c_{z''z}\, \sigma_{yz} \\ &\quad + c_{x''z}\, c_{z''x}\, \sigma_{zx} + c_{x''z}\, c_{z''y}\, \sigma_{zy} + c_{x''z}\, c_{z''z}\, \sigma_{zz}\end{aligned}$$

$$\begin{aligned}\sigma_{x''z''} &= (0.9397)(0.1962)(50\,\text{ksi}) + (0.9397)(-0.5390)(-10\,\text{ksi}) \\ &\quad + (0.9397)(0.8192)(15\,\text{ksi}) \\ &\quad + (0.3420)(0.1962)(-10\,\text{ksi}) + (0.3420)(-0.5390)(25\,\text{ksi}) \\ &\quad + (0.3420)(0.8192)(30\,\text{ksi}) \\ &\quad + (0)(0.1962)(15\,\text{ksi}) + (0)(-0.5390)(30\,\text{ksi}) + (0)(0.8192)(-5\,\text{ksi})\end{aligned}$$

$$\sigma_{x''z''} = 28.95\,\text{ksi}$$

Alternatively, if matrix notation is used, then Equation 2.12b becomes:

$$\begin{bmatrix}\sigma_{x''x''} & \sigma_{x''y''} & \sigma_{x''z''} \\ \sigma_{y''x''} & \sigma_{y''y''} & \sigma_{y''z''} \\ \sigma_{z''x''} & \sigma_{z''y''} & \sigma_{z''z''}\end{bmatrix} = \begin{bmatrix}0.9397 & 0.3420 & 0 \\ -0.2802 & 0.7698 & 0.5736 \\ 0.1962 & -0.5390 & 0.8192\end{bmatrix}\begin{bmatrix}50 & -10 & 15 \\ -10 & 25 & 30 \\ 15 & 30 & -5\end{bmatrix} \times \begin{bmatrix}0.9397 & -0.2802 & 0.1962 \\ 0.3420 & 0.7698 & -0.5390 \\ 0 & 0.5736 & 0.8192\end{bmatrix}$$

Completing the matrix multiplication indicated, there results:

$$\begin{bmatrix}\sigma_{x''x''} & \sigma_{x''y''} & \sigma_{x''z''} \\ \sigma_{y''x''} & \sigma_{y''y''} & \sigma_{y''z''} \\ \sigma_{z''x''} & \sigma_{z''y''} & \sigma_{z''z''}\end{bmatrix} = \begin{bmatrix}40.65 & 1.113 & 28.95 \\ 1.113 & 43.08 & -10.60 \\ 28.95 & -10.60 & -13.72\end{bmatrix}(\text{ksi})$$

Notice that the value of $\sigma_{x''z''}$ determined through matrix multiplication is identical to that obtained using indicial notation, as previously described. The stress element is shown in the original and final coordinate systems in Figure 2.11.

b. The first, second, and third stress invariants will now be calculated using components of both σ_{ij} and $\sigma_{i''j''}$. It is expected

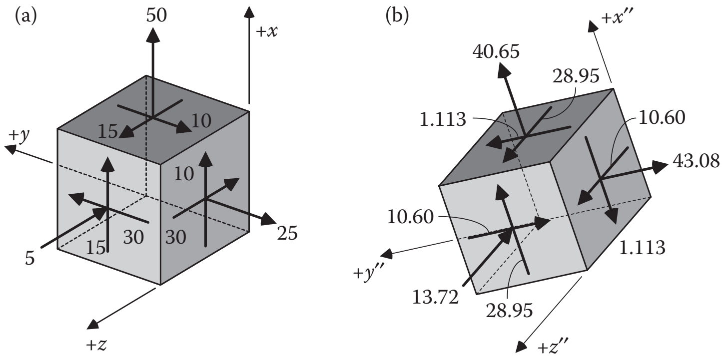

FIGURE 2.11
Stress tensor of Example Problem 2.3, referenced to two different coordinate systems (magnitude of all stress components in ksi). (a) Referenced to *x–y–z* coordinate system; (b) referenced to *x″–y″–z″* coordinate system.

that identical values will be obtained, as the stress invariants are independent of coordinate system.

First Stress Invariant:
x–y–z coordinate system:

$$\Theta = \sigma_{ii} = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$$

$$\Theta = (50 + 25 - 5)°\text{ksi}$$

$$\Theta = 70°\,\text{ksi}$$

x″–y″–z″ coordinate system:

$$\Theta = \sigma_{i''i''} = \sigma_{x''x''} + \sigma_{y''y''} + \sigma_{z''z''}$$

$$\Theta = (40.65 + 43.08 - 13.72)$$

$$\Theta = 70\,\text{ksi}$$

As expected, the first stress invariant is independent of coordinate system.

Second Stress Invariant:
x–y–z coordinate system:

$$\Phi = \frac{1}{2}\left(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ij}\right) = \sigma_{xx}\sigma_{yy} + \sigma_{xx}\sigma_{zz} + \sigma_{yy}\sigma_{zz} - \left(\sigma_{xy}^2 + \sigma_{xz}^2 + \sigma_{yz}^2\right)$$

$$\Phi = \left\{(50)(25) + (50)(-5) + (25)(-5) - [(-10)^2 + (15)^2 + (30)^2]\right\}(\text{ksi})^2$$

$$\Phi = -350(\text{ksi})^2$$

x″–y″–z″ coordinate system:

$$\Phi = \frac{1}{2}\left(\sigma_{i''i''}\sigma_{j''j''} - \sigma_{i''j''}\sigma_{i''j''}\right)$$

$$\Phi = \sigma_{x''x''}\sigma_{y''y''} + \sigma_{x''x''}\sigma_{z''z''} + \sigma_{y''y''}\sigma_{z''z''} - \left(\sigma^2_{x''y''} + \sigma^2_{x''z''} + \sigma^2_{y''z''}\right)$$

$$\Phi = \left\{(40.65)(43.08) + (40.65)(-13.72) + (43.08)(-13.72) - [(1.113)^2 + (28.95)^2 + (-10.60)^2]\right\}(\text{ksi})^2$$

$$\Phi = -350(\text{ksi})^2$$

As expected, the second stress invariant is independent of coordinate system.

Third Stress Invariant:

x–y–z coordinate system:

$$\Psi = \frac{1}{6}\left(\sigma_{ii}\sigma_{jj}\sigma_{kk} - 3\sigma_{ii}\sigma_{jk}\sigma_{jk} + 2\sigma_{ij}\sigma_{jk}\sigma_{ki}\right)$$

$$\Psi = \sigma_{xx}\sigma_{yy}\sigma_{zz} - \sigma_{xx}\sigma^2_{yz} - \sigma_{yy}\sigma^2_{xz} - \sigma_{zz}\sigma^2_{xy} + 2\sigma_{xy}\sigma_{xz}\sigma_{yz}$$

$$\Psi = [(50)(25)(-5) - (50)(30)^2 - (25)(15)^2 - (-5)(-10)^2 + 2(-10)(15)(30)](\text{ksi})^3$$

$$\Psi = -65375(\text{ksi})^3$$

x″–y″–z″ coordinate system:

$$\Psi = \frac{1}{6}\left(\sigma_{i''i''}\sigma_{j''j''}\sigma_{k''k''} - 3\sigma_{i''i''}\sigma_{j''k''}\sigma_{j''k''} + 2\sigma_{i''j''}\sigma_{j''k''}\sigma_{k''i''}\right)$$

$$\Psi = \sigma_{x''x''}\sigma_{y''y''}\sigma_{z''z''} - \sigma_{x''x''}\sigma^2_{y''z''} - \sigma_{y''y''}\sigma^2_{x''z''} - \sigma_{z''z''}\sigma^2_{x''y''} + 2\sigma_{x''y''}\sigma_{x''z''}\sigma_{y''z''}$$

$$\Psi = [(40.65)(43.08)(-13.72) - (40.65)(-10.60)^2 - (43.08)(28.95)^2$$
$$- (-13.72)(1.113)^2 + 2(1.113)(28.95)(-10.60)](\text{ksi})^3$$

$$\Psi = -65375(\text{ksi})^3$$

As expected, the third stress invariant is independent of coordinate system.

2.7 Principal Stresses

It is always possible to rotate the stress tensor to a special coordinate system in which no shear stresses exist. This coordinate system is called the *principal stress coordinate system*, and the normal stresses that exist in this coordinate system are called *principal stresses*. Principal stresses can be used to predict failure of isotropic materials. Therefore, knowledge of the principal stresses and orientation of the principal stress coordinate system is of vital importance during design and analysis of isotropic metal structures.

This is not the case for anisotropic composite materials. Failure of composite material is *not* governed by principal stresses. Principal stresses are only of occasional interest to the composite engineer and are reviewed here only in the interests of completeness.

Principal stresses are usually denoted as σ_1, σ_2, and σ_3. However, in the study of composites, the labels "1", "2", and "3" are used to label a special coordinate system called the *principal material* coordinate system. Therefore, in this chapter the axes associated with the principal stress coordinate system will be labeled the "p_1", "p_2", and "p_3" axes, and the principal stresses will be denoted as σ_{p1}, σ_{p2}, and σ_{p3}.

Principal stresses may be related to stress components in an x–y–z coordinate system using the free-body diagram shown in Figure 2.12. It is assumed that plane ABC is one of the three principal planes (i.e., $n = 1, 2$, or 3), and therefore no shear stress exists on this plane. The line-of-action of principal stress σ_{pn} defines one axis of the principal stress coordinate system. The direction cosines between this principal axis and the x-, y-, and z-axes are c_{pnx}, c_{pny}, and c_{pnz}, respectively. The surface area of triangle ABC is denoted A_{ABC}. The normal force acting over triangle ABC therefore equals $(\sigma_{pn})(A_{ABC})$. The components of this normal force acting in the x-, y-, and z-directions equal $(c_{pnx})(\sigma_{pn})(A_{ABC})$, $(c_{pny})(\sigma_{pn})(A_{ABC})$ and $(c_{pnz})(\sigma_{pn})(A_{ABC})$, respectively.

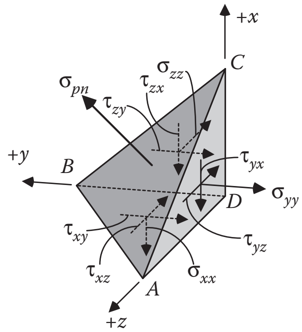

FIGURE 2.12
Free-body diagram used to relate strress components in the x–y–z coordinate system to a principal stress.

The area of the other triangular faces are given by

Area of triangle $ABD = (c_{pnx})(A_{ABC})$
Area of triangle $ACD = (c_{pny})(A_{ABC})$
Area of triangle $BCD = (c_{pnz})(A_{ABC})$.

Summing forces in the x-direction and equating to zero, we obtain

$$c_{pnx}\sigma_{pn}A_{ABC} - \sigma_{xx}c_{pnx}A_{ABC} - \tau_{xy}c_{pny}A_{ABC} - \tau_{xz}c_{pnz}A_{ABC} = 0$$

which can be reduced and simplified to

$$(\sigma_{pn} - \sigma_{xx})c_{pnx} - \tau_{xy}c_{pny} - \tau_{xz}c_{pnz} = 0 \tag{2.16a}$$

Similarly, summing forces in the y- and z-directions results in

$$-\tau_{xy}c_{pnx} + (\sigma_{pn} - \sigma_{yy})c_{pny} - \tau_{yz}c_{pnz} = 0 \tag{2.16b}$$

$$-\tau_{xz}c_{pnx} - \tau_{yz}c_{pny} + (\sigma_{pn} - \sigma_{zz})c_{pnz} = 0 \tag{2.16c}$$

Equation 2.16 represent three linear homogeneous equations which must be satisfied simultaneously. As direction cosines c_{pnx}, c_{pny}, and c_{pnz} must also satisfy Equation 2.8, and therefore cannot all equal zero, the solution can be obtained by requiring that the determinant of the coefficients of c_{pnx}, c_{pny}, and c_{pnz} equal zero:

$$\begin{vmatrix} (\sigma_{pn} - \sigma_{xx}) & -\tau_{xy} & -\tau_{xz} \\ -\tau_{xy} & (\sigma_{pn} - \sigma_{yy}) & -\tau_{yz} \\ -\tau_{xz} & -\tau_{yz} & (\sigma_{pn} - \sigma_{zz}) \end{vmatrix} = 0$$

Equating the determinant to zero results in the following cubic equation:

$$\sigma_{pn}^3 - \Theta\sigma_{pn}^2 + \Phi\sigma_{pn} - \Psi = 0 \tag{2.17}$$

where Θ, Φ, and Ψ are the first, second, and third stress invariants, respectively, and have been previously listed as Equations 2.14 and 2.15. The three roots of this cubic equation represent the three principal stresses and may be found by application of the standard approach [3]. By convention, the principal stresses are numbered such that σ_{p1} is the algebraically greatest principal stress, whereas σ_{p3} is the algebraically least. That is, $\sigma_{p1} > \sigma_{p2} > \sigma_{p3}$.

Once the principal stresses are determined the three sets of direction cosines (which define the principal coordinate directions) are found by substituting the three principal stresses given by Equation 2.17 into Equation 2.16 in turn. As only two of Equation 2.16 are independent, Equation 2.8 is used as a third independent equation involving the three unknown constants, c_{pnx}, c_{pny}}, and c_{pnz}.

The process of finding principal stresses and direction cosines will be demonstrated in the following Example Problem.

Example Problem 2.4

Given: A state of stress referenced to an *x*–*y*–*z* coordinate is known to be:

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} 50 & -10 & 15 \\ -10 & 25 & 30 \\ 15 & 30 & -5 \end{bmatrix} (\text{ksi})$$

PROBLEM

Find (a) the principal stresses and (b) the direction cosines that define the principal stress coordinate system.

SOLUTION

This is the same stress tensor considered in Example Problem 2.3. As a part of that problem the first, second, and third stress invariants were found to be

$$\Theta = 70 \text{ ksi}$$

$$\Phi = -350 \, (\text{ksi})^2$$

$$\Psi = -65375 \, (\text{ksi})^3$$

a. *Determining the Principal Stresses*: In accordance with Equation 2.17, the three principal stresses are the roots of the following cubic equation:

$$\sigma^3 - 70\sigma^2 - 350\sigma + 65375 = 0$$

The three roots of this equation represent the three principal stresses, and are given by

$$\sigma_{p1} = 54.21\text{ ksi}, \quad \sigma_{p2} = 43.51\text{ ksi}, \quad \text{and} \quad \sigma_{p3} = -27.72\text{ ksi}$$

b. *Determining the Direction Cosines*: The first two of Equations 2.16 and 2.8 are used to form three independent equations in three unknowns. We have

$$(\sigma_{pn} - \sigma_{xx})c_{pnx} - \tau_{xy}c_{pny} - \tau_{xz}c_{pnz} = 0$$

$$-\tau_{xy}c_{pnx} + (\sigma_{pn} - \sigma_{yy})c_{pny} - \tau_{yz}c_{pnz} = 0$$

$$(c_{pnx})^2 + (c_{pny})^2 + (c_{pnz})^2 = 1$$

Direction cosines for σ_{p1}: The three independent equations become:

$$(54.21 - 50)c_{p1x} + 10c_{p1y} - 15c_{p1z} = 0$$

$$10c_{p1x} + (54.21 - 25)c_{p1y} - 30c_{p1z} = 0$$

$$(c_{p1x})^2 + (c_{p1y})^2 + (c_{p1z})^2 = 1$$

Solving simultaneously, we obtain:

$$c_{p1x} = -0.9726$$

$$c_{p1y} = 0.1666$$

$$c_{p1z} = -0.1620$$

Direction cosines for σ_{p2}: The three independent equations become:

$$(43.51 - 50)c_{p2x} + 10c_{p2y} - 15c_{p2z} = 0$$

$$10c_{p2x} + (43.51 - 25)c_{p2y} - 30c_{p2z} = 0$$

$$(c_{p2x})^2 + (c_{p2y})^2 + (c_{p2z})^2 = 1$$

Solving simultaneously, we obtain

$$c_{p2x} = -0.05466$$

$$c_{p2y} = -0.8416$$

$$c_{p2z} = -0.5738$$

Direction cosines for σ_{p3}: The three independent equations become:

$$(-27.72 - 50)c_{p3x} + 10c_{p3y} - 15c_{p3z} = 0$$

$$10c_{p3x} + (-27.72 - 25)c_{p3y} - 30c_{p3z} = 0$$

$$(c_{p3x})^2 + (c_{p3y})^2 + (c_{p3z})^2 = 1$$

Solving simultaneously, we obtain:

$$c_{p3x} = -0.8276$$

$$c_{p3y} = 0.2259$$

$$c_{p3z} = 0.5138$$

2.8 Plane Stress

A stress tensor is *always* defined by six components of stress: three normal stress components and three shear stress components. However, in practice a state of stress is often encountered in which the magnitudes of three stress components in one coordinate direction are known to be zero *a priori*. For example, suppose $\sigma_{zz} = \tau_{xz} = \tau_{yz} = 0$, as shown in Figure 2.13a. As the three remaining nonzero stress components (σ_{xx}, σ_{yy}, and τ_{xy}), all lie within the x–y plane, such a condition is called a state of *plane stress*. Plane stress conditions occur most often because of the geometry of the structure of interest. Specifically, the plane stress condition usually exists in relatively thin, plate-like structures. Examples include the web of an I-beam, the body panel of an automobile, or the skin of an airplane fuselage. In these instances the stresses induced normal to the plane of the structure are very small compared with those induced within the plane of the structure. Hence, the small out-of-plane stresses are assumed to be zero, and attention is focused on the relatively high stress components acting within the plane of the structure.

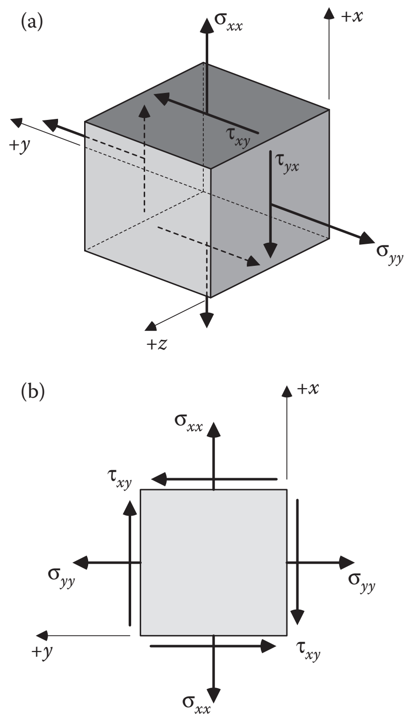

FIGURE 2.13
Stress elements subjected to a state of plane stress. (a) 3-D stress element subjected to a plane stress state (all stress components shown in a positive sense); (b) plane stress element drawn as a square rather than a cube (positive *z*-axis out of the plane of the figure; all stress components shown in a positive sense).

As laminated composites are often used in the form of thin plates or shells, the plane stress assumption is widely applicable in composite structures and will be used throughout most of the analyses discussed in this book. As the out-of-plane stresses are negligibly small, for convenience an infinitesimal stress element subjected to plane stress will usually be drawn as a square rather than a cube, as shown in Figure 2.13b.

Results discussed in earlier sections for general 3D state of stress will now be specialized for the plane stress condition. It will be assumed that the nonzero stresses lie in the x–y plane (i.e., $\sigma_{zz} = \tau_{xz} = \tau_{yz} = 0$). This allows the remaining components of stress to be written in the form of a column array, rather than a 3×3 array:

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & 0 \\ \tau_{xy} & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix}$$

Note that when a plane stress state is described, stress *appears* to be a first-order tensor, as (apparently) only three components of stress (σ_{xx}, σ_{yy}, and τ_{xy}) need be specified to describe the state of stress. This is, of course, not the case. Stress is a second-order tensor in all instances, and six components of stress must *always* be specified to define a state of stress. When we invoke the plane stress assumption, we have simply assumed *a priori* that the magnitude of three stress components (σ_{zz}, τ_{xz}, and τ_{yz}) are zero.

Recall that either Equation 2.12a or Equation 2.12b governs the transformation of a stress tensor from one coordinate system to another. Equation 2.12b is repeated here for convenience:

$$\begin{bmatrix} \sigma_{x'x'} & \sigma_{x'y'} & \sigma_{x'z'} \\ \sigma_{y'x'} & \sigma_{y'y'} & \sigma_{y'z'} \\ \sigma_{z'x'} & \sigma_{z'y'} & \sigma_{z'z'} \end{bmatrix} = \begin{bmatrix} c_{x'x} & c_{x'y} & c_{x'z} \\ c_{y'x} & c_{y'y} & c_{y'z} \\ c_{z'x} & c_{z'y} & c_{z'z} \end{bmatrix} \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$

$$\times \begin{bmatrix} c_{x'x} & c_{y'x} & c_{z'x} \\ c_{x'y} & c_{y'y} & c_{z'y} \\ c_{x'z} & c_{y'z} & c_{z'z} \end{bmatrix} \qquad \text{(2.12b) (repeated)}$$

When transformation of a plane stress tensor is considered, it will be assumed that the x'–y'–z' coordinate system is generated from the x–y–z system by a rotation θ about the z-axis. That is, the z- and z'-axes are coincident, as shown in Figure 2.14. In this case, the direction cosines are

$$c_{x'x} = \cos(\theta)$$

$$c_{x'y} = \cos(90° - \theta) = \sin(\theta)$$

$$c_{x'z} = \cos(90°) = 0$$

$$c_{y'x} = \cos(90° + \theta) = -\sin(\theta)$$

$$c_{y'y} = \cos(\theta)$$

$$c_{y'z} = \cos(90°) = 0$$

$$c_{z'x} = \cos(90°) = 0$$

$$c_{z'y} = \cos(90°) = 0$$

$$c_{z'z} = \cos(0°) = 1$$

If we now (a) substitute these direction cosines into Equation 2.12b, (b) label the shear stresses using the symbol using "τ" rather "σ,"

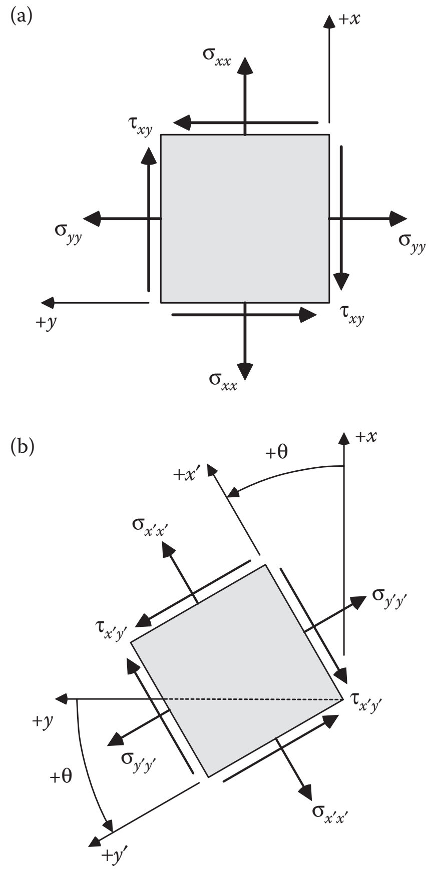

FIGURE 2.14
Transformation of a plane stress element from one coordinate system to another. (a) Plane stress element referenced to the x–y–z coordinate system; (b) plane stress element referenced to the x'–y'–z' coordinate system, oriented θ-degrees counter-clockwise from the x–y–z coordinate system.

and (c) note that $\sigma_{zz} = \tau_{xz} = \tau_{yz} = 0$ by assumption, then Equation 2.12b becomes:

$$\begin{bmatrix} \sigma_{x'x'} & \tau_{x'y'} & \tau_{x'z'} \\ \tau_{y'x'} & \sigma_{y'y'} & \tau_{y'z'} \\ \tau_{z'x'} & \tau_{z'y'} & \sigma_{z'z'} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{xx} & \tau_{xy} & 0 \\ \tau_{yx} & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\times \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Completing the matrix multiplication indicated results in:

$$\begin{bmatrix} \sigma_{x'x'} & \tau_{x'y'} & \tau_{x'z'} \\ \tau_{y'x'} & \sigma_{y'y'} & \tau_{y'z'} \\ \tau_{z'x'} & \tau_{z'y'} & \sigma_{z'z'} \end{bmatrix}$$
$$= \begin{bmatrix} \cos^2\theta\sigma_{xx} + \sin^2\theta\sigma_{yy} + 2\cos\theta\sin\theta\tau_{xy} & -\cos\theta\sin\theta\sigma_{xx} + \cos\theta\sin\theta\sigma_{yy} + (\cos^2\theta - \sin^2\theta)\tau_{xy} & 0 \\ -\cos\theta\sin\theta\sigma_{xx} + \cos\theta\sin\theta\sigma_{yy} + (\cos^2\theta - \sin^2\theta)\tau_{xy} & \sin^2\theta\sigma_{xx} + \cos^2\theta\sigma_{yy} - 2\cos\theta\sin\theta\tau_{xy} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.18}$$

As would be expected, the out-of-plane stresses are zero: $\sigma_{z'z'} = \tau_{x'z'} = \tau_{y'z'} = 0$. The remaining stress components are

$$\begin{aligned} \sigma_{x'x'} &= \cos^2(\theta)\sigma_{xx} + \sin^2(\theta)\sigma_{yy} + 2\cos(\theta)\sin(\theta)\tau_{xy} \\ \sigma_{y'y'} &= \sin^2(\theta)\sigma_{xx} + \cos^2(\theta)\sigma_{yy} - 2\cos(\theta)\sin(\theta)\tau_{xy} \\ \tau_{x'y'} &= -\cos(\theta)\sin(\theta)\sigma_{xx} + \cos(\theta)\sin(\theta)\sigma_{yy} + [\cos^2(\theta) - \sin^2(\theta)]\tau_{xy} \end{aligned} \tag{2.19}$$

Equation 2.19 can be written using matrix notation as

$$\begin{Bmatrix} \sigma_{x'x'} \\ \sigma_{y'y'} \\ \tau_{x'y'} \end{Bmatrix} = \begin{bmatrix} \cos^2(\theta) & \sin^2(\theta) & 2\cos(\theta)\sin(\theta) \\ \sin^2(\theta) & \cos^2(\theta) & -2\cos(\theta)\sin(\theta) \\ -\cos(\theta)\sin(\theta) & \cos(\theta)\sin(\theta) & \cos^2(\theta) - \sin^2(\theta) \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} \tag{2.20}$$

It should be kept in mind that these results are valid only for a state of plane stress. More precisely, Equations 2.19 and 2.20 represent stress transformations within the x–y plane, *and are only valid if the z-axis is a principal stress axis*.

The 3×3 array that appears in Equation 2.20 is called the *transformation matrix*, and is abbreviated as $[T]$:

$$[T] = \begin{bmatrix} \cos^2(\theta) & \sin^2(\theta) & 2\cos(\theta)\sin(\theta) \\ \sin^2(\theta) & \cos^2(\theta) & -2\cos(\theta)\sin(\theta) \\ -\cos(\theta)\sin(\theta) & \cos(\theta)\sin(\theta) & \cos^2(\theta) - \sin^2(\theta) \end{bmatrix} \tag{2.21}$$

The stress invariants (given by Equation 2.14 or 2.15) are considerably simplified in the case of plane stress. Since by definition $\sigma_{zz} = \tau_{xz} = \tau_{yz} = 0$, the stress invariants become:

$$\text{First stress invariant} = \Theta = \sigma_{xx} + \sigma_{yy}$$
$$\text{Second stress invariant} = \Phi = \sigma_{xx}\sigma_{yy} - \tau_{xy}^2 \tag{2.22}$$
$$\text{Third stress invariant} = \Psi = 0$$

The principal stresses equal the roots of the cubic equation previously listed as Equation 2.17. In the case of plane stress, this cubic equation becomes (since $\Psi = 0$):

$$\sigma^3 - \Theta\sigma^2 + \Phi\sigma = 0 \tag{2.23}$$

Obviously, one root of Equation 2.23 is $\sigma = 0$. This root corresponds to σ_{zz}, and for present purposes will be labeled σ_{p3} even though it may not be the algebraically least principal stress. Thus, in the case of plane stress the z-axis is a principal stress direction, and $\sigma_{zz} = \sigma_{p3} = 0$ is one of the three principal stresses. As the three principal stress directions are orthogonal, this implies that the remaining two principal stress directions must lie within the x–y plane.

Removing the known root from Equation 2.23, we have the following quadratic equation:

$$\sigma^2 - \Theta\sigma + \Phi = 0 \tag{2.24}$$

The two roots of this quadratic equation are found using the standard approach [3], and are given by

$$\sigma_{p1}, \sigma_{p2} = \frac{1}{2}\left[\Theta \pm \sqrt{\Theta^2 - 4\Phi}\right] \tag{2.25}$$

Substituting Equation 2.22 into 2.25 and simplifying, there results:

$$\sigma_{p1}, \sigma_{p2} = \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2} \tag{2.26}$$

The angle θ_p between the x-axis and either the p_1 or p_2 axis is given by

$$\theta_p = \frac{1}{2}\arctan\left(\frac{2\tau_{xy}}{\sigma_{xx} - \sigma_{yy}}\right) \tag{2.27}$$

Example Problem 2.5

Given: The plane stress element shown in Figure 2.15a.

PROBLEM

a. Rotate the stress element to a new coordinate system oriented 15° clockwise from the *x*-axis, and redraw the stress element with all stress components properly oriented.

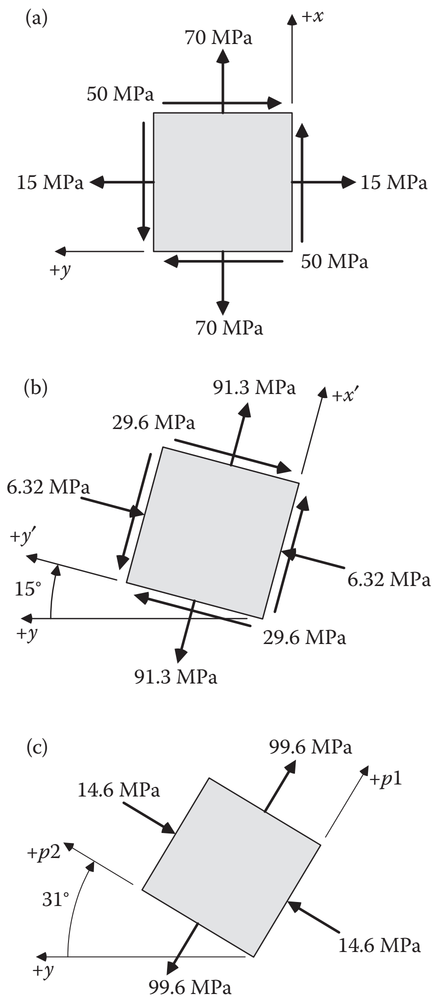

FIGURE 2.15
Plane stress elements associated with Example Problem 2.5. (a) Plane stress element in the x–y coordinate system; (b) plane stress element in the x'–y' coordinate system; (c) plane stress element in the principal stress coordinate system.

b. Determine the principal stresses and principal stress coordinate system, and redraw the stress element with the principal stress components properly oriented.

SOLUTION

a. The following components of stress are implied by the stress element shown (note that the shear stress is algebraically negative, in accordance with the sign convention discussed in Section 2.5):

$$\sigma_{xx} = 70\,\text{MPa}$$

$$\sigma_{yy} = 15\,\text{MPa}$$

$$\tau_{xy} = -50\,\text{MPa}$$

The stress element is to be rotated clockwise. That is, the $+x'$-axis is rotated away from the $+y$-axis. Applying the right-hand rule it is clear that this is a negative rotation:

$$\theta = -15°$$

Equation 2.20 becomes:

$$\begin{Bmatrix} \sigma_{x'x'} \\ \sigma_{y'y'} \\ \tau_{x'y'} \end{Bmatrix} = \begin{bmatrix} \cos^2(-15°) & \sin^2(-15°) & 2\cos(-15°)\sin(-15°) \\ \sin^2(-15°) & \cos^2(-15°) & -2\cos(-15°)\sin(-15°) \\ -\cos(-15°)\sin(-15°) & \cos(-15°)\sin(-15°) & \cos^2(-15°) - \sin^2(-15°) \end{bmatrix} \times \begin{Bmatrix} 70 \\ 15 \\ -50 \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_{x'x'} \\ \sigma_{y'y'} \\ \tau_{x'y'} \end{Bmatrix} = \begin{bmatrix} 0.9330 & 0.0670 & -0.5000 \\ 0.0670 & 0.9330 & 0.5000 \\ 0.2500 & -0.2500 & 0.8660 \end{bmatrix} \begin{Bmatrix} 70 \\ 15 \\ -50 \end{Bmatrix} = \begin{Bmatrix} 91.3 \\ -6.32 \\ -29.6 \end{Bmatrix} \text{MPa}$$

The rotated stress element is shown in Figure 2.15b.

b. The principal stresses are found through application of Equation 2.26:

$$\sigma_{p1}, \sigma_{p2} = \frac{70 + 15}{2} \pm \sqrt{\left(\frac{70 - 15}{2}\right)^2 + (-50)^2} = 42.5 \pm 57.1\,\text{MPa}$$

$$\sigma_{p1} = 99.6\,\text{MPa}$$

$$\sigma_{p2} = -14.6\,\text{MPa}$$

The orientation of the principal stress coordinate system is given by Equation 2.27:

$$\theta_p = \frac{1}{2}\arctan\left(\frac{2(-50)}{70-15}\right) = -31°$$

Since θ_p is negative, the +*p*1-axis is oriented 31° clockwise from the *x*-axis. The stress element is shown in the principal stress coordinate system in Figure 2.15c.

2.9 Definition of Strain

All materials deform to some extent when subjected to external forces and/or environmental changes. In essence, the *state of strain* is a measure of the magnitude and orientation of the deformations induced by these effects. As in the case of stress, there are two types of strain: *normal* strain and *shear* strain.

The two types of strain can be visualized using the strain element shown in Figure 2.16. Imagine that a perfect square has been drawn on a flat surface of interest. Initially, angle $\angle ABC$ is exactly $\pi/2$ radians (i.e., initially $\angle ABC = 90°$), and sides *AB* and *BC* are of exactly equal lengths. Now suppose that some mechanism(s) causes the surface to deform. The mechanism(s) which cause the surface to deform need not be defined at this point, but might be external loading (i.e., stresses), a change in temperature, and/or (in the case of polymeric-based materials such as composites) the adsorption or desorption of water molecules. In any event, as the surface is deformed the initially square element drawn on the surface is deformed as well. As shown in Figure 2.16, point *A* moves to point *A′*, and point *C* moves to point *C′*. It is

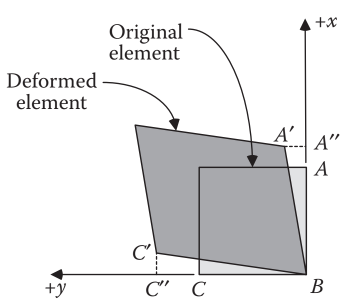

FIGURE 2.16
2-D element used to illustrate normal and shear strains (deformations are shown greatly exaggerated for clarity).

assumed that the element remains a parallelogram, that is, it is assumed that sides $A'B$ and $C'B$ remain straight lines after deformation. This assumption is valid if the element is *infinitesimally small*. In the present context "infinitesimally small" implies that lengths AB and CB are small enough such that the deformed element may be treated as a parallelogram.

Normal strain ε_{xx} is defined as the change in length of AB divided by the original length of AB:

$$\varepsilon_{xx} = \frac{\Delta AB}{AB} \tag{2.28}$$

The change in length AB is given by

$$\Delta AB = (A'B - AB)$$

From the figure it can be seen that the projection of length $A'B$ in the x-direction, that is length $A''B$, is given by

$$A''B = A'B\cos(\angle A'BA) \tag{2.29}$$

If we now assume that $\angle A'BA$ is "small" then we can invoke the *small angle approximation*,* which states that if $\angle A'BA$ is expressed in radians and is less than about 0.1745 radians (about 10°), then:

$$\sin(\angle A'BA) \approx \angle A'BA \quad \tan(\angle A'BA) \approx \angle A'BA \quad \cos(\angle A'BA) \approx 1$$

Based on the small angle approximation Equation 2.29 implies that $A''B \approx A'B$, and therefore that the change in length of AB is approximately given by

$$\Delta AB \approx (A''B - AB) = A''A$$

Equation 2.28 can now be written:

$$\varepsilon_{xx} = \frac{A''A}{AB} \tag{2.30}$$

In an entirely analogous manner, normal strain ε_{yy} is defined as the change in length of CB divided by the original length of CB:

* The reader is encouraged to personally verify the "small angle approximation." For example, use a calculator to demonstrate that an angle of 5 degs equals 0.08727 radians, and that sin(0.08727 rad) = 0.08716, tan(0.08727 rad) = 0.08749 and cos(0.08727 rad) = 0.99619. Therefore, in this example the small angle approximation results in a maximum error of less than 1%.

$$\varepsilon_{yy} = \frac{\Delta CB}{CB}$$

Based on the small angle approximation, the change in length of *CB* is approximately given by

$$\Delta CB = (C'B - CB) \approx C''C$$

and therefore

$$\varepsilon_{yy} = \frac{C''C}{CB} \tag{2.31}$$

As before, the approximation for change in length *CB* is valid if angle $\angle C''BC$ is small.

Recall that the original element shown in Figure 2.16 was assumed to be perfectly square, and in particular that angle $\angle ABC$ is exactly $\pi/2$ radians (i.e., initially $\angle ABC = 90°$). *Engineering shear strain* is defined as the change in angle $\angle ABC$, expressed in radians:

$$\gamma_{xy} = \Delta(\angle ABC) = \angle A'BA + \angle C'BC \tag{2.32}$$

The subscripts associated with a shear strain (e.g., subscripts "*xy*" in Equation 2.32) indicate that the shear strain represents the change in angle defined by line segments originally aligned with the *x*- and *y*-axes.

As discussed in the following sections, it is very convenient to describe a *state of strain* as a second-order tensor. To do so, we must use a slightly different definition of shear strain. Specifically, *tensoral shear strain* is defined as

$$\varepsilon_{xy} = \frac{1}{2}\gamma_{xy} \tag{2.33}$$

As engineering shear strain has been defined as the total change in angle $\angle ABC$, tensoral shear strain is simply half this change in angle. The use of tensoral shear strain is convenient because it greatly simplifies the transformation of a state of strain from one coordinate system to another. However, the use of engineering shear strain is far more common in practice. In this chapter, tensoral shear strain will be used during initial mathematical manipulations of the strain tensor, but all final results will be converted to relations involving engineering shear strain.

Although strains are unit-less quantities, normal strains are usually reported in units of (length/length), and shear strains are usually reported in units of radians. The values of a strain is independent of the system of units used, for example, 1 (meter/meter) = 1 (inch/inch). Common abbreviations used throughout this chapter are as follows:

$$1 \times 10^{-6} \text{ meter/meter} = 1 \text{ micrometer/meter} = 1\ \mu\text{m/m} = 1\ \mu\text{in/in}$$

$$1 \times 10^{-6} \text{ radians} = 1 \text{ microradians} = 1\mu\text{rad}$$

We must next define the algebraic sign convention used to describe individual strain components. The sign convention for normal strains is very straightforward and intuitive: a positive (or "tensile") normal strain is associated with an increase in length, while a negative (or "compressive") normal strain is associated with a decrease in length.

To define the algebraic sign of a shear strain, we first identify the algebraic sign of each face of the infinitesimal strain element (the algebraic sign of face was defined in Section 2.5). An algebraically positive shear strain corresponds to a *decrease* in the angle between two positive faces, or equivalently, to a *decrease* in the angle between two negative faces.

The above sign conventions can be used to confirm that all strains shown in Figure 2.16 are algebraically positive.

Example Problem 2.6

Given: The following two sets of strain components:
Set 1:

$$\varepsilon_{xx} = 1000\ \mu\text{m/m}$$

$$\varepsilon_{yy} = -500\ \mu\text{m/m}$$

$$\gamma_{xy} = 1500\ \mu\text{rad}$$

Set 2:

$$\varepsilon_{xx} = 1000\ \mu\text{m/m}$$

$$\varepsilon_{yy} = -500\ \mu\text{m/m}$$

$$\gamma_{xy} = -1500\ \mu\text{rad}$$

Determine: Prepare sketches (not to scale) of the deformed strain elements represented by the two sets of strain components.

SOLUTION

The required sketches are shown in Figure 2.17. Note that the only difference between the two sets of strain components is that in set 1 γ_{xy} is algebraically positive, whereas in set 2 γ_{xy} is algebraically negative.

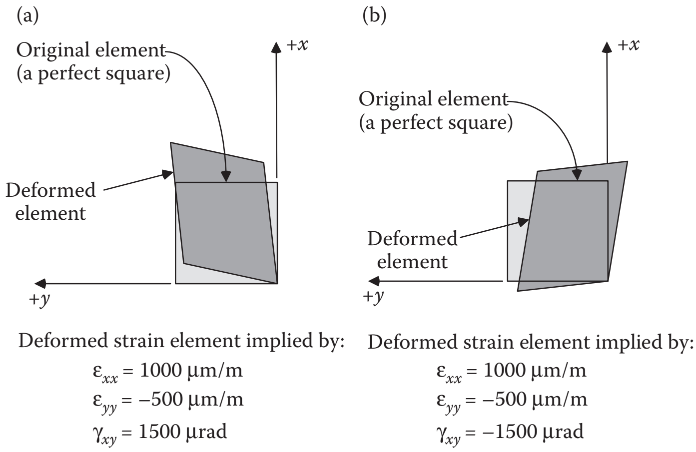

FIGURE 2.17
Strain elements associated with Example Problem 2.6 (not to scale).

2.10 The Strain Tensor

A general 3D solid body is shown in Figure 2.18a. An infinitesimally small cube isolated from an interior region of the body is shown in Figure 2.18b. The cube is referenced to an x–y–z coordinate system, and the cube edges are aligned with these axes.

Now assume that the body is subjected to some mechanism(s) which cause the body to deform. The mechanism(s) which causes this deformation need not be defined at this point, but might be external loading (i.e., stresses), a change in temperature, the adsorption or desorption of water molecules (in the case of polymeric-based materials such as composites), or any combination thereof.

As the entire body is deformed, the internal infinitesimal cube is deformed into a parallelepiped, as shown in Figure 2.18c. It can be shown [1,2] that the state of strain experienced by the cube can represented as a symmetric second-order tensor, involving six components of strain: three normal strains (ε_{xx}, ε_{yy}, ε_{zz}) and three tensoral shear strains (ε_{xy}, ε_{xz}, ε_{yz}). These six strain components are defined in the same manner as those discussed in the preceding section. Normal strains ε_{xx}, ε_{yy}, and ε_{zz} represent the change in length in the x-, y-, and z-directions, respectively. Tensoral shear strains ε_{xy}, ε_{xz}, and ε_{yz} represent the change in angle between cube edges initially aligned with the (x-, y-), (x-, z-), and (y-, z-) axes, respectively. Using matrix notation the strain tensor may be written as

$$\begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{xy} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_{zz} \end{bmatrix} \quad (2.34)$$

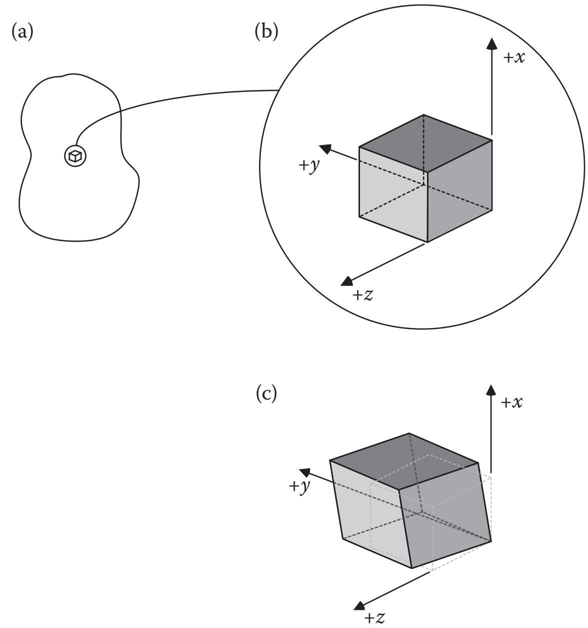

FIGURE 2.18
Infinitesimal element used to illustrate the strain tensor. (a) General 3-D solid body; (b) infinitesimal cube removed from the body, prior to deformation; (c) infinitesimal cube removed from the body, after deformation.

Alternatively, the strain tensor can be succinctly written using indicial notation as

$$\varepsilon_{ij},\ i,j = x, y, \text{ or } z \tag{2.35}$$

Note that if engineering shear strain is used, then Equation 2.34 becomes

$$\begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{xy} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_{zz} \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} & (\gamma_{xy}/2) & (\gamma_{xz}/2) \\ (\gamma_{xy}/2) & \varepsilon_{yy} & (\gamma_{yz}/2) \\ (\gamma_{xz}/2) & (\gamma_{yz}/2) & \varepsilon_{zz} \end{bmatrix}$$

If engineering shear strain is used, the strain tensor *cannot* be written using indicial notation (as in Equation 2.35), due to the 1/2 factor that appears in all off-diagonal positions.

In Section 2.1 it was noted that a force vector is a first-order tensor, as only one subscript is required to describe a force tensor, F_i. The fact that strain is a second-order tensor is evident from Equation 2.35, as two subscripts are necessary to describe a state of strain.

2.11 Transformation of the Strain Tensor

As both stress and strain are second-order tensors, transformation of the strain tensor from one coordinate system to another is analogous to transformation of the stress tensor, as discussed in Section 2.6. For example, it can be shown [1,2] that the strain components in the $x'–y'–z'$ coordinate system ($\varepsilon_{i'j'}$) are related to the components in $x–y–z$ coordinate system (ε_{ij}) according to

$$\varepsilon_{i'j'} = c_{i'k}\, c_{j'l}\, \varepsilon_{kl} \quad \text{where} \quad k, l = x, y, z \tag{2.36a}$$

Alternatively, using matrix notation the strain tensor transforms according to

$$[\varepsilon_{i'j'}] = [c_{i'j}][\varepsilon_{ij}][c_{i'j}]^T$$

which expands as follows:

$$\begin{bmatrix} \varepsilon_{x'x'} & \varepsilon_{x'y'} & \varepsilon_{x'z'} \\ \varepsilon_{y'x'} & \varepsilon_{y'y'} & \varepsilon_{y'z'} \\ \varepsilon_{z'x'} & \varepsilon_{z'y'} & \varepsilon_{z'z'} \end{bmatrix} = \begin{bmatrix} c_{x'x} & c_{x'y} & c_{x'z} \\ c_{y'x} & c_{y'y} & c_{y'z} \\ c_{z'x} & c_{z'y} & c_{z'z} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} \begin{bmatrix} c_{x'x} & c_{y'x} & c_{z'x} \\ c_{x'y} & c_{y'y} & c_{z'y} \\ c_{x'z} & c_{y'z} & c_{z'z} \end{bmatrix} \tag{2.36b}$$

The terms $c_{i'j}$ which appear in Equation 2.36a,b are direction cosines and equal the cosine of the angle between the axes of the $x'–y'–z'$ and $x–y–z$ coordinate systems.

As was the case for the stress tensor, there are certain features of the strain tensor that do not vary when the tensor is transformed from one coordinate system to another. These features are called the *strain invariants*. Three independent strain invariants exist, and are defined as follows:

$$\text{First strain invariant} = \Theta_\varepsilon = \varepsilon_{ii} \tag{2.37a}$$

$$\text{Second strain invariant} = \Phi_\varepsilon = \frac{1}{2}(\varepsilon_{ii}\varepsilon_{jj} - \varepsilon_{ij}\varepsilon_{ij}) \tag{2.37b}$$

$$\text{Third strain invariant} = \Psi_\varepsilon = \frac{1}{6}(\varepsilon_{ii}\varepsilon_{jj}\varepsilon_{kk} - 3\varepsilon_{ii}\varepsilon_{jk}\varepsilon_{jk} + 2\varepsilon_{ij}\varepsilon_{jk}\varepsilon_{ki}) \tag{2.37c}$$

Alternatively, by expanding these equations over the range $i,j,k = x,y,z$ and simplifying, the strain invariants can be written as

$$\text{First strain invariant} = \Theta_\varepsilon = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} \tag{2.38a}$$

$$\text{Second strain invariant} = \Phi_\varepsilon = \varepsilon_{xx}\varepsilon_{yy} + \varepsilon_{xx}\varepsilon_{zz} + \varepsilon_{yy}\varepsilon_{zz} - (\varepsilon_{xy}^2 + \varepsilon_{xz}^2 + \varepsilon_{yz}^2) \tag{2.38b}$$

$$\text{Third strain invariant} = \Psi_\varepsilon = \varepsilon_{xx}\varepsilon_{yy}\varepsilon_{zz} - \varepsilon_{xx}\varepsilon_{yz}^2 - \varepsilon_{yy}\varepsilon_{xz}^2 - \varepsilon_{zz}\varepsilon_{xy}^2 + 2\varepsilon_{xy}\varepsilon_{xz}\varepsilon_{yz} \tag{2.38c}$$

Example Problem 2.7

Given: A state of strain referenced to an x–y–z coordinate is known to be

$$\begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} = \begin{bmatrix} 1000\ \mu\text{m/m} & 500\ \mu\text{rad} & 250\ \mu\text{rad} \\ 500\ \mu\text{rad} & 1500\ \mu\text{m/m} & 750\ \mu\text{rad} \\ 250\ \mu\text{rad} & 750\ \mu\text{rad} & 2000\ \mu\text{m/m} \end{bmatrix}$$

It is desired to express this state of strain in a x''–y''–z'' coordinate system, generated by

i. Rotation of $\theta = 20°$ about the original z-axis (which defines an intermediate x'–y'–z' coordinate system), followed by
ii. Rotation of $\beta = 35°$ about the x'-axis (which defines the final x''–y''–z'' coordinate system)

(this coordinate transformation has been previously considered in Example Problem 2.1, and is shown in Figure 2.2).

PROBLEM

a. Rotate the strain tensor to the x''–y''–z'' coordinate system.
b. Calculate the first, second, and third invariants of the strain tensor using (i) elements of the strain tensor referenced to the x–y–z coordinate system, ε_{ij}, and (ii) elements of the strain tensor referenced to the x''–y''–z'' coordinate system, $\varepsilon_{i''j''}$.

SOLUTION

a. General expressions for direction cosines relating the x–y–z and x''–y''–z'' coordinate systems were determined as a part of Example Problem 2.1. Furthermore, numerical values for the particular rotation $\theta = 20°$ and $\beta = 35°$ were determined in Example Problem 2.3 and found to be:

$$c_{x''x} = \cos(20°) = 0.9397$$

$$c_{x''y} = \sin(20°) = 0.3420$$

$$c_{x''z} = 0$$

$$c_{y''x} = -\cos(35°)\sin(20°) = -0.2802$$

$$c_{y''y} = \cos(35°)\cos(20°) = 0.7698$$

$$c_{y''z} = \sin(35°) = 0.5736$$

$$c_{z''x} = \sin(35°)\sin(20°) = 0.1962$$

$$c_{z''y} = -\sin(35°)\cos(20°) = -0.5390$$

$$c_{z''z} = \cos(35°) = 0.8192$$

Each component of the transformed strain tensor can now be found through application of Equation 2.36a or Equation 2.36b. For example, setting $i' = x''$, $j' = x''$ and expanding Equation 2.36a, strain component $\varepsilon_{x''x''}$ is given by

$$\begin{aligned}\varepsilon_{x''x''} = {} & c_{x''x}\, c_{x''x}\, \varepsilon_{xx} + c_{x''x}\, c_{x''y}\, \varepsilon_{xy} + c_{x''x}\, c_{x''z}\, \varepsilon_{xz} \\ & + c_{x''y}\, c_{x''x}\, \varepsilon_{yx} + c_{x''y}\, c_{x''y}\, \varepsilon_{yy} + c_{x''y}\, c_{x''z}\, \varepsilon_{yz} \\ & + c_{x''z}\, c_{x''x}\, \varepsilon_{zx} + c_{x''z}\, c_{x''y}\, \varepsilon_{zy} + c_{x''z}\, c_{x''z}\, \varepsilon_{zz}\end{aligned}$$

$$\begin{aligned}\varepsilon_{x''x''} = {} & (0.9397)(0.9397)(1000) + (0.9397)(0.3420)(500) + (0.9397)(0)(250) \\ & + (0.3420)(0.9397)(500) + (0.3420)(0.3420)(1500) + (0.3420)(0)(750) \\ & + (0)(0.9397)(250) + (0)(0.3420)(750) + (0)(0)(2000)\end{aligned}$$

$$\varepsilon_{x''x''} = 1380\ \mu\text{m/m}$$

Alternatively, if matrix notation is used, then Equation 2.36b becomes

$$\begin{bmatrix} \varepsilon_{x''x''} & \varepsilon_{x''y''} & \varepsilon_{x''z''} \\ \varepsilon_{y''x''} & \varepsilon_{y''y''} & \varepsilon_{y''z''} \\ \varepsilon_{z''x''} & \varepsilon_{z''y''} & \varepsilon_{z''z''} \end{bmatrix} = \begin{bmatrix} 0.9397 & 0.3420 & 0 \\ -0.2802 & 0.7698 & 0.5736 \\ 0.1962 & -0.5390 & 0.8192 \end{bmatrix} \begin{bmatrix} 1000 & 500 & 250 \\ 500 & 1500 & 750 \\ 250 & 750 & 2000 \end{bmatrix}$$

$$\times \begin{bmatrix} 0.9397 & -0.2802 & 0.1962 \\ 0.3420 & 0.7698 & -0.5390 \\ 0 & 0.5736 & 0.8192 \end{bmatrix}$$

Completing the matrix multiplication indicated, there results:

$$\begin{bmatrix} \varepsilon_{x''x''} & \varepsilon_{x''y''} & \varepsilon_{x''z''} \\ \varepsilon_{y''x''} & \varepsilon_{y''y''} & \varepsilon_{y''z''} \\ \varepsilon_{z''x''} & \varepsilon_{z''y''} & \varepsilon_{z''z''} \end{bmatrix} = \begin{bmatrix} 1380\,\mu\text{m/m} & 727\,\mu\text{rad} & 91\,\mu\text{rad} \\ 727\,\mu\text{rad} & 1991\,\mu\text{m/m} & 625\,\mu\text{rad} \\ 91\,\mu\text{rad} & 625\,\mu\text{rad} & 1129\,\mu\text{m/m} \end{bmatrix}$$

Notice that the value of $\varepsilon_{x''x''}$ determined through matrix multiplication is identical to that obtained using indicial notation, as expected.

b. The first, second, and third strain invariants will now be calculated using components of both ε_{ij} and $\varepsilon_{i''j''}$. It is expected that identical values will be obtained, as the strain invariants are independent of coordinate system.

First Strain Invariant

x–y–z coordinate system:

$$\Theta_\varepsilon = \varepsilon_{ii} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}$$

$$\Theta_\varepsilon = (1000 + 1500 + 2000)\,\mu\text{m/m}$$

$$\Theta_\varepsilon = 4500\,\mu\text{m/m} = 0.004500\,\text{m/m}$$

x″–y″–z″ coordinate system:

$$\Theta_\varepsilon = \varepsilon_{i''i''} = \varepsilon_{x''x''} + \varepsilon_{y''y''} + \varepsilon_{z''z''}$$

$$\Theta_\varepsilon = (1380 + 1991 + 1129)\,\mu\text{m/m}$$

$$\Theta_\varepsilon = 4500\,\mu\text{m/m} = 0.004500\,\text{m/m}$$

As expected, the first strain invariant is independent of coordinate system.

Second Strain Invariant

x–y–z coordinate system:

$$\Phi_\varepsilon = \frac{1}{2}\left(\varepsilon_{ii}\varepsilon_{jj} - \varepsilon_{ij}\varepsilon_{ij}\right) = \varepsilon_{xx}\varepsilon_{yy} + \varepsilon_{xx}\varepsilon_{zz} + \varepsilon_{yy}\varepsilon_{zz} - \left(\varepsilon_{xy}^2 + \varepsilon_{xz}^2 + \varepsilon_{yz}^2\right)$$

$$\Phi_\varepsilon = \{(1000)(1500) + (1000)(2000) + (1500)(2000) - [(500)^2 + (250)^2 + (750)^2]\}(\mu\text{m/m})^2$$

$$\Phi = 5.625 \times 10^6\ (\mu\text{m/m})^2 = 5.625 \times 10^{-6}\ (\text{m/m})^2$$

x''–y''–z'' coordinate system:

$$\Phi_\varepsilon = \frac{1}{2}\left(\varepsilon_{i''i''}\varepsilon_{j''j''} - \varepsilon_{i''j''}\varepsilon_{i''j''}\right)$$

$$\Phi_\varepsilon = \varepsilon_{x''x''}\varepsilon_{y''y''} + \varepsilon_{x''x''}\varepsilon_{z''z''} + \varepsilon_{y''y''}\varepsilon_{z''z''} - \left(\varepsilon^2_{x''x''} + \varepsilon^2_{x''z''} + \varepsilon^2_{y''z''}\right)$$

$$\Phi_\varepsilon = \{(1380)(1991) + (1380)(1129) + (1991)(1129) - [(727)^2 + (91)^2 + (625)^2]\}\,(\mu\text{m/m})^2$$

$$\Phi_\varepsilon = 5.625 \times 10^6\ (\mu\text{m/m})^2 = 5.625 \times 10^{-6}(\text{m/m})^2$$

As expected, the second strain invariant is independent of coordinate system.

Third Strain Invariant

x–y–z coordinate system:

$$\Psi_\varepsilon = \frac{1}{6}\left(\varepsilon_{ii}\varepsilon_{jj}\varepsilon_{kk} - 3\varepsilon_{ii}\varepsilon_{jk}\varepsilon_{jk} + 2\varepsilon_{ij}\varepsilon_{jk}\varepsilon_{ki}\right)$$

$$\Psi_\varepsilon = \varepsilon_{xx}\varepsilon_{yy}\varepsilon_{zz} - \varepsilon_{xx}\varepsilon^2_{yz} - \varepsilon_{yy}\varepsilon^2_{xz} - \varepsilon_{zz}\varepsilon^2_{xy} + 2\varepsilon_{xy}\varepsilon_{xz}\varepsilon_{yz}$$

$$\Psi_\varepsilon = [(1000)(1500)(2000) - (1000)(750)^2 - (1500)(250)^2 - (2000)(500)^2 + 2(500)(250)(750)]\,(\mu\text{m/m})^3$$

$$\Psi = 2.031 \times 10^9\ (\mu\text{m/m})^3 = 2.031 \times 10^{-9}(\text{m/m})^3$$

x''–y''–z'' coordinate system:

$$\Psi_\varepsilon = \frac{1}{6}\left(\varepsilon_{i''i''}\varepsilon_{j''j''}\varepsilon_{k''k''} - 3\varepsilon_{i''i''}\varepsilon_{j''k''}\varepsilon_{j''k''} + 2\varepsilon_{i''j''}\varepsilon_{j''k''}\varepsilon_{k''i''}\right)$$

$$\Psi_\varepsilon = \varepsilon_{x''x''}\varepsilon_{y''y''}\varepsilon_{z''z''} - \varepsilon_{x''x''}\varepsilon^2_{y''z''} - \varepsilon_{y''y''}\varepsilon^2_{x''z''} - \varepsilon_{z''z''}\varepsilon^2_{x''y''} + 2\varepsilon_{x''y''}\varepsilon_{x''z''}\varepsilon_{y''z''}$$

$$\Psi_\varepsilon = [(1380)(1991)(1129) - (1380)(625)^2 - (1991)(91)^2 - (1129)(727)^2 + 2(727)(91)(625)]\,(\mu\text{m/m})^3$$

$$\Psi = 2.031 \times 10^9(\mu\text{m/m})^3 = 2.031 \times 10^{-9}(\text{m/m})^3$$

As expected, the third stress invariant is independent of coordinate system.

2.12 Principal Strains

It is always possible to rotate the strain tensor to a special coordinate system in which no shear strains exist. This coordinate system is called the *principal strain coordinate system*, and the normal strains that exist in this coordinate system are called *principal strains*. Calculation of principal strains (and principal stresses) is important during the study of traditional isotropic materials and structures because principal strains or stresses can be used to predict failure of isotropic materials. This is not the case for anisotropic composite materials. Failure of composite material is *not* governed by principal strains or stresses. Principal strains are only of occasional interest to the composite engineer and are reviewed here only in the interests of completeness.

Principal strains are usually denoted ε_1, ε_2, and ε_3. However, in the study of composites the labels "1", "2", and "3" are used to refer to the principal material coordinate system rather than the directions of principal strain. Therefore, in this chapter the axes associated with the principal strain coordinate system will be labeled the "$p1$", "$p2$", and "$p3$" axes, and the principal strains will be denoted ε_{p1}, ε_{p2} and ε_{p3}.

As both stress and strain are second-order tensors, the principal strains can be found using an approach analogous to that used to find principal stresses. Specifically, it can be shown [1,2] that the principal strains must satisfy the following three simultaneous equations:

$$(\varepsilon_{pn} - \varepsilon_{xx})c_{pnx} - \varepsilon_{xy}c_{pny} - \varepsilon_{xz}c_{pnz} = 0 \tag{2.39a}$$

$$-\varepsilon_{xy}c_{pnx} + (\varepsilon_{pn} - \varepsilon_{yy})c_{pny} - \varepsilon_{yz}c_{pnz} = 0 \tag{2.39b}$$

$$-\varepsilon_{xz}c_{pnx} - \varepsilon_{yz}c_{pny} + (\varepsilon_{pn} - \varepsilon_{zz})c_{pnz} = 0 \tag{2.39c}$$

As direction cosines c_{pnx}, c_{pny}, and c_{pnz} must also satisfy Equation 2.8, and therefore cannot all equal zero, the solution can be obtained by requiring that the determinant of the coefficients of c_{pnx}, c_{pny}, and c_{pnz} equal zero:

$$\begin{vmatrix} (\varepsilon_{pn} - \varepsilon_{xx}) & -\varepsilon_{xy} & -\varepsilon_{xz} \\ -\varepsilon_{xy} & (\varepsilon_{pn} - \varepsilon_{yy}) & -\varepsilon_{yz} \\ -\varepsilon_{xz} & -\varepsilon_{yz} & (\varepsilon_{pn} - \varepsilon_{zz}) \end{vmatrix} = 0$$

Equating the determinant to zero results in the following cubic equation:

$$\varepsilon_{pn}^3 - \Theta_\varepsilon \varepsilon_{pn}^2 + \Phi_\varepsilon \varepsilon_{pn} - \Psi_\varepsilon = 0 \tag{2.40}$$

where Θ_ε, Φ_ε, and Ψ_ε are the first, second, and third strain invariants, respectively, and have been previously listed as Equations 2.37 and 2.38. The three roots of the cubic equation represent the three principal strains and may be found by application of the standard approach [3]. By convention, the principal stresses are numbered such that ε_{p1} is the algebraically greatest principal stress, whereas ε_{p3} is the algebraically least. That is, $\varepsilon_{p1} > \varepsilon_{p2} > \varepsilon_{p3}$.

Once the principal strains are determined, the three sets of direction cosines (which define the principal coordinate directions) are found by substituting the three principal strains given by Equation 2.40 into Equation 2.39 in turn. As only two of Equation 2.39 are independent, Equation 2.8 is used as a third independent equation involving the three unknown constants, c_{pnx}, c_{pny}, and c_{pnz}.

The process of finding principal strains and direction cosines will be demonstrated in the following Example Problem.

Example Problem 2.8

Given: A state of strain referenced to an *x–y–z* coordinate is known to be

$$\begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} = \begin{bmatrix} 1000\,\mu\text{m/m} & 500\,\mu\text{rad} & 250\,\mu\text{rad} \\ 500\,\mu\text{rad} & 1500\,\mu\text{m/m} & 750\,\mu\text{rad} \\ 250\,\mu\text{rad} & 750\,\mu\text{rad} & 2000\,\mu\text{m/m} \end{bmatrix}$$

PROBLEM

Find (a) the principal strains and (b) the direction cosines that define the principal strain coordinate system.

SOLUTION

This is the same strain tensor considered in Example Problem 2.7. As a part of that problem, the first, second, and third strain invariants were found to be

$$\Theta_\varepsilon = 0.004500\,\text{m/m}$$

$$\Phi = 5.625 \times 10^{-6}(\text{m/m})^2$$

$$\Psi = 2.031 \times 10^{-9}(\text{m/m})^3$$

a. *Determining the Principal Strains*: In accordance with Equation 2.40, the three principal strains are the roots of the following cubic equation:

$$\varepsilon_{pn}^3 - (0.004500)\varepsilon_{pn}^2 + (5.625 \times 10^{-6})\varepsilon_{pn} - (2.031 \times 10^{-9}) = 0$$

The three roots of this equation represent the three principal strains, and are given by

$$\varepsilon_{p1} = 2689\,\mu\text{m/m}, \quad \varepsilon_{p2} = 1160\,\mu\text{m/m}, \quad \text{and} \quad \varepsilon_{p3} = 651\,\mu\text{m/m}$$

b. *Determining the Direction Cosines:* The first two of Equations 2.39 and 2.8 will be used to form three independent equations in three unknowns. We have

$$(\varepsilon_{pn} - \varepsilon_{xx})c_{pnx} - \varepsilon_{xy}c_{pny} - \varepsilon_{xz}c_{pnz} = 0$$

$$-\varepsilon_{xy}c_{pnx} + (\varepsilon_{pn} - \varepsilon_{yy})c_{pny} - \varepsilon_{yz}c_{pnz} = 0$$

$$(c_{pnx})^2 + (c_{pny})^2 + (c_{pnz})^2 = 1$$

Direction cosines for ε_{p1}: The three independent equations become

$$(2689 - 1000)c_{p1x} - 500c_{p1y} - 250c_{p1z} = 0$$

$$-500c_{p1x} + (2689 - 1500)c_{p1y} - 750c_{p1z} = 0$$

$$(c_{p1x})^2 + (c_{p1y})^2 + (c_{p1z})^2 = 1$$

Solving simultaneously, we obtain

$$c_{p1x} = 0.2872$$

$$c_{p1y} = 0.5945$$

$$c_{p1z} = 0.7511$$

Direction cosines for ε_{p2}: The three independent equations become

$$(1160 - 1000)c_{p2x} - 500c_{p2y} - 250c_{p2z} = 0$$

$$-500c_{p2x} + (1160 - 1500)c_{p2y} - 750c_{p2z} = 0$$

$$(c_{p2x})^2 + (c_{p2y})^2 + (c_{p2z})^2 = 1$$

Solving simultaneously, we obtain

$$c_{p2x} = 0.5960$$

$$c_{p2y} = 0.5035$$

$$c_{p2z} = -0.6256$$

Direction cosines for ε_{p3}: The three independent equations become

$$(651 - 1000)c_{p3x} - 500c_{p3y} - 250c_{p3z} = 0$$

$$-500c_{p3x} + (651 - 1500)c_{p3y} - 750c_{p3z} = 0$$

$$(c_{p3x})^2 + (c_{p3y})^2 + (c_{p3z})^2 = 1$$

Solving simultaneously, we obtain

$$c_{p3x} = -0.7481$$

$$c_{p3y} = 0.6286$$

$$c_{p3z} = -0.2128$$

2.13 Strains within a Plane Perpendicular to a Principal Strain Direction

It has been seen that a strain tensor is defined by six components of strain: three normal strain components and three shear strain components. However, in practice there are circumstances in which it is known *a priori* that both shear strain components in one direction are zero: $\varepsilon_{xz} = \varepsilon_{yz} = 0$, say (or equivalently, $\gamma_{xz} = \gamma_{yz} = 0$). *This implies that the z-axis is a principal strain axis*. In these instances, we are primarily interested in the strains induced within the x–y plane, ε_{xx}, ε_{yy}, ε_{xy}. Two different circumstances are encountered in which it is known *a priori* that the z-axis is a principal strain axis.

In the first case, *all three out-of-plane strain components in the z-direction* are known *a priori* to equal zero. That is, it is known *a priori* that $\varepsilon_{zz} = \varepsilon_{xz} = \varepsilon_{yz} = 0$. Not only is the z-axis a principal strain axis in this case, but in addition the principal strain equals zero: $\varepsilon_{zz} = \varepsilon_{p3} = 0$. As the three remaining nonzero strain components (ε_{xx}, ε_{yy}, and ε_{xy}) all lie within the x–y plane, it is natural to call this condition a *state of plane strain*. Plane strain conditions occur most often because of the geometry of the structure of interest. Specifically, the plane strain condition usually exists in internal regions of very long (or very thick) structures. Examples include solid shafts or long dams. In these instances the strains induced along the long axis of the structure are often negligibly small compared with those induced within the transverse plane of the structure.

The second case in which the out-of-plane z-axis *may* be a principal axis is when a structure is subjected to a state of *plane stress*. As has been discussed in Section 2.8, the state of plane stress occurs most often in thin, plate-like structures. In this case, the z-axis is a principal strain axis, and ε_{zz} is again one of the principal strains. However, in this second case the out of plane normal strain does not, in general, equal zero: $\varepsilon_{zz} \neq 0$.

It is emphasized that a state of plane stress usually, *but not always*, causes a state of strain in which the z-axis is a principal strain axis. This point will be further discussed in Chapter 4. It will be seen that it is possible for a material to exhibit a coupling between in-plane stresses and out-of-plane shear strains. That is, in some cases stresses acting within the x–y plane (σ_{xx}, σ_{yy}, and/or τ_{xy}) can cause out-of-plane shear strains (ε_{xz} and/or ε_{yz}). In these instances, the out-of-plane z-axis is not a principal strain axis, even though the out-of-plane stresses all equal zero.

In any event, for present purposes assume that it is known *a priori* that the out-of-plane z-axis is a principal strain axis, and we are primarily interested in the strains induced within the x–y plane, ε_{xx}, ε_{yy}, and ε_{xy}. We will write these strains in the form of a column array, rather than a 3×3 array:

$$\begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & 0 \\ \varepsilon_{xy} & \varepsilon_{yy} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} & (\gamma_{xy}/2) & 0 \\ (\gamma_{xy}/2) & \varepsilon_{yy} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{bmatrix} \rightarrow \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy}/2 \end{Bmatrix} \quad (2.41)$$

Note that ε_{zz} does not appear in the column array. This is not of concern in the case of plane strain, as in this case $\varepsilon_{zz} = 0$. However, in the case of plane stress it is important to remember that (in general) $\varepsilon_{zz} \neq 0$. Although in following chapters we will be primarily interested in strains induced within the x–y plane, the reader is advised to remember that an out-of-plane strain ε_{zz} is also induced by a state of plane stress.

The transformation of a general 3D strain tensor has already been discussed in Section 2.11. The relations presented there will now be simplified for the case of transformation of strains within a plane.

Recall that either Equation 2.36a or Equation2.36b governs the transformation of a strain tensor from one coordinate system to another. Equation 2.36b is repeated here for convenience:

$$\begin{bmatrix} \varepsilon_{x'x'} & \varepsilon_{x'y'} & \varepsilon_{x'z'} \\ \varepsilon_{y'x'} & \varepsilon_{y'y'} & \varepsilon_{y'z'} \\ \varepsilon_{z'x'} & \varepsilon_{z'y'} & \varepsilon_{z'z'} \end{bmatrix} = \begin{bmatrix} c_{x'x} & c_{x'y} & c_{x'z} \\ c_{y'x} & c_{y'y} & c_{y'z} \\ c_{z'x} & c_{z'y} & c_{z'z} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} \begin{bmatrix} c_{x'x} & c_{y'x} & c_{z'x} \\ c_{x'y} & c_{y'y} & c_{z'y} \\ c_{x'z} & c_{y'z} & c_{z'z} \end{bmatrix}$$

(2.36b) (repeated)

Assuming that the x'–y'–z' coordinate system is generated from the x–y–z system by a rotation θ about the z-axis, the direction cosines are

$$
\begin{aligned}
c_{x'x} &= \cos(\theta) \\
c_{x'y} &= \cos(90° - \theta) = \sin(\theta) \\
c_{x'z} &= \cos(90°) = 0 \\
c_{y'x} &= \cos(90° + \theta) = -\sin(\theta) \\
c_{y'y} &= \cos(\theta) \\
c_{y'z} &= \cos(90°) = 0 \\
c_{z'x} &= \cos(90°) = 0 \\
c_{z'y} &= \cos(90°) = 0 \\
c_{z'z} &= \cos(0°) = 1
\end{aligned}
$$

Substituting these direction cosines into Equation 2.36b and noting that by assumption $\varepsilon_{xz} = \varepsilon_{yz} = 0$, we have

$$
\begin{bmatrix} \varepsilon_{x'x'} & \varepsilon_{x'y'} & \varepsilon_{x'z'} \\ \varepsilon_{y'x'} & \varepsilon_{y'y'} & \varepsilon_{y'z'} \\ \varepsilon_{z'x'} & \varepsilon_{z'y'} & \varepsilon_{z'z'} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & 0 \\ \varepsilon_{yx} & \varepsilon_{yy} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

Completing the matrix multiplication indicated results in:

$$
\begin{bmatrix} \varepsilon_{x'x'} & \varepsilon_{x'y'} & \varepsilon_{x'z'} \\ \varepsilon_{y'x'} & \varepsilon_{y'y'} & \varepsilon_{y'z'} \\ \varepsilon_{z'x'} & \varepsilon_{z'y'} & \varepsilon_{z'z'} \end{bmatrix}
$$

$$
= \begin{bmatrix} \cos^2\theta\varepsilon_{xx} + \sin^2\theta\varepsilon_{yy} + 2\cos\theta\sin\theta\varepsilon_{xy} & -\cos\theta\sin\theta\varepsilon_{xx} + \cos\theta\sin\theta\varepsilon_{yy} + (\cos^2\theta - \sin^2\theta)\varepsilon_{xy} & 0 \\ -\cos\theta\sin\theta\varepsilon_{xx} + \cos\theta\sin\theta\varepsilon_{yy} + (\cos^2\theta - \sin^2\theta)\varepsilon_{xy} & \sin^2\theta\varepsilon_{xx} + \cos^2\theta\varepsilon_{yy} - 2\cos\theta\sin\theta\varepsilon_{xy} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{bmatrix}
$$

As would be expected $\varepsilon_{x'z'} = \varepsilon_{y'z'} = 0$. The remaining strain components are:

$$\begin{aligned}
\varepsilon_{x'x'} &= \cos^2(\theta)\varepsilon_{xx} + \sin^2(\theta)\varepsilon_{yy} + 2\cos(\theta)\sin(\theta)\varepsilon_{xy} \\
\varepsilon_{y'y'} &= \sin^2(\theta)\varepsilon_{xx} + \cos^2(\theta)\varepsilon_{yy} - 2\cos(\theta)\sin(\theta)\varepsilon_{xy} \\
\varepsilon_{x'y'} &= -\cos(\theta)\sin(\theta)\varepsilon_{xx} + \cos(\theta)\sin(\theta)\varepsilon_{yy} + [\cos^2(\theta) + \sin^2(\theta)]\varepsilon_{xy} \\
\varepsilon_{z'z'} &= \varepsilon_{zz}
\end{aligned} \tag{2.42}$$

Tensoral shear strains were used in Equation 2.36a,b so that rotation of the strain tensor could be accomplished using the normal transformation law for a second-order tensor. As engineering shear strains are commonly used in practice, we will convert our final results, Equation 2.42, to ones which involve engineering shear strain (γ_{xy}). Recall from Section 2.9 that $\varepsilon_{xy} = (1\backslash 2)\gamma_{xy}$. Hence, to convert Equation 2.42, simply replace ε_{xy} with $(1\backslash 2)\gamma_{xy}$ everywhere, resulting in:

$$\begin{aligned}
\varepsilon_{x'x'} &= \cos^2(\theta)\varepsilon_{xx} + \sin^2(\theta)\varepsilon_{yy} + \cos(\theta)\sin(\theta)\gamma_{xy} \\
\varepsilon_{y'y'} &= \sin^2(\theta)\varepsilon_{xx} + \cos^2(\theta)\varepsilon_{yy} - \cos(\theta)\sin(\theta)\gamma_{xy} \\
\frac{\gamma_{x'y'}}{2} &= -\cos(\theta)\sin(\theta)\varepsilon_{xx} + \cos(\theta)\sin(\theta)\varepsilon_{yy} + [\cos^2(\theta) + \sin^2(\theta)]\frac{\gamma_{xy}}{2} \\
\varepsilon_{z'z'} &= \varepsilon_{zz}
\end{aligned} \tag{2.43}$$

Equation 2.43 relates the components of strain in two different coordinate systems within a single plane, and will be used extensively throughout the remainder of this chapter. It is important to remember that these equations are only valid *if the out-of-plane z-axis is a principal strain axis*.

The first three of Equation 2.43 can be written using matrix notation as

$$\begin{Bmatrix} \varepsilon_{x'x'} \\ \varepsilon_{y'y'} \\ \dfrac{\gamma_{x'y'}}{2} \end{Bmatrix} = \begin{bmatrix} \cos^2(\theta) & \sin^2(\theta) & 2\cos(\theta)\sin(\theta) \\ \sin^2(\theta) & \cos^2(\theta) & -2\cos(\theta)\sin(\theta) \\ -\cos(\theta)\sin(\theta) & \cos(\theta)\sin(\theta) & \cos^2(\theta) - \sin^2(\theta) \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \dfrac{\gamma_{xy}}{2} \end{Bmatrix} \tag{2.44}$$

Compare Equation 2.44 with Equation 2.20. In particular, note that the transformation matrix, [*T*], which was previously encountered during the discussion of plane stress in Section 2.8, also appears in Equation 2.44.

The strain invariants (given by Equation 2.37 or 2.38) are considerably simplified when the out-of-plane z-axis is a principal axis. As by definition $\varepsilon_{xz} = \gamma_{xz}/2 = \varepsilon_{yz} = \gamma_{yz}/2 = 0$, the strain invariants become:

$$\text{First strain invariant} = \Theta_\varepsilon = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}$$

$$\text{Second strain invariant} = \Phi_\varepsilon = \varepsilon_{xx}\varepsilon_{yy} + \varepsilon_{xx}\varepsilon_{zz} + \varepsilon_{yy}\varepsilon_{zz} - \frac{\gamma_{xy}^2}{4} \tag{2.45}$$

$$\text{Third strain invariant} = \Psi_\varepsilon = \varepsilon_{xx}\varepsilon_{yy}\varepsilon_{zz} - \varepsilon_{zz}\frac{\gamma_{xy}^4}{4}$$

The principal strains equal the roots of the cubic equation previously listed as Equation 2.40. Substituting Equation 2.45 into Equation 2.40, there results:

$$\varepsilon_{p_n}^3 - (\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz})\varepsilon_{pn}^2 + \left(\varepsilon_{xx}\varepsilon_{yy} + \varepsilon_{xx}\varepsilon_{zz} + \varepsilon_{yy}\varepsilon_{zz} - \frac{\gamma_{xy}^2}{4}\right)\varepsilon_{pn}$$
$$- \left(\varepsilon_{xx}\varepsilon_{yy}\varepsilon_{zz} - \varepsilon_{zz}\frac{\gamma_{xy}^2}{4}\right) = 0 \tag{2.46}$$

One root of Equation 2.23 is $\varepsilon_{pn} = \varepsilon_{zz}$. For present purposes, this root will be labeled ε_{p3} even though it may not be the algebraically least principal strain. In the case of plane strain $\varepsilon_{p3} = \varepsilon_{zz} = 0$.

Removing the known root from Equation 2.46, we have the following quadratic equation:

$$\varepsilon_{pn}^2 - (\varepsilon_{xx} + \varepsilon_{yy})\varepsilon_{pn} + \left(\varepsilon_{xx}\varepsilon_{yy} - \frac{\gamma_{xy}^2}{4}\right) = 0$$

The two roots of this quadratic equation (i.e., the two remaining principal strains, ε_{p1} and ε_{p2}) may be found by application of the standard approach [3], and are given by

$$\varepsilon_{p1}, \varepsilon_{p2} = \frac{\varepsilon_{xx} + \varepsilon_{yy}}{2} \pm \sqrt{\left(\frac{\varepsilon_{xx} - \varepsilon_{yy}}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} \tag{2.47}$$

The angle θ_{p_ε} between the x-axis and either the $p1$ or $p2$ axis is given by

$$\theta_{p_\varepsilon} = \frac{1}{2}\arctan\left(\frac{\gamma_{xy}}{\varepsilon_{xx} - \varepsilon_{yy}}\right) \tag{2.48}$$

Example Problem 2.9

Given: A state of plane strain is known to consist of:

$$\varepsilon_{xx} = 500\ \mu\text{m/m}$$

$$\varepsilon_{yy} = -1000\ \mu\text{m/m}$$

$$\gamma_{xy} = -2500\ \mu\text{rad}$$

PROBLEM

a. Prepare a rough sketch (not to scale) of the deformed strain element in the x–y coordinate system.
b. Determine the strain components which correspond to an x'–y' coordinate system, oriented 25° CCW from the x–y coordinate system, and prepare a rough sketch (not to scale) of the deformed strain element in the x'–y' coordinate system.
c. Determine the principal strain components that exist within the x–y plane, and prepare a rough sketch (not to scale) of the deformed strain element in the principal strain coordinate system.

SOLUTION

a. A sketch showing the deformed strain element (not to scale) in the x–y coordinate system is shown in Figure 2.19a. Note that
   - The length of the element side parallel to the x-axis has increased (corresponding to the tensile strain $\varepsilon_{xx} = 500\ \mu\text{m/m}$).
   - The length of the element side parallel to the y-axis has decreased (corresponding to the compressive strain $\varepsilon_{yy} = -1000\ \mu\text{m/m}$).
   - The angle defined by x–y axes has increased (corresponding to the negative shear strain $\gamma_{xy} = -2500\ \mu\text{rad}$).
b. Since the x'-axis is oriented 25° CCW from the x-axis, in accordance with the right-hand rule the angle of rotation is *positive*, that is, $\theta = +25°$. Substituting this angle and the given strain components in Equation 2.44:

$$\begin{Bmatrix} \varepsilon_{x'x'} \\ \varepsilon_{y'y'} \\ \dfrac{\gamma_{x'y'}}{2} \end{Bmatrix} = \begin{bmatrix} \cos^2(25°) & \sin^2(25°) & 2\cos(25°)\sin(25°) \\ \sin^2(25°) & \cos^2(25°) & -2\cos(25°)\sin(25°) \\ -\cos(25°)\sin(25°) & \cos(25°)\sin(25°) & \cos^2(25°) - \sin^2(25°) \end{bmatrix}$$

$$\times \begin{Bmatrix} 500 \\ -1000 \\ \dfrac{-2500}{2} \end{Bmatrix}$$

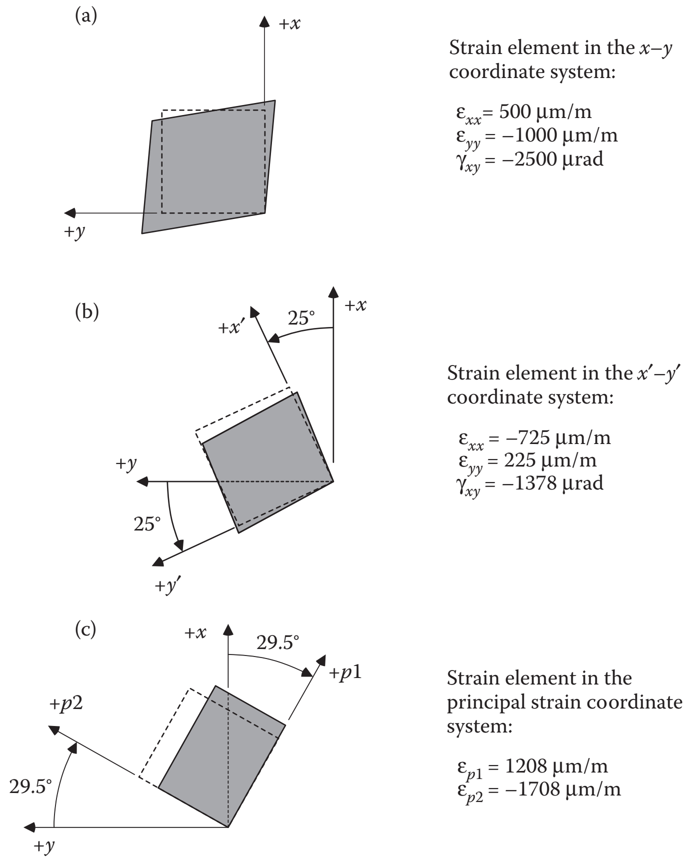

FIGURE 2.19
Strain elements associated with Example Problem 2.9 (all deformations shown greatly exaggerated for clarity).

Completing the matrix multiplication indicated results in

$$\begin{Bmatrix} \varepsilon_{x'x'} \\ \varepsilon_{y'y'} \\ \dfrac{\gamma_{x'y'}}{2} \end{Bmatrix} = \begin{Bmatrix} -725\,\mu\text{m/m} \\ 225\,\mu\text{m/m} \\ -1378\,\mu\text{rad} \end{Bmatrix}$$

A sketch showing the deformed strain element (not to scale) in the x'–y' coordinate system is shown in Figure 2.19b. Note that

- The length of the element side parallel to the x'-axis has decreased (corresponding to the compressive strain $\varepsilon_{x'x'} = -725$ μm/m).

- The length of the element side parallel to the y'-axis has increased (corresponding to the tensile strain $\varepsilon_{y'y'} = 225$ μm/m).
- The angle defined by the x'–y' axes has increased (corresponding to the negative shear strain $\gamma_{x'y'} = -2756$ μrad).

c. The principal strains are found through application of Equation 2.47:

$$\varepsilon_{p1}, \varepsilon_{p2} = \frac{500 - 1000}{2} \pm \sqrt{\left(\frac{500 + 1000}{2}\right)^2 + \left(\frac{(-2500)}{2}\right)^2}$$

$$\varepsilon_{p1} = 1208\,\mu\text{m/m}$$

$$\varepsilon_{p2} = -1708\,\mu\text{m/m}$$

The orientation of the principal strain coordinate system is given by Equation 2.48:

$$\theta_{p\varepsilon} = \frac{1}{2}\arctan\left(\frac{-2500}{500 + 1000}\right) = -29.5°$$

A sketch showing the deformed strain element (not to scale) in the principal strain coordinate system is shown in Figure 2.19c. Note that

- The length of the element side parallel to the $p1$-axis has increased (corresponding to the tensile principal strain $\varepsilon_{p1} = 1208$ μm/m).
- The length of the element side parallel to the $p2$-axis has decreased (corresponding to the compressive principal strain $\varepsilon_{p2} = -1708$ μm/m).
- The angle defined by the principal strain axes has remained precisely π/2 radians (i.e., 90°), as in the principal strain coordinate system the shear strain is zero.

2.14 Relating Strains to Displacement Fields

Most analyses considered in later chapters begin with consideration of the *displacement fields* induced in the structure of interest. That is, mathematical expressions that describe the displacements induced at all points within a structure by external loading and/or environmental changes will be assumed or otherwise specified. Strains induced in the structure will then be inferred from these displacement fields.

In the most general case three displacement fields are involved. Specifically, displacements in the *x*-, *y*-, and *z*-directions, typically denoted as the *u*-, *v*-, and *w*-displacement fields, respectively. In general, all three displacement fields are functions of *x*, *y*, and *z*:

Displacements in the *x*-direction: $u = u(x,y,z)$
Displacements in the *y*-direction: $v = v(x,y,z)$
Displacements in the *z*-direction: $w = w(x,y,z)$.

However, if the out-of-plane *z*-axis is a principal strain axis, then *u* and *v* are (at most) functions of *x* and *y* only, while *w* is (at most) a function of *z* only. In this case:

Displacements in the *x*-direction: $u = u(x,y)$
Displacements in the *y*-direction: $v = v(x,y)$
Displacements in the *z*-direction: $w = w(z)$.

A detailed derivation of the relationship between displacements and strains is beyond the scope of this review, and the interested reader is referred to References 1, 2 for details. It can be shown that the relationship between displacement fields and the strain tensor depends upon the magnitude of *derivatives* of displacement fields (also called *displacement gradients*). If displacement gradients are arbitrarily large then the associated level of strain is said to be "finite," and each component of the strain tensor is related *nonlinearly* to displacement gradients as follows:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right]$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right]$$

$$\varepsilon_{zz} = \frac{\partial w}{\partial z} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right]$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right) + \left(\frac{\partial v}{\partial x}\right)\left(\frac{\partial v}{\partial y}\right) + \left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right)$$

$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} + \left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial z}\right) + \left(\frac{\partial v}{\partial x}\right)\left(\frac{\partial v}{\partial z}\right) + \left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial z}\right)$$

$$\gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} + \left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial u}{\partial z}\right) + \left(\frac{\partial v}{\partial y}\right)\left(\frac{\partial v}{\partial z}\right) + \left(\frac{\partial w}{\partial y}\right)\left(\frac{\partial w}{\partial y}\right)$$

The expressions listed above define what is known as *Green's strain tensor* (also known as the *Lagrangian* strain tensor).

In most cases encountered in practice, however, displacement gradients are very small, and consequently the *products* of displacement gradients are negligibly small and can be discarded. For example, it can usually be assumed that

$$\left(\frac{\partial u}{\partial x}\right)^2 \approx 0 \quad \left(\frac{\partial v}{\partial x}\right)^2 \approx 0$$

$$\left(\frac{\partial w}{\partial x}\right)^2 \approx 0 \quad \left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right) \approx 0, \quad \text{etc}$$

When displacement gradients are very small the level of strain is said to be "infinitesimal," and each component of the strain tensor is *linearly* related to displacement gradients as follows:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} \tag{2.49a}$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} \tag{2.49b}$$

$$\varepsilon_{zz} = \frac{\partial w}{\partial z} \tag{2.49c}$$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \tag{2.49d}$$

$$\gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \tag{2.49e}$$

$$\gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \tag{2.49f}$$

For most analyses considered in later chapters we will assume that strains are infinitesimal and are related to displacement fields in accordance with Equation 2.49. The one exception occurs in Chapter 11, where it will be necessary to include nonlinear terms in the strain–displacement relationships.

At stated above, most analyses begin with consideration of the displacement fields induced in a structure of interest. Strains fields implied by these displacements are then calculated in accordance with Equation 2.49. This process insures that strain fields are consistent with displacements. Consider the opposite approach. Specifically, suppose that mathematical expressions for *strain fields* are assumed, perhaps on the basis of engineering judgment. In this case, it is possible that the *assumed* strain fields correspond to physically unrealistic displacement fields. For example, displacement fields inferred from assumed strain fields may imply that the solid body has voids and/or overlapping regions, a physically unrealistic circumstance. A system of six equations known as the *compatibility conditions* can be developed that guarantee that assumed expressions for the six components of strain do, in fact, correspond to physically reasonable displacement fields $u(x,y,z)$, $v(x,y,z)$, and $w(x,y,z)$. To develop the compatibility conditions, differentiate Equation 2.49d twice, once with respect to x and once with respect to y. We obtain

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^3 u}{\partial x \partial y^2} + \frac{\partial^3 v}{\partial x^2 \partial y}$$

From Equations 2.49a,b is easily seen that

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} = \frac{\partial^3 u}{\partial x \partial y^2} \qquad \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{\partial^3 v}{\partial x^2 \partial y}$$

Combining these results, we see that assumed expressions for the strain components ε_{xx}, ε_{yy}, and γ_{xy} correspond to physically reasonable displacement fields (i.e., "are compatible") only if they satisfy:

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} \tag{2.50a}$$

Equation 2.50a is the first compatibility condition. Following a similar procedure using Equations 2.49e and 2.49f, we obtain

$$\frac{\partial^2 \gamma_{yz}}{\partial y \partial z} = \frac{\partial^2 \varepsilon_{yy}}{\partial z^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial y^2} \tag{2.50b}$$

$$\frac{\partial^2 \gamma_{xz}}{\partial x \partial z} = \frac{\partial^2 \varepsilon_{xx}}{\partial z^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial x^2} \tag{2.50c}$$

These are the second and third compatibility conditions. Next, the following expressions are obtained using Equations 2.49a through 2.49f), respectively:

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = \frac{\partial^2 \varepsilon_{xx}}{\partial y \partial z}$$

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial z} - \frac{\partial^3 v}{\partial x^2 \partial z}$$

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = \frac{\partial^2 \gamma_{xz}}{\partial x \partial y} - \frac{\partial^3 w}{\partial x^2 \partial y}$$

$$\frac{\partial^3 w}{\partial x^2 \partial y} + \frac{\partial^3 v}{\partial x^2 \partial z} = \frac{\partial^2 \gamma_{yz}}{\partial x^2}$$

Combining these four expressions, we find that *assumed* expressions for strain components ε_{xx}, γ_{xy}, γ_{xz}, and γ_{yz} are compatible if

$$2\frac{\partial^2 \varepsilon_{xx}}{\partial y \partial z} = \frac{\partial}{\partial x}\left(\frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{xz}}{\partial y} - \frac{\gamma_{yz}}{\partial x}\right) \tag{2.50d}$$

This is the fourth compatibility condition. The final two compatibility conditions are developed using a similar process, and are given by

$$2\frac{\partial^2 \varepsilon_{yy}}{\partial x \partial z} = \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{xz}}{\partial y} + \frac{\gamma_{xy}}{\partial z}\right) \tag{2.50e}$$

$$2\frac{\partial^2 \varepsilon_{zz}}{\partial x \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} - \frac{\gamma_{xy}}{\partial z}\right) \tag{2.50f}$$

2.15 Computer Programs *3DROTATE* and *2DROTATE*

A review of force, stress, and strain tensors has been presented in this chapter. These concepts will be applied routinely in later chapters, as we develop a macromechanics-based analysis of structural composite materials and structures. It will be seen that transformation of stress and strain tensors is of particular importance. Indeed, nearly all analyses of composite materials and structures presented herein require multiple transformations of stress and strain tensors from one coordinate system to another.

Two computer programs, *3DROTATE* and *2DROTATE*, that can be used to perform transformations of force, stress, or strain tensors can be downloaded at no cost from the following website:

http://depts.washington.edu/amtas/computer.html

Program *3DROTATE* performs the calculations necessary to transform a force, stress, or strain tensor from the x–y–z coordinate system to the x'''–y'''–z''' coordinate system, where the x'''–y'''–z''' coordinate system is generated from the x–y–z coordinate system by (up to) three successive rotations. Derivation of the direction cosines that relate these two coordinate systems is left as a student exercise (see Homework Problem 2.2). Program *3DROTATE* also calculates the angles between the x–y–z and x'''–y'''–z''' coordinate axes, invariants of the force, stress, or strain tensors, and principal stresses and strains. All of the numerical results discussed in Example Problems 2.1, 2.3, and 2.7 can be obtained through the use of program *3DROTATE*.

The second program, *2DROTATE*, can be used to rotate stresses within a plane (as discussed in Section 2.8) and/or strains within a plane (as discussed in Section 2.13). It is important to remember that these transformations are only valid if the out-of-plane direction is a principal stress or principal strain axis. For the most part, thin plate-like composite structures will be considered in this book. Therefore, it can usually be assumed that the direction normal to the surface of the composite is a direction of principal stress or strain. Hence, most of the stress or strain transformations considered in this chapter involve rotations within a plane. Most of the numerical results discussed in Example Problems 2.5 and 2.9 can be obtained through the use of program *2DROTATE*.

HOMEWORK PROBLEMS

In the following problem statements the phrase “solve by hand” means that numerical solutions should be obtained using a pencil, paper, and nonprogrammable calculator. Solutions obtained by hand will then be compared with numerical results returned by appropriate computer programs. This process will insure understanding of the mathematical processes involved.

2.1. Solve part (c) of Example Problem 2.1 by hand, based on the rotation angles listed below. In each case calculate that the magnitude of the transformed force vector. Confirm your calculations using program *3DROTATE*.

(a) $\theta = 60°$	$\beta = -45°$
(b) $\theta = 60°$	$\beta = 45°$
(c) $\theta = -60°$	$\beta = -45°$
(d) $\theta = -60°$	$\beta = 45°$
(e) $\theta = -45°$	$\beta = 60°$
(f) $\theta = -45°$	$\beta = -60°$
(g) $\theta = 45°$	$\beta = 60°$
(h) $\theta = 45°$	$\beta = -60°$

2.2. Consider an x'''–y'''–z''' coordinate system, which is generated from an x–y–z coordinate system by the following three rotations:

a. A rotation of θ-degrees about the original z-axis, which defines an intermediate x'–y'–z' coordinate system (see Figure 2.2a).
b. A rotation of β-degrees about the x'-axis, which defines an intermediate x''–y''–z'' coordinate system (see Figure 2.2b).
c. A rotation of ψ-degrees about the y''-axis, which defines the final x'''–y'''–z''' coordinate system.

Show that the x'''–y'''–z''' and x–y–z coordinate systems are related by the following direction cosines:

$$\begin{bmatrix} c_{x'''x} & c_{x'''y} & c_{x'''z} \\ c_{y'''x} & c_{y'''y} & c_{y'''z} \\ c_{z'''x} & c_{z'''y} & c_{z'''z} \end{bmatrix}$$

$$= \begin{bmatrix} \cos\psi\cos\theta - \sin\psi\sin\beta\sin\theta & \cos\psi\sin\theta + \sin\psi\sin\beta\cos\theta & -\sin\psi\cos\beta \\ -\cos\beta\sin\theta & \cos\beta\cos\theta & \sin\beta \\ \sin\psi\cos\theta + \cos\psi\sin\beta\sin\theta & \sin\psi\sin\theta - \cos\psi\sin\beta\cos\theta & \cos\psi\cos\beta \end{bmatrix}$$

2.3. The force vector discussed in Example Problem 2.1 is given by

$$\bar{F} = 1000\hat{i} + 200\hat{j} + 600\hat{k}$$

Using Equation 2.6c, express $\bar{F}$ in a new coordinate system defined by three successive rotations, as listed below, using the direction cosines listed in Problem 2.2. In each case compare the magnitude of the transformed force vector to the magnitudes calculated in

Example Problem 2.1. Solve these problems by hand, and then confirm your calculations using program *3DROTATE*.

(a) $\theta = 60°$	$\beta = -45°$	$\psi = 25°$
(b) $\theta = 60°$	$\beta = -45°$	$\psi = -25°$
(c) $\theta = 60°$	$\beta = -45°$	$\psi = 25°$
(d) $\theta = 60°$	$\beta = 45°$	$\psi = 25°$
(e) $\theta = -60°$	$\beta = -45°$	$\psi = 25°$

2.4. Solve Example Problem 2.3 by hand, except use the following rotation angles:

(a) $\theta = 20°$	$\beta = -35°$
(b) $\theta = -20°$	$\beta = 35°$
(c) $\theta = -20°$	$\beta = -35°$

Confirm your calculations using program *3DROTATE*.

2.5. Use Equation 2.12a to obtain an expression (in expanded form) for the following stress component (in each case the expanded expression will be similar to Equation 2.13):

a. $\sigma_{x'x'}$
b. $\sigma_{x'y'}$
c. $\sigma_{y'y'}$
d. $\sigma_{y'z'}$
e. $\sigma_{z'z'}$

2.6. Use program *3DROTATE* to determine the stress invariants for the stress tensor listed below, and compare to those determined in Example Problem 2.3. (Note: this stress tensor is similar to the one considered in Example Problem 2.3, except that the algebraic sign of all three normal stresses has been reversed.):

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} -50 & -10 & 15 \\ -10 & -25 & 30 \\ 15 & 30 & 5 \end{bmatrix} (\text{ksi})$$

2.7. Use program *3DROTATE* to determine the stress invariants for the stress tensor listed below, and compare with those determined in Example Problem 2.3. (Note: this stress tensor is similar to the one considered in Example Problem 2.3, except that the algebraic sign of all three shear stresses has been reversed.):

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} 50 & 10 & -15 \\ 10 & 25 & -30 \\ -15 & -30 & -5 \end{bmatrix} (\text{ksi})$$

2.8. Use program *3DROTATE* to determine the stress invariants for the stress tensor listed below, and compare with those determined in Example Problem 2.3. (Note: this stress tensor is similar to the one considered in Example Problem 2.3, except that the algebraic sign of all stress components has been reversed.):

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} -50 & 10 & -15 \\ 10 & -25 & -30 \\ -15 & -30 & 5 \end{bmatrix} (\text{ksi})$$

2.9. Use program *3DROTATE* to determine the strain invariants for the strain tensor listed below, and compare with those determined in Example Problem 2.7. (Note: this strain tensor is similar to the one considered in Example Problem 2.7, except that the algebraic sign of all shear strain components has been reversed.):

$$\begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} = \begin{bmatrix} 1000\,\mu\text{m/m} & -500\,\mu\text{rad} & -250\,\mu\text{rad} \\ -500\,\mu\text{rad} & 1500\,\mu\text{m/m} & -750\,\mu\text{rad} \\ -250\,\mu\text{rad} & -750\,\mu\text{rad} & 2000\,\mu\text{m/m} \end{bmatrix}$$

2.10. Use program *3DROTATE* to determine the strain invariants for the strain tensor listed below, and compare with those determined in Example Problem 2.7. (Note: this strain tensor is similar to the one considered in Example Problem 2.7, except that the algebraic sign of all normal strain components has been reversed.):

$$\begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} = \begin{bmatrix} -1000\,\mu\text{m/m} & 500\,\mu\text{rad} & 250\,\mu\text{rad} \\ 500\,\mu\text{rad} & -1500\,\mu\text{m/m} & 750\,\mu\text{rad} \\ 250\,\mu\text{rad} & 750\,\mu\text{rad} & -2000\,\mu\text{m/m} \end{bmatrix}$$

2.11. Use program *3DROTATE* to determine the strain invariants for the strain tensor listed below, and compare with those determined in Example Problem 2.7. (Note: this strain tensor is similar to the one considered in Example Problem 2.7, except that the algebraic sign of all strain components has been reversed.):

$$\begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} = \begin{bmatrix} -1000\,\mu\text{m/m} & -500\,\mu\text{rad} & -250\,\mu\text{rad} \\ -500\,\mu\text{rad} & -1500\,\mu\text{m/m} & -750\,\mu\text{rad} \\ -250\,\mu\text{rad} & -750\,\mu\text{rad} & -2000\,\mu\text{m/m} \end{bmatrix}$$

References

1. Frederick, D., and Chang, T.S., *Continuum Mechanics*, Scientific Publishers, Cambridge, MA, 1972.
2. Fung, Y.C., *A First Course In Continuum Mechanics*, Prentice-Hall, Englewood Cliffs, NJ, 1969.
3. Consult any handbook of mathematical functions and tables, for example, *CRC Basic Mathematical Tables*, S.M. Selby, ed., The Chemical Company, Cleveland, OH, p. 59. 1970.