Elementary proof of the Routh-Hurwitz test

Gjerrit Meinsma

Department of Electrical and Computer Engineering
The University of Newcastle, University Drive, Callaghan
N.S.W. 2308, Australia, E-mail: meinsma@pascal.newcastle.edu.au

Abstract

This note presents an elementary proof of the familiar Routh-Hurwitz test. The proof is basically one continuity argument, it does not rely on Sturm chains, Cauchy index and the principle of the argument and it is fully self contained. In the same style an extended Routh-Hurwitz test is derived, which finds the inertia of polynomials.

Keywords: Routh-Hurwitz test, stability theory.

1 Introduction

One of the most famous results from stability theory is the Routh-Hurwitz test (R-H-test) which states that all zeros of a polynomial $p(s) = p_0 s^n + p_1 s^{n-1} + \cdots + p_n$ $(p_i \in \mathbb{R})$ lie in the open left-half plane iff a certain set of algebraic combinations of its coefficients have the same sign. In full:

Theorem 1.1 (Routh-Hurwitz test). A polynomial $p(s) = p_0 s^n + p_1 s^{n-1} + \cdots + p_{n-1}s + p_n$, $(p_i \in \mathbb{R},\ p_0 \neq 0)$, is stable iff all $n+1$ elements of the first column of the Routh table

$$\begin{array}{ccccc} p_0 & p_2 & p_4 & p_6 & \cdots \\ p_1 & p_3 & p_5 & p_7 & \cdots \\ r_{3,1} & r_{3,2} & r_{3,3} & \cdots & \\ r_{4,1} & r_{4,2} & r_{4,3} & \cdots & \\ \vdots & \vdots & \vdots & & \\ r_{n+1,1} & & & & \end{array}$$

are nonzero and have the same sign. The Routh table has $n+1$ rows, its first two rows are given by p as shown above and the other rows are defined successively as

$$\begin{aligned}(r_{i,1}\ r_{i,2}\ \cdots) &:= (r_{i-2,2}\ r_{i-2,3}\ \cdots) \\ &\quad - \frac{r_{i-2,1}}{r_{i-1,1}}(r_{i-1,2}\ r_{i-1,3} \cdots), \quad (i > 2).\end{aligned}$$

In most cases the first column of the table is well defined and has no zero elements, even if p is not stable. In this case p has no imaginary zeros and the number of unstable zeros of p equals the number of sign changes in the first column of the table. □

In some unlikely cases—such as when p has imaginary zeros—the R-H-test fails to come to an end due to a division by zero. This is commonly referred to as the *singular* case. Routh himself has devised some techniques to cope with singular cases, leading to an *extended Routh-Hurwitz test* (see [1]).

The authoritative reference for the R-H-test and extensions is Gantmacher [1]. In [1] the proof depends on Cauchy indices and Sturm Chains. In most papers on related issues, at some stage a Sturm chain, Cauchy index and a principle of the argument enter the story (see, for example, [2–4]). In [3,5] Sturm chains are not required, however, the proofs in [3,5] are still rather elaborate, and in [5] the results are only applicable to stable polynomials. Our presentation has links with [3] and the root locus arguments of [6].

We derive in this note an easier proof of the R-H-test and an extended R-H-test, using only a continuity argument. The R-H-test is proved in Section 2, and in Section 3 we derive in a similar style an extended R-H-test that may be used to find the *inertia* of a polynomial p, that is, the triple of integers

$$n_-(p),\ n_0(p),\ n_+(p)$$

denoting the number of stable zeros, $n_-(p)$, the number of zeros on the imaginary axis, $n_0(p)$, and the number of antistable zeros[1], $n_+(p)$ of p. The results resemble that of [3] and are not new, but to our knowledge the proofs are new and are much easier than other known proofs.

We consider only polynomials whose coefficients are real-valued. The expression $p(s) = q(s)$ usually means that $p(s) = q(s)$ for all complex s. A polynomial p is *odd* if it may be written as $p(s) = sk(s^2)$ and p is *even* if it may be written as $p(s) = h(s^2)$. Given a polynomial p, we use p_{even} and p_{odd} to denote the even and odd polynomials such that $p = p_{\text{even}} + p_{\text{odd}}$.

2 The Routh-Hurwitz test

The R-H-test is a result about polynomials, that, for the sake of computation, is written in terms of operations on the coefficients of polynomials. In order to understand the R-H-test we have to re-translate the results in terms of polynomials.

Let p be a polynomial of degree n. In the same way as

[1] A zero is *stable* if it lies in the open left-half plane, and *antistable* if it lies in the open right-half plane.

the first two rows of the Routh table of p

$$\begin{array}{cccc} p_0 & p_2 & p_4 & \cdots \\ p_1 & p_3 & p_5 & \cdots \\ r_{3,1} & r_{3,2} & \cdots & \end{array}$$

contain the coefficients of p, also the second and third row of this table may be seen as containing the coefficients of a polynomial: Define the polynomial q of degree $n-1$ as

$$q(s) := (p_1 s^{n-1} + p_3 s^{n-3} + \cdots) + (r_{3,1}s^{n-2} + r_{3,2}s^{n-4} + \cdots).$$

Using the definition of the third row, we may rewrite this as

$$\begin{aligned} q(s) &= p_1 s^{n-1} + (p_2 - \frac{p_0}{p_1} p_3) s^{n-2} + p_3 s^{n-3} \cdots \\ &= p(s) - \frac{p_0}{p_1}(p_1 s^n + p_3 s^{n-2} + \cdots). \end{aligned}$$

The Routh table of q is precisely that of p minus its first row. Hence the first column of the Routh table of p has no sign changes iff the Routh table of q has no sign changes and p_0 and p_1 have the same sign. The degree of q is less than that of p, and because the R-H-test is certainly correct for polynomials of degree one, we have by induction that the R-H-test is nothing but:

Theorem 2.1 (R-H-test). A nonconstant polynomial $p(s) = p_0 s^n + p_1 s^{n-1} + \cdots + p_n$, $(p_i \in \mathbb{R}, p_0 \neq 0)$ is stable iff p_1 is nonzero, p_0 and p_1 have the same sign, and the polynomial of degree $n-1$

$$q(s) := p(s) - \frac{p_0}{p_1}(p_1 s^n + p_3 s^{n-2} + p_5 s^{n-4} + \cdots)$$

is stable.

Furthermore, in most cases p_1 is nonzero even if p is not stable. In this case the inertia of p equals that of q with one extra stable (antistable) zero if p_0 and p_1 have the same (different) sign. In fact, p and q have the same imaginary zeros, counting multiplicities. □

Proof. Assume p_1 is nonzero. Define the nth degree polynomial q_η depending $\eta \in \mathbb{R}$ as

$$\begin{aligned} q_\eta(s) &:= p(s) - \eta(p_1 s^n + p_3 s^{n-2} + \cdots) \\ &= \begin{cases} p(s) - \eta\, s\, p_{\text{odd}}(s) & \text{if } n \text{ is even,} \\ p(s) - \eta\, s\, p_{\text{even}}(s) & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

For $\eta = 0$ we have $q_\eta = p$ and for η equal to

$$\eta_* := p_0/p_1$$

we have $q_\eta = q_{\eta*} = q$. A remarkable property of the family of polynomials $\{q_\eta\}$ is that they all have the same imaginary zeros, counting multiplicities. So, in particular, p and q have the same imaginary zeros.

(Proof: Suppose n is even and write q_η as the sum of an even and odd polynomial:

$$q_\eta = [p_{\text{even}} - \eta\, s\, p_{\text{odd}}] + p_{\text{odd}}.$$

A $j\omega$ is a zero of q_η of at least multiplicity k iff it is a zero of at least multiplicity k of both its odd and even part. This is because on the imaginary axis the even part takes only real values and the odd part only imaginary values. From the above expression for q_η it is clear that this is the case iff $j\omega$ is a zero of at least multiplicity k of both p_{odd} and p_{even}. This is independent of η, and, hence, the proof is complete when n is even. The same arguments work for the case that n is odd, in which case the even and odd part of q_η are as in $q_\eta = p_{\text{even}} + [p_{\text{odd}} - \eta\, s\, p_{\text{even}}]$.)

Therefore, by a continuity argument, as η varies no zero of q_η can cross the imaginary axis. The only way the inertia of q_η can change as a function of η is when at a certain point q_η drops degree. The only value of η for which $q_\eta(s) = (p_0 - \eta p_1)s^n + \cdots$ drops degree is $\eta = \eta_* = p_0/p_1$. As η approaches η_* from the origin—so the inertia of q_η equals that of $p = q_0$—one and only one zero of

$$q_\eta(s) = (p_0 - \eta p_1)s^n + p_1 s^{n-1} + \cdots$$

approaches $-p_1/(p_0 - \eta p_1)$ which goes to infinity, and the other $n-1$ zeros approach the zeros of $q_{\eta*} = q$. (Remember that the imaginary zeros of q_η are fixed, so the only zeros that wander around are the non-imaginary ones, and they do not reach the imaginary axis.) The zero that goes to infinity is stable iff $p_0 - \eta p_1$ and p_1 have the same sign, that is, iff p_0 and p_1 have the same sign. □

We silently assumed in the proof that p_1 is nonzero if p is stable. This is a well known fact and follows directly from an expansion of $p(s) = \prod_{i=1}^n (s - \lambda_i)$.

3 Extensions

In this section we extend the R-H-test so that it can handle every polynomial, revealing its inertia. The results in this section are practically a copy of [3].

It is convenient from this point on to call two polynomials *equivalent* if they have the same inertia (notation: $p \overset{\text{in}}{\simeq} q$). A polynomial $p_0 s^n + p_1 s^{n-1} + \cdots + p_n$ with $p_0 \neq 0$ is *regular* if $p_1 \neq 0$ and *singular* if $p_1 = 0$.

The crucial step in the proof of Theorem 2.1 is to identify a family of polynomials equivalent to p and then to pick from this family a degenerate one that is essentially of lower degree. This works as long as p is regular. If p is not regular it makes sense to switch to another *equivalent* polynomial that *is* regular and then continue the usual procedure with that polynomial. We show in this section that this can easily be done if p is not even or odd. The case that p is even or odd is dealt with separately. Both cases use the following rather general set of equivalent polynomials:

Lemma 3.1 ([3]). Let a polynomial $p = p_{\text{odd}} + p_{\text{even}}$ be given.

1. If α is an even polynomial with $\alpha(j\omega) > 0$ for all $\omega \in \mathbb{R}$, and $\deg(\alpha p_{\text{odd}}) < \deg p_{\text{even}}$, then

$$p \overset{\text{in}}{\simeq} p_{\text{even}} + \alpha\, p_{\text{odd}}.$$

2. If α is an even polynomial with $\alpha(j\omega) > 0$ for all $\omega \in \mathbb{R}$, and $\deg(\alpha p_{\text{even}}) < \deg p_{\text{odd}}$, then

$$p \overset{\text{in}}{\simeq} \alpha\, p_{\text{even}} + p_{\text{odd}}.$$

□

Proof. (This is practically the same as what we did in the proof of Theorem 2.1.) We proof Item 1; Item 2 is essentially equivalent.

Given α, define q_λ as $q_\lambda = p_{\text{even}} + ((1-\lambda) + \lambda\alpha)p_{\text{odd}}$. Then $q_0 = p$ and $q_1 = p_{\text{even}} + \alpha p_{\text{odd}}$. For all λ, q_λ has the same degree as p, so the proof is complete if we can show that as λ varies in the closed interval $[0, 1]$ no zero of q_λ can cross the imaginary axis. This we do by showing that the imaginary zeros of q_λ (counting multiplicities) are independent of $\lambda \in [0, 1]$.

Fix a $\lambda \in [0, 1]$ and suppose $j\omega$ is a zero of q_λ of at least multiplicity k. Then it is also zero of at least multiplicity k of p_{even} and $[(1-\lambda) + \lambda\alpha]p_{\text{odd}}$ because on the imaginary axis p_{even} takes only real values and $[(1-\lambda) + \lambda\alpha]p_{\text{odd}}$ takes only imaginary values. We know that on the imaginary axis $[(1-\lambda) + \lambda\alpha]$ is positive, nonzero for every λ in $[0, 1]$, hence, $j\omega$ is a zero of q_λ of at least multiplicity k iff it is a zero of at least multiplicity k of both p_{even} and p_{odd}. This is independent of λ. □

Case I: When singular p is not odd or even. Assume first that the degree n of singular polynomial p is even. Singularity of p implies that the odd part p_{odd} has degree $n - 1 - 2k$ for some $k > 0$. Then by Lemma 3.1 the polynomial

$$\begin{aligned} r(s) &:= p_{\text{even}}(s) + (1 + (-s^2)^k)p_{\text{odd}}(s) \\ &= p(s) + (-s^2)^k p_{\text{odd}}(s) \\ &= p_0 s^n \pm p_{1+2k}s^{n-1} + \cdots \end{aligned}$$

is equivalent to p because $\alpha(s) := 1 + (-s^2)^k$ is real and strictly positive on the imaginary axis. The polynomial r is regular and we may therefore proceed with the degree reduction step with this r.

Similarly, if p is singular and has *odd* degree, then p is equivalent to the regular polynomial $r(s) := (1 + (-s^2)^k)p_{\text{even}}(s) + p_{\text{odd}}(s)$, where k is defined through $\deg p_{\text{even}} + 2k = \deg p_{\text{odd}} - 1$.

Case II: When p is even or odd. The regularization as described above fails only if p is even or odd. An even or odd polynomial p has as many stable zeros as it has antistable zeros because $p(s) =: s^k h(s^2) = \pm p(-s)$. This symmetry implies that the inertia of an even or odd polynomial is completely determined by its degree and its number of, say, antistable zeros. As an introduction to the following lemma we assume p is even and we define r_ϵ depending on $\epsilon > 0$ as

$$r_\epsilon(s) = p(s + \epsilon).$$

This amounts to a shift of the zeros to the left. If $\epsilon > 0$ is small enough we have that r_ϵ is regular and that

$$n_+(p) = n_+(r_\epsilon).$$

Therefore by checking the inertia of r_ϵ for sufficiently small $\epsilon > 0$ we have in principle a means to perform the degree reduction step while keeping track of the inertia of p. The following shows we need not know how small ϵ must be if all imaginary zeros of p are simple (remember that p is assumed even):

$$\begin{aligned} r_\epsilon &= \underbrace{(p + \frac{\epsilon^2}{2!}p'' + \cdots)}_{\text{even}} + \epsilon\,\underbrace{(p' + \frac{\epsilon^2}{3!}p''' + \cdots)}_{\text{odd}} \\ &\overset{\text{in}}{\simeq} (p + \frac{\epsilon^2}{2!}p'' + \cdots) + (p' + \frac{\epsilon^2}{3!}p''' + \cdots) \\ &\qquad\qquad \text{(by Lemma 3.1).} \end{aligned}$$

The last expression converges to $p + p'$ when ϵ goes to zero. So if $p + p'$ does not have zeros on the imaginary axis we get that for small enough $\epsilon > 0$, $r_\epsilon \overset{\text{in}}{\simeq} p + p'$. If p has multiple zeros on the imaginary axis a similar result holds:

Lemma 3.2 (Case II, [3]). Suppose p is odd or even and that its degree is n. Then $r := p + p'$ is regular and $n_+(r) = n_+(p)$. That is, the inertia $\{n_-(p), n_0(p), n_+(p)\}$ of p is equal to

$$\{n_+(r),\ n - 2n_+(r),\ n_+(r)\}.$$

□

Proof. Let p be even or odd. Regularity of $p + p'$ is trivial. We examine the inertia of $r := p + p'$. By Lemma 3.1 the inertia of $p + p'$ equals that of $q_\epsilon := p + \epsilon p'$ for every $\epsilon > 0$. We may therefore as well examine the inertia of q_ϵ for an arbitrary small positive ϵ, and, hence, we need only worry about the root locus of the imaginary zeros of q_ϵ around $\epsilon = 0$.

If $j\omega$ is a zero of $q_0 = p$ of multiplicity k, then it is a zero of $q_\epsilon \neq p$ of multiplicity $k - 1$. Therefore as ϵ increases from 0, only one zero of q_ϵ moves (continuously as function of ϵ) away from $j\omega$. A Taylor series expansion of q_ϵ around $s = j\omega$ is

$$\begin{aligned} q_\epsilon(j\omega + \delta) &= \epsilon\frac{\delta^{k-1}}{(k-1)!}p^k(j\omega) + \frac{\delta^k}{k!}p^k(j\omega) + \\ &\quad \epsilon\frac{\delta^k}{k!}p^{k+1}(j\omega) + \text{higher order terms} \\ &= \frac{\delta^{k-1}}{k!}p^k(j\omega) \times \\ &\quad [\epsilon k + \delta + \epsilon\delta\frac{p^{k+1}(j\omega)}{p^k(j\omega)} + \text{h.o.t.}]. \end{aligned}$$

Solving $q_\epsilon(j\omega + \delta) = 0$ for δ shows—apart from the obvious $k - 1$ fixed zeros $\delta = 0$—that the remaining zero δ is approximately $-k\epsilon$ for small ϵ. This means that the remaining zero moves into the open left-half plane as ϵ increases from zero. Hence $n_+(p) = n_+(q_\epsilon) = n_+(p + p')$. □

This completes the second singular case, and in combination with Case I we may now formulate an extended

Routh-Hurwitz-test. We do this in the form of a Matlab macro. The validity of this extended test is easily checked by translating the operations on the coefficients as demonstrated in this macro in terms of the polynomial manipulations derived earlier. The macro is meant to be easy to read, it is not very sophisticated.

```
function inertia=erh(p)
% Finds the inertia of p=[p_0 p_1 .. ]
ind=find(abs(p) > 1e-11);
p(1:ind(1)-1)=[];
degree=max(size(p))-1;
inert=[0 0 0];
wehavehadcase2=0;
for n=degree:-1:1 % Reduce the degree to 1
    k=find(abs(p(2:2:n+1)) > 1e-11);
    if k == []    % Case-2: Differentiate.
       p(2:2:n+1)=p(1:2:n) .* (n:-2:1);
       wehavehadcase2=1;
    elseif k(1)>1 % Case-1: Add polynomial.
       ind=0:2:(n+1-2*k(1));
       f=(-1)^k(1);
       p(ind+2)=p(ind+2)-f*p(ind+2*k(1));
    end
    eta=p(1)/p(2);
    if wehavehadcase2
         inert=inert+[(eta<0) 0 (eta<0)];
    else inert=inert+[(eta>0) 0 (eta<0)];
    end
    p(1:2:n)=p(1:2:n)-eta*p(2:2:n+1);
    p(1)=[];      % Reduce degree to n-1
end
inertia=inert+[0 degree-sum(inert) 0];
```

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