ME500/AA535 Advanced Composite Structural Analysis

A brief review of Chapters 1-7

(particularly Chapter 4-7)

Hooke's Law for an Anisotropic Material

(neglecting thermal or moisture effects)

$$\varepsilon_{ij} = S_{ijkl}\sigma_{kl}, \quad i, j, k, l = x, y, z$$

(or)

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}, \quad i, j, k, l = x, y, z$$

where:

$\sigma_{ij}, \varepsilon_{ij}$ = 2nd order stress and strain tensors, respectively

S_{ijkl} = 4th order compliance matrix

C_{ijkl} = 4th order stiffness matrix

Hooke's Law for an Anisotropic Material
(neglecting thermal or moisture effects)

•Would like to write Hooke's law using 2-D arrays, but cannot because of 4th order compliance and stiffness tensors

•In the absence of body forces, stress and strain tensors are symmetric…allows use of "contracted notation"

$$\begin{aligned}
\varepsilon_{xx} &\rightarrow \varepsilon_1 \\
\varepsilon_{yy} &\rightarrow \varepsilon_2 \\
\varepsilon_{zz} &\rightarrow \varepsilon_3 \\
\gamma_{yz} = \gamma_{zy} &\rightarrow \varepsilon_4 \\
\gamma_{xz} = \gamma_{zx} &\rightarrow \varepsilon_5 \\
\gamma_{xy} = \gamma_{yx} &\rightarrow \varepsilon_6
\end{aligned}$$

$$\begin{aligned}
\sigma_{xx} &\rightarrow \sigma_1 \\
\sigma_{yy} &\rightarrow \sigma_2 \\
\sigma_{zz} &\rightarrow \sigma_3 \\
\sigma_{yz} = \sigma_{zy} &\rightarrow \sigma_4 \\
\sigma_{xz} = \sigma_{zx} &\rightarrow \sigma_5 \\
\sigma_{xy} = \sigma_{yx} &\rightarrow \sigma_6
\end{aligned}$$

Hooke's Law for an Anisotropic Material
(neglecting thermal or moisture effects)

$$\varepsilon_i = S_{ij}\sigma_j, \quad i, j = 1 \ldots 6$$

(or)

$$\sigma_i = C_{ij}\varepsilon_j, \quad i, j = 1 \ldots 6$$

where:

σ_i, ε_i = (disguised) 2nd order stress and strain tensors, respectively

S_{ij} = (disguised) 4th order compliance matrix

C_{ij} = (disguised) 4th order stiffness matrix

Hooke's Law for an Anisotropic Material

(neglecting thermal or moisture effects)

$$\varepsilon_i = S_{ij}\sigma_j, \quad i, j = 1\ldots6$$

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix}$$

Hooke's Law for an Anisotropic Material
(neglecting thermal or moisture effects)

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix}$$

Hooke's Law for an Anisotropic Material

(neglecting thermal or moisture effects)

$$\sigma_i = C_{ij}\varepsilon_j, \quad i, j = 1\ldots 6$$

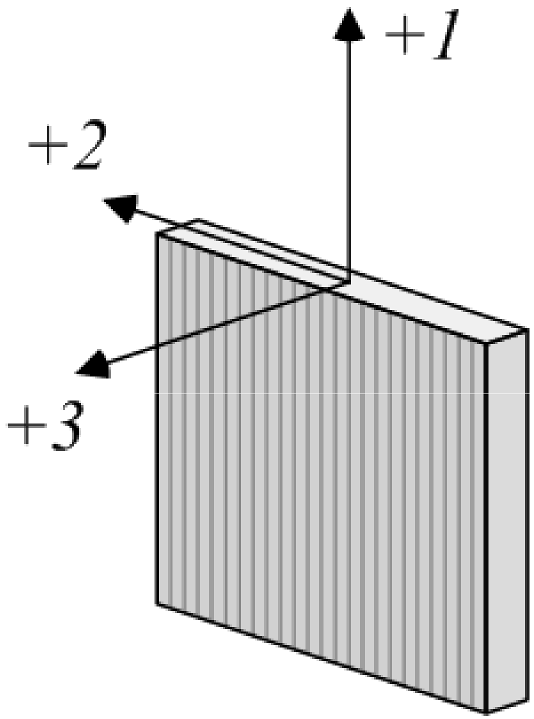
$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix}$$

Hooke's Law for an Anisotropic Material
(neglecting thermal or moisture effects)

$$\sigma_i = C_{ij}\varepsilon_j, \quad i, j = 1...6$$

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix}$$

Principal Material Coordinate System



- Defined by the symmetry associated with fiber orientation
- Usually labeled the 1-2-3 coordinate system
- No "unusual" coupling in PMCS
- Elastic & failure properties measured in the 1-2-3 coordinate system:

$$E_{11}, E_{22}, E_{33}, \nu_{12}, \nu_{13}, \nu_{23}, G_{12}, G_{13}, G_{23}$$

$$\sigma_{11}^{fT}, \sigma_{11}^{fC}, \sigma_{22}^{fT}, \sigma_{22}^{fC}, \sigma_{33}^{fT}, \sigma_{33}^{fC}, \tau_{12}^{f}, \tau_{13}^{f}, \tau_{23}^{f}$$

Hooke's Law In Principal Material Coordinate System

- Since no "unusual" couplings, all of the following compliances equal zero in the principal material coordinate system:

$$S_{14} = S_{41}, S_{15} = S_{51}, S_{16} = S_{61}$$
$$S_{24} = S_{42}, S_{25} = S_{52}, S_{26} = S_{62}$$
$$S_{34} = S_{43}, S_{35} = S_{53}, S_{36} = S_{63}$$
$$S_{45} = S_{54}, S_{46} = S_{64}, S_{56} = S_{65}$$

Hooke's Law In Principal Material Coordinate System

- Since no "unusual" couplings, in the principal material coordinate system:

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix}$$

Hooke's Law In Principal Material Coordinate System

- Similarly:

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix}$$

Called "Orthotropic" if $E_{33} \neq E_{22}$:

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} \left(\frac{1}{E_{11}}\right) & \left(\frac{-\nu_{21}}{E_{22}}\right) & \left(\frac{-\nu_{31}}{E_{33}}\right) & 0 & 0 & 0 \\ \left(\frac{-\nu_{12}}{E_{11}}\right) & \left(\frac{1}{E_{22}}\right) & \left(\frac{-\nu_{32}}{E_{33}}\right) & 0 & 0 & 0 \\ \left(\frac{-\nu_{13}}{E_{11}}\right) & \left(\frac{-\nu_{23}}{E_{22}}\right) & \left(\frac{1}{E_{33}}\right) & 0 & 0 & 0 \\ 0 & 0 & 0 & \left(\frac{1}{G_{23}}\right) & 0 & 0 \\ 0 & 0 & 0 & 0 & \left(\frac{1}{G_{13}}\right) & 0 \\ 0 & 0 & 0 & 0 & 0 & \left(\frac{1}{G_{12}}\right) \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix}$$

Orthotropic Materials

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix}$$

Called “Transversely Isotropic” if $E_{33} = E_{22}$

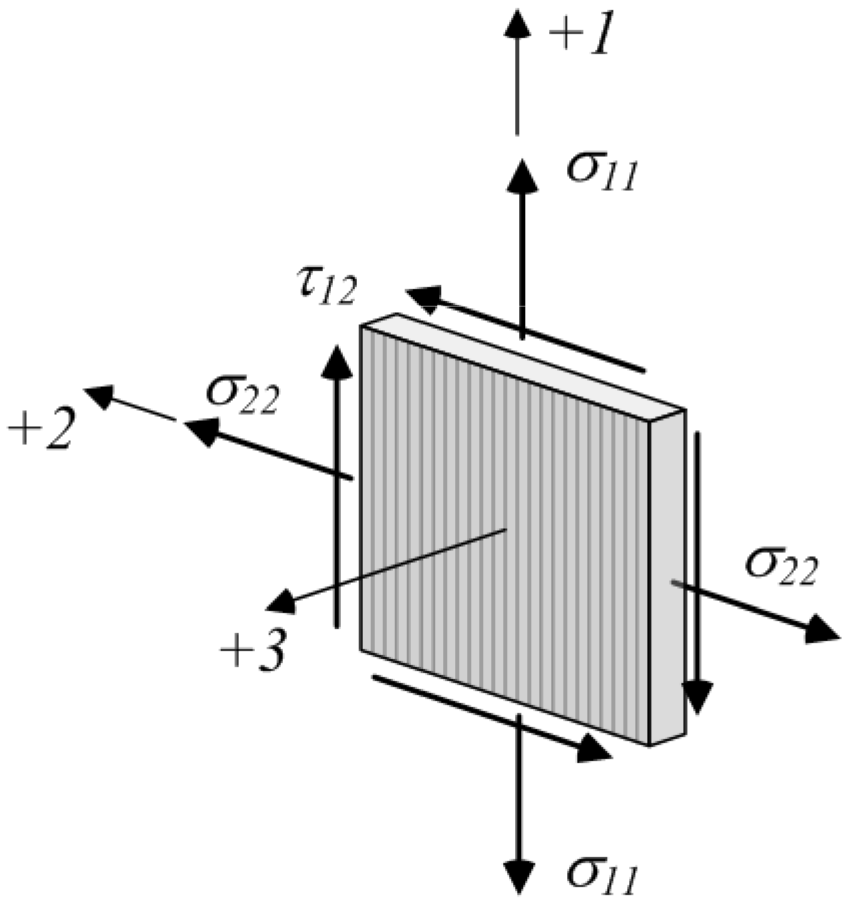
$$
\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix}
=
\begin{bmatrix}
\left(\frac{1}{E_{11}}\right) & \left(\frac{-\nu_{21}}{E_{22}}\right) & \left(\frac{-\nu_{21}}{E_{22}}\right) & 0 & 0 & 0 \\
\left(\frac{-\nu_{12}}{E_{11}}\right) & \left(\frac{1}{E_{22}}\right) & \left(\frac{-\nu_{23}}{E_{22}}\right) & 0 & 0 & 0 \\
\left(\frac{-\nu_{12}}{E_{11}}\right) & \left(\frac{-\nu_{23}}{E_{22}}\right) & \left(\frac{1}{E_{22}}\right) & 0 & 0 & 0 \\
0 & 0 & 0 & \left(\frac{2(1+\nu_{23})}{E_{22}}\right) & 0 & 0 \\
0 & 0 & 0 & 0 & \left(\frac{1}{G_{12}}\right) & 0 \\
0 & 0 & 0 & 0 & 0 & \left(\frac{1}{G_{12}}\right)
\end{bmatrix}
\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix}
$$

Transversely Isotropic Materials

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{12} & S_{23} & S_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(S_{22}-S_{23}) & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{66} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix}$$

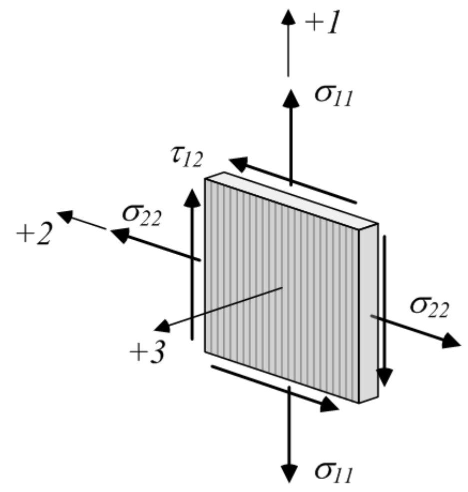
$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{12} & C_{23} & C_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & (C_{22}-C_{23})/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{66} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix}$$

Plane Stress Conditions



- For thin, plate-like structures it is usually appropriate to assume all out-of-plane stresses are negligibly small (i.e., assume $\sigma_{33} = \tau_{13} = \tau_{23} = 0$)

Plane Stress Conditions



- For plane stress conditions:

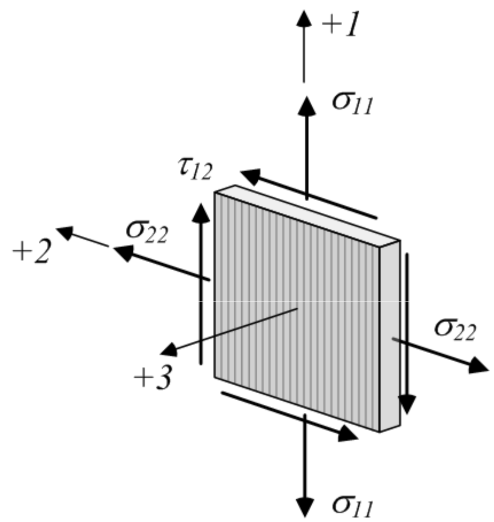
$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ 0 \\ 0 \\ 0 \\ \tau_{12} \end{Bmatrix}$$

Plane Stress Conditions

- Hooke's law can be "reduced" to:

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{Bmatrix}$$

$$\varepsilon_{33} = S_{13}\sigma_{11} + S_{23}\sigma_{22}$$

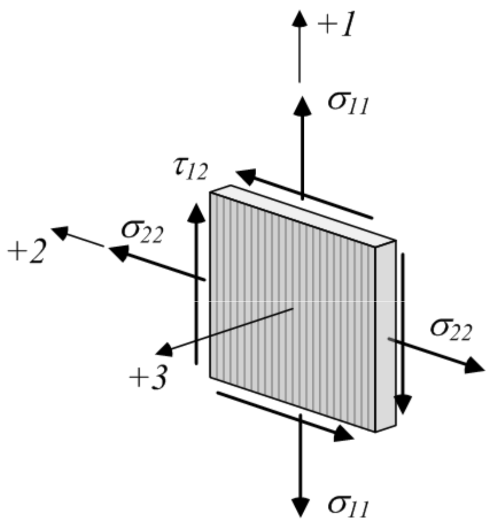


$$[S] = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} = \text{the "reduced compliance matrix"}$$

Plane Stress Conditions

- Inverting:

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{Bmatrix}$$



$$[\mathbf{Q}] = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} = \text{the "reduced stiffness matrix"}$$

Strains Caused by Stress and/or Changes in Temperature and/or Moisture Content

- Three-dimensional stress-states (orthotropic mat'ls):

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix} + \Delta T \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ \alpha_{33} \\ 0 \\ 0 \\ 0 \end{Bmatrix} + \Delta M \begin{Bmatrix} \beta_{11} \\ \beta_{22} \\ \beta_{33} \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$\Delta T, \Delta M$ = change in temperature and moisture content, respectively

α_{ii}, β_{ii} = thermal and moisture expansion coefficients, respectively

…Inverting

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} - \Delta T \alpha_{11} - \Delta M \beta_{11} \\ \varepsilon_{22} - \Delta T \alpha_{22} - \Delta M \beta_{22} \\ \varepsilon_{33} - \Delta T \alpha_{33} - \Delta M \beta_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix}$$

$\Delta T, \Delta M$ = change in temperature and moisture content, respectively

α_{ii}, β_{ii} = thermal and moisture expansion coefficients, respectively

Strains Caused by Stress and/or Changes in Temperature and/or Moisture Content

- Reducing to plane stress conditions:

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{Bmatrix} + \Delta T \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ 0 \end{Bmatrix} + \Delta M \begin{Bmatrix} \beta_{11} \\ \beta_{22} \\ 0 \end{Bmatrix}$$

$$\varepsilon_{33} = S_{13}\sigma_{11} + S_{23}\sigma_{22} + \Delta T \alpha_{33} + \Delta M \beta_{33}$$

$\Delta T, \Delta M$ = change in temperature and moisture content, respectively

α_{ii}, β_{ii} = thermal and moisture expansion coefficients, respectively

Strains Caused by Stress and/or Changes in Temperature and/or Moisture Content

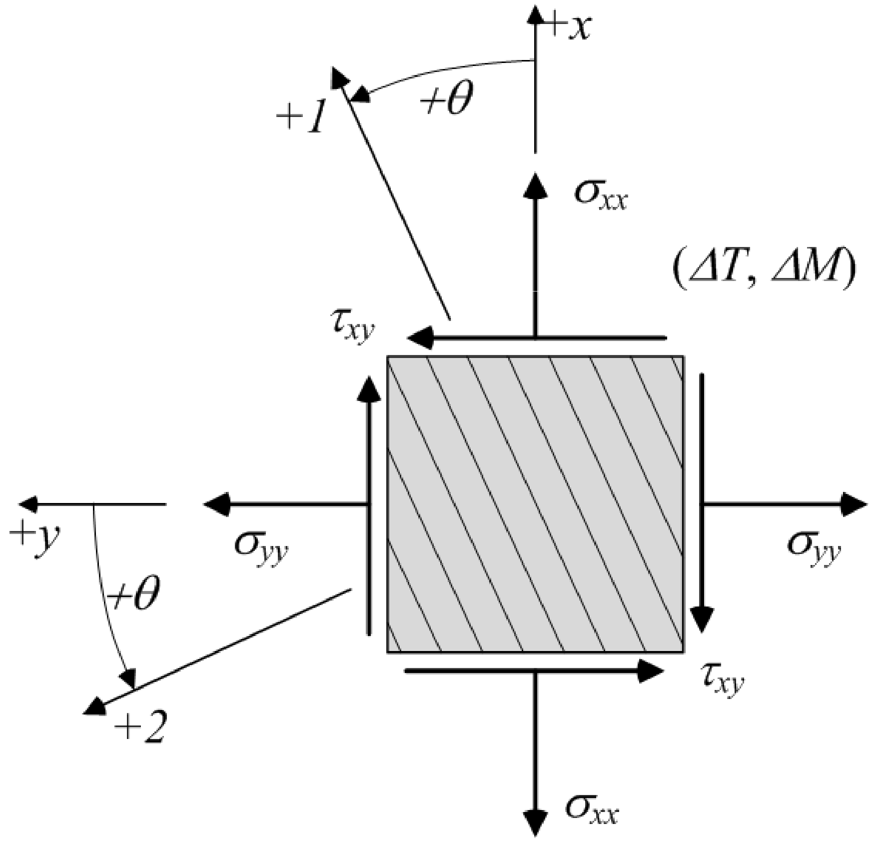
- Inverting:

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} - \Delta T \alpha_{11} - \Delta M \beta_{11} \\ \varepsilon_{22} - \Delta T \alpha_{22} - \Delta M \beta_{22} \\ \gamma_{12} \end{Bmatrix}$$

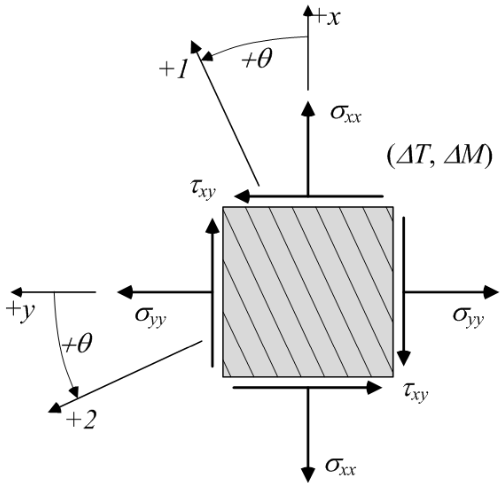
$\Delta T, \Delta M$ = change in temperature and moisture content, respectively

α_{ii}, β_{ii} = thermal and moisture expansion coefficients, respectively

Hooke's Law for Orthotropic Composites Referenced to an Arbitrary *x-y-z* Coordinate System (Plane Stress)



Hooke's Law for Orthotropic Composites Referenced to an Arbitrary *x-y-z* Coordinate System (Plane Stress)



$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16} \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{26} \\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{12} \end{Bmatrix} + \Delta T \begin{Bmatrix} \alpha_{xx} \\ \alpha_{yy} \\ \alpha_{xy} \end{Bmatrix} + \Delta M \begin{Bmatrix} \beta_{xx} \\ \beta_{yy} \\ \beta_{xy} \end{Bmatrix}$$

$$\varepsilon_{33} = \varepsilon_{zz} = S_{13}\sigma_{11} + S_{23}\sigma_{22} + \Delta T\alpha_{33} + \Delta M\beta_{33}$$

$[\bar{S}_{ij}]$ = Transformed, reduced compliance matrix

Hooke's Law for Orthotropic Composites Referenced to an Arbitrary *x-y-z* Coordinate System (Plane Stress)

$$\bar{S}_{11} = S_{11}\cos^4\theta + (2S_{12} + S_{66})\cos^2\theta\sin^2\theta + S_{22}\sin^4\theta$$

$$\bar{S}_{12} = \bar{S}_{21} = S_{12}(\cos^4\theta + \sin^4\theta) + (S_{11} + S_{22} - S_{66})\cos^2\theta\sin^2\theta$$

$$\bar{S}_{16} = \bar{S}_{61} = (2S_{11} - 2S_{12} - S_{66})\cos^3\theta\sin\theta - (2S_{22} - 2S_{12} - S_{66})\cos\theta\sin^3\theta$$

$$\bar{S}_{22} = S_{11}\sin^4\theta + (2S_{12} + S_{66})\cos^2\theta\sin^2\theta + S_{22}\cos^4\theta$$

$$\bar{S}_{26} = \bar{S}_{62} = (2S_{11} - 2S_{12} - S_{66})\cos\theta\sin^3\theta - (2S_{22} - 2S_{12} - S_{66})\cos^3\theta\sin\theta$$

$$\bar{S}_{66} = 2(2S_{11} + 2S_{22} - 4S_{12} - S_{66})\cos^2\theta\sin^2\theta + S_{66}(\cos^4\theta + \sin^4\theta)$$

Hooke's Law for Orthotropic Composites Referenced to an Arbitrary *x-y-z* Coordinate System (Plane Stress)

$$\alpha_{xx} = \alpha_{11}\cos^2(\theta) + \alpha_{22}\sin^2(\theta)$$

$$\alpha_{yy} = \alpha_{11}\sin^2(\theta) + \alpha_{22}\cos^2(\theta)$$

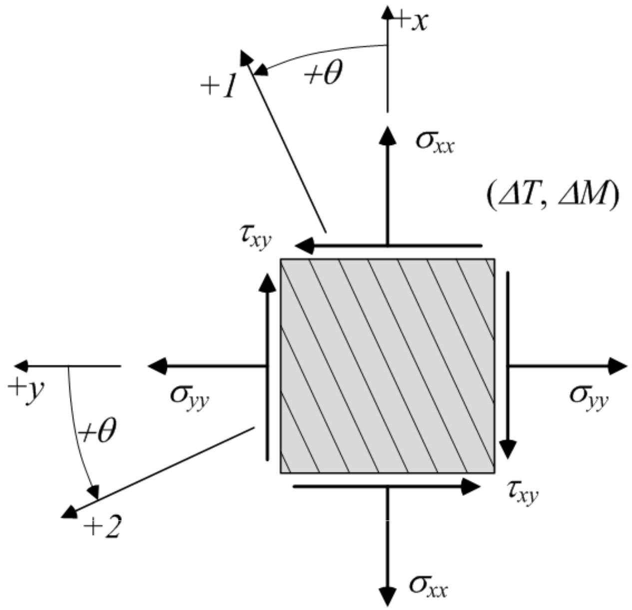
$$\alpha_{xy} = 2\cos(\theta)\sin(\theta)(\alpha_{11} - \alpha_{22})$$

$$\beta_{xx} = \beta_{11}\cos^2(\theta) + \beta_{22}\sin^2(\theta)$$

$$\beta_{yy} = \beta_{11}\sin^2(\theta) + \beta_{22}\cos^2(\theta)$$

$$\beta_{xy} = 2\cos(\theta)\sin(\theta)(\beta_{11} - \beta_{22})$$

…Inverting



$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} - \Delta T \alpha_{xx} - \Delta M \beta_{xx} \\ \varepsilon_{yy} - \Delta T \alpha_{yy} - \Delta M \beta_{yy} \\ \gamma_{xy} - \Delta T \alpha_{xy} - \Delta M \beta_{xy} \end{Bmatrix}$$

$$\left[\overline{Q}_{ij}\right] = \text{Transformed, reduced stiffness matrix}$$

…Where

$$\overline{Q}_{11} = Q_{11}\cos^4\theta + 2(Q_{12} + 2Q_{66})\cos^2\theta\sin^2\theta + Q_{22}\sin^4\theta$$

$$\overline{Q}_{12} = \overline{Q}_{21} = Q_{12}(\cos^4\theta + \sin^4\theta) + (Q_{11} + Q_{22} - 4Q_{66})\cos^2\theta\sin^2\theta$$

$$\overline{Q}_{16} = \overline{Q}_{61} = (Q_{11} - Q_{12} - 2Q_{66})\cos^3\theta\sin\theta - (Q_{22} - Q_{12} - 2Q_{66})\cos\theta\sin^3\theta$$

$$\overline{Q}_{22} = Q_{11}\sin^4\theta + 2(Q_{12} + 2Q_{66})\cos^2\theta\sin^2\theta + Q_{22}\cos^4\theta$$

$$\overline{Q}_{26} = \overline{Q}_{62} = (Q_{11} - Q_{12} - 2Q_{66})\cos\theta\sin^3\theta - (Q_{22} - Q_{12} - 2Q_{66})\cos^3\theta\sin\theta$$

$$\overline{Q}_{66} = (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})\cos^2\theta\sin^2\theta + Q_{66}(\cos^4\theta + \sin^4\theta)$$

An Observation Regarding Stress Distributions

assuming linear-elastic conditions with no body forces

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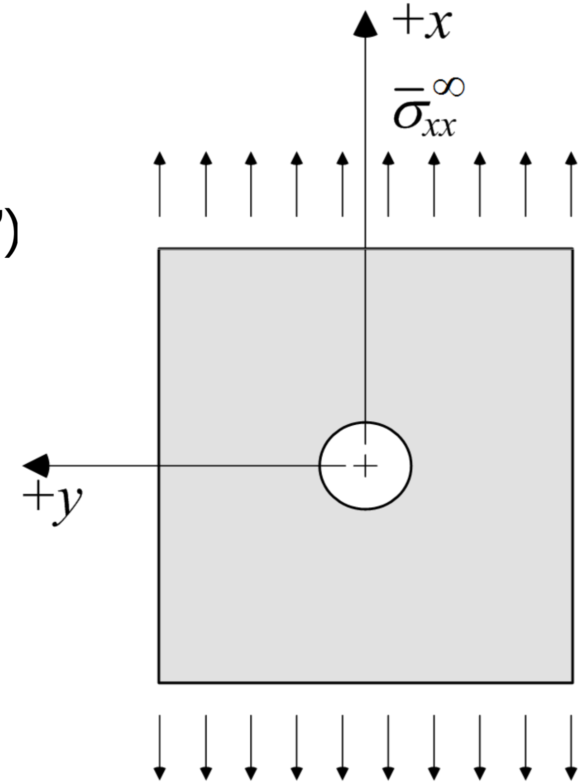

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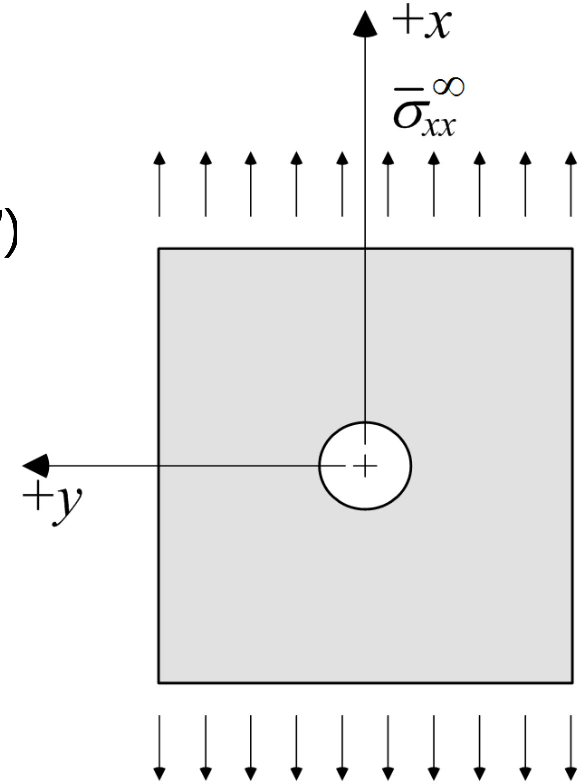

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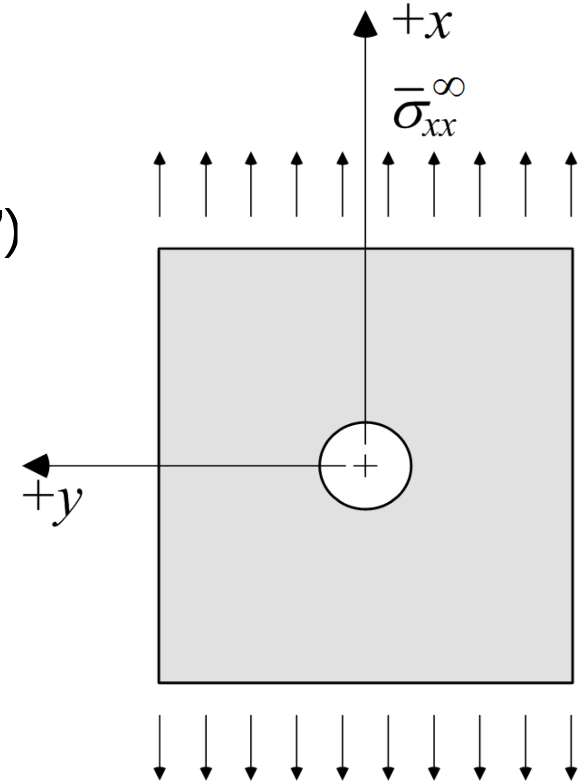

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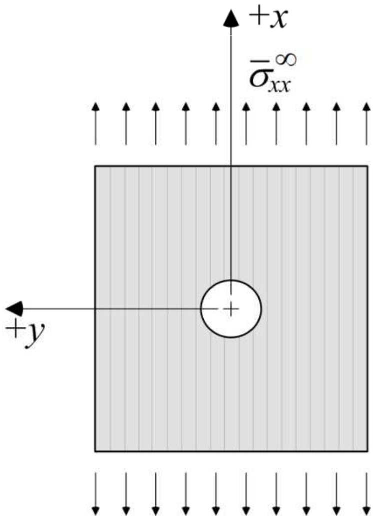

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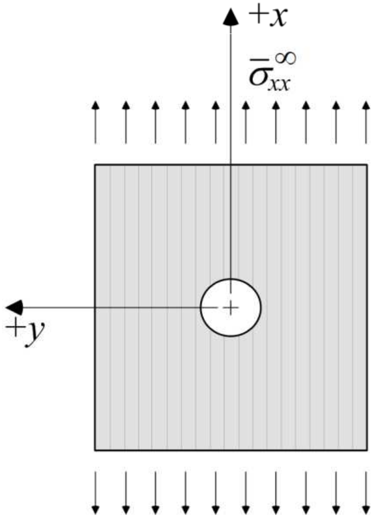
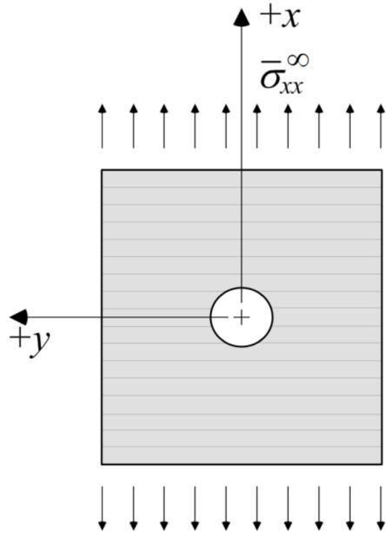


• SCF for circular hole in large [90º] panel: ~ 1.5 - 2.5



Macroscopic Failure Theories

- Dozens of failure theories have been proposed…none are universally accepted
  - •Hinton, M.J., Kaddour, A.S., and Soden, P.D., *Failure Criteria in Fibre Reinforced Polymer Composites: The World-Wide failure Exercise*, Elsevier, ISBN 0-08-044475-X (2004)

•Three common failure criterion described in textbook:

- •Maximum stress failure criterion
- •Tsai-Hill failure criterion
- •Tsai-Wu failure criterion

Macroscopic Failure Theories

Maximum Stress Failure Criterion (plane stress form)

- Failure does not occur if:

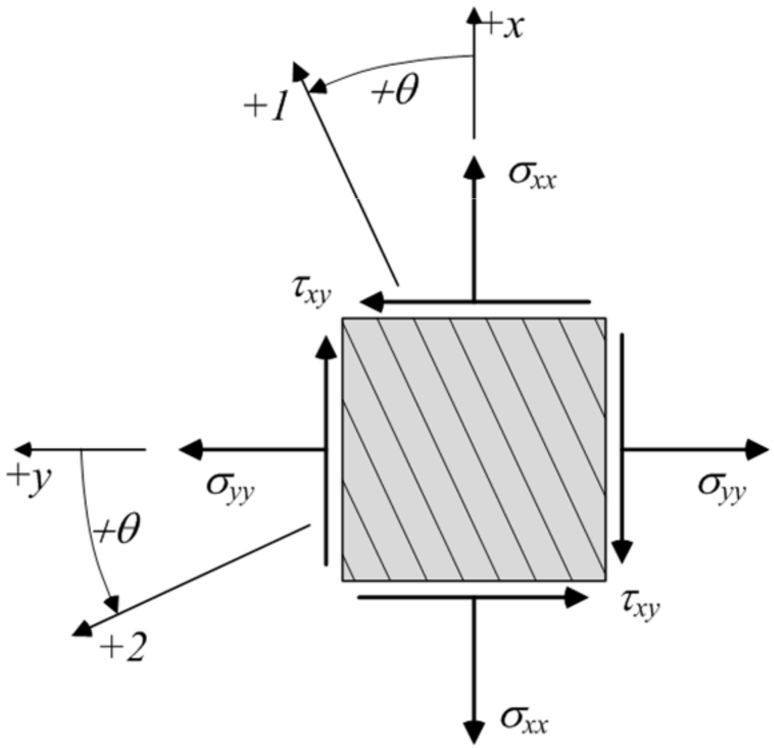
$$-1*\sigma_{11}^{fC} < \sigma_{11} < \sigma_{11}^{fT}$$

(and)

$$-1*\sigma_{22}^{fC} < \sigma_{22} < \sigma_{22}^{fT}$$

(and)

$$|\tau_{12}| < \tau_{12}^{f}$$

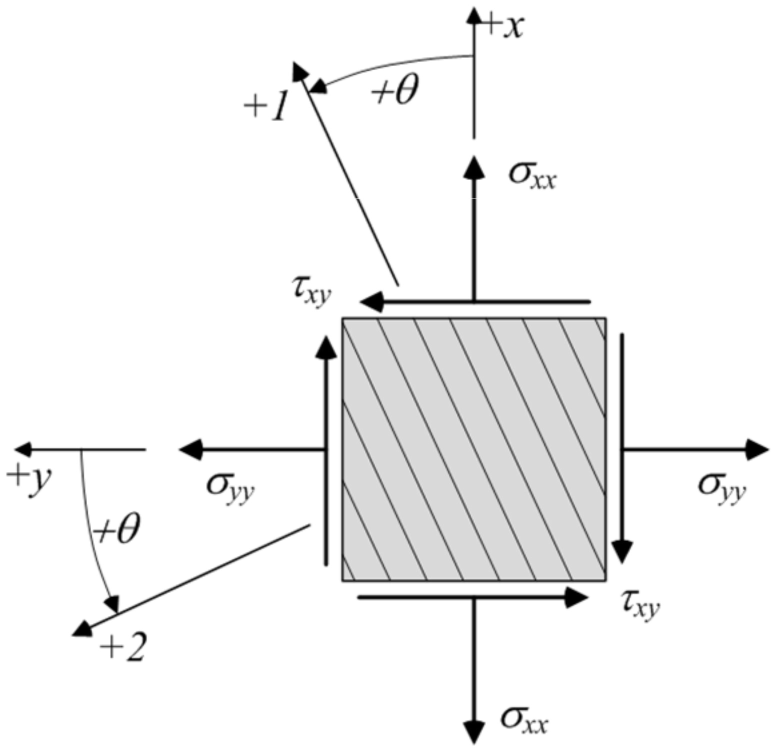


Macroscopic Failure Theories

Tsai-Hill Failure Criterion (plane stress form)

- Failure does not occur if:

$$\frac{(\sigma_{11})^2}{\left(\sigma_{11}^{fT}\right)^2} + \frac{(\sigma_{22})^2}{\left(\sigma_{22}^{fT}\right)^2} + \frac{(\tau_{12})^2}{\left(\tau_{12}^{f}\right)^2} - \frac{\sigma_{11}\sigma_{22}}{\left(\sigma_{11}^{fT}\right)^2} < 1$$

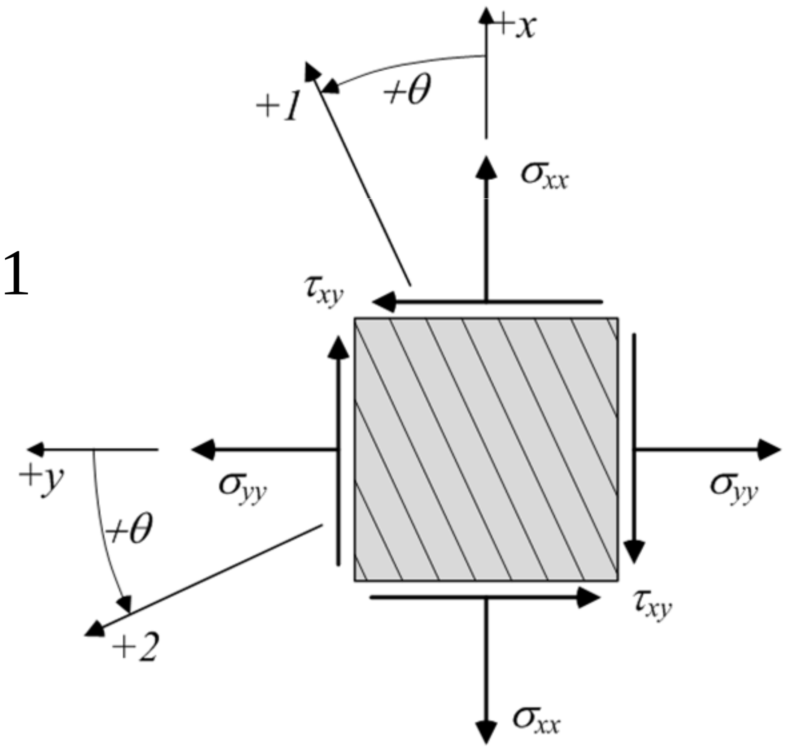


Macroscopic Failure Theories

Tsai-Wu Failure Criterion (plane stress form)

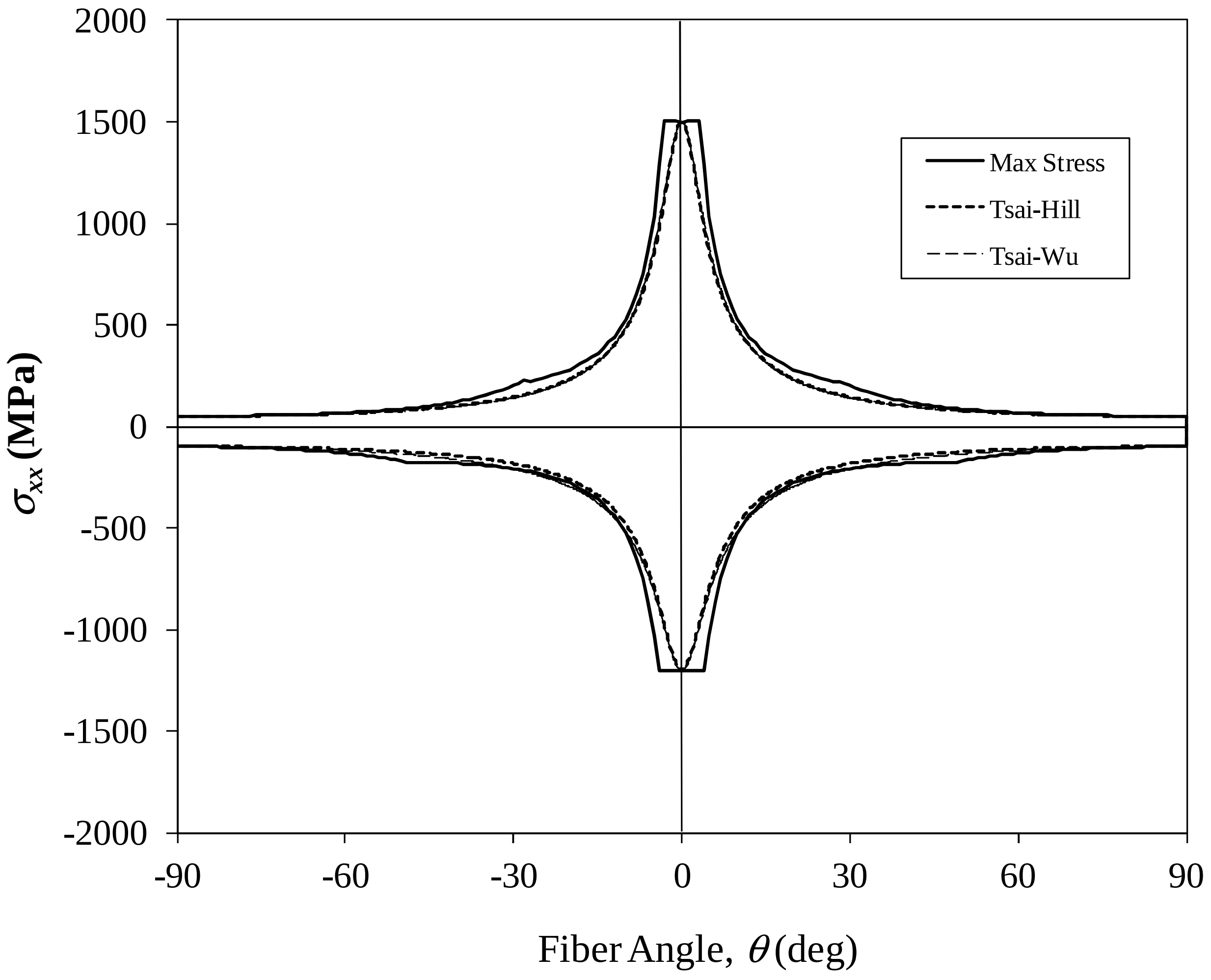
- Failure does not occur if:

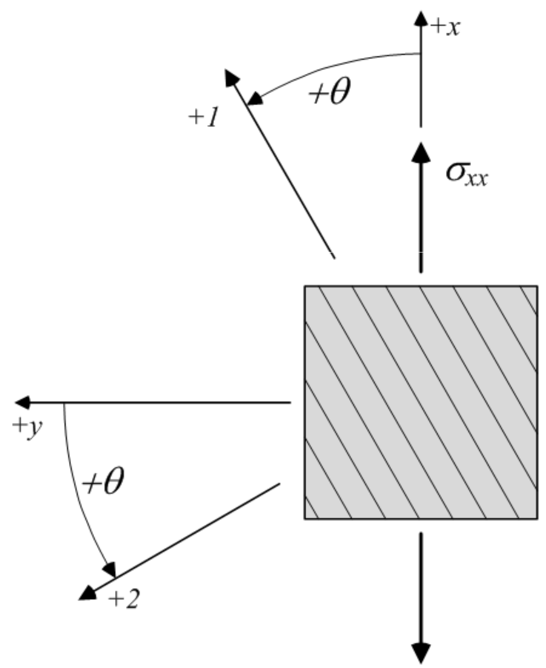
$$X_1\sigma_{11} + X_2\sigma_{22} + X_{11}\sigma_{11}^2$$
$$+ X_{22}\sigma_{22}^2 + X_{66}\tau_{12}^2 + 2X_{12}\sigma_{11}\sigma_{22} < 1$$



Macroscopic Failure Theories: Comparisons

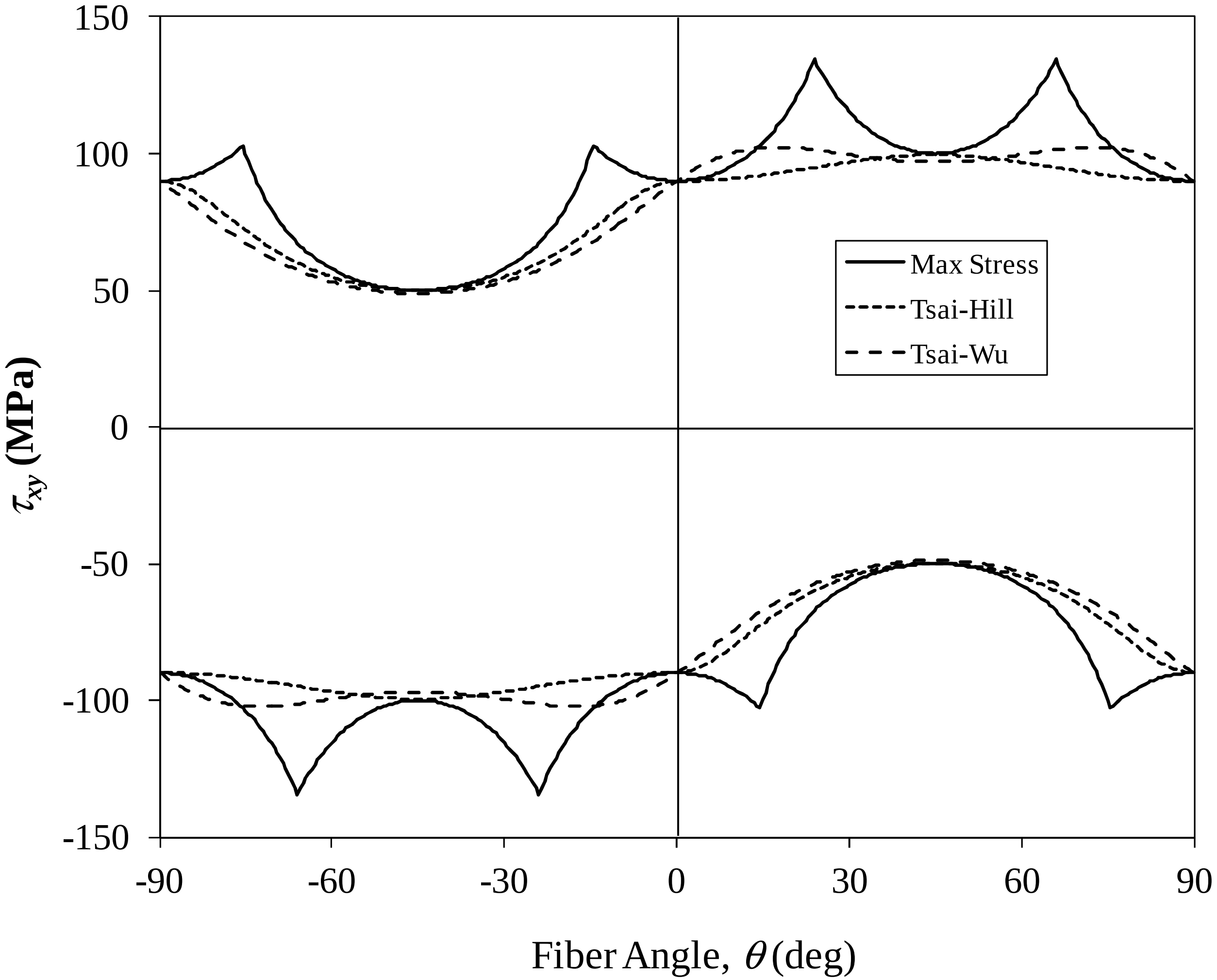
Off-axis Gr/Ep specimen subject to uniaxial stress

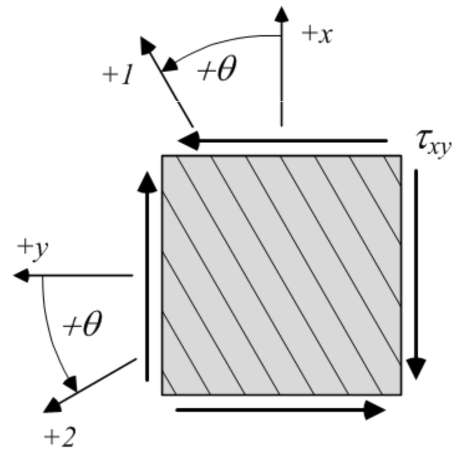




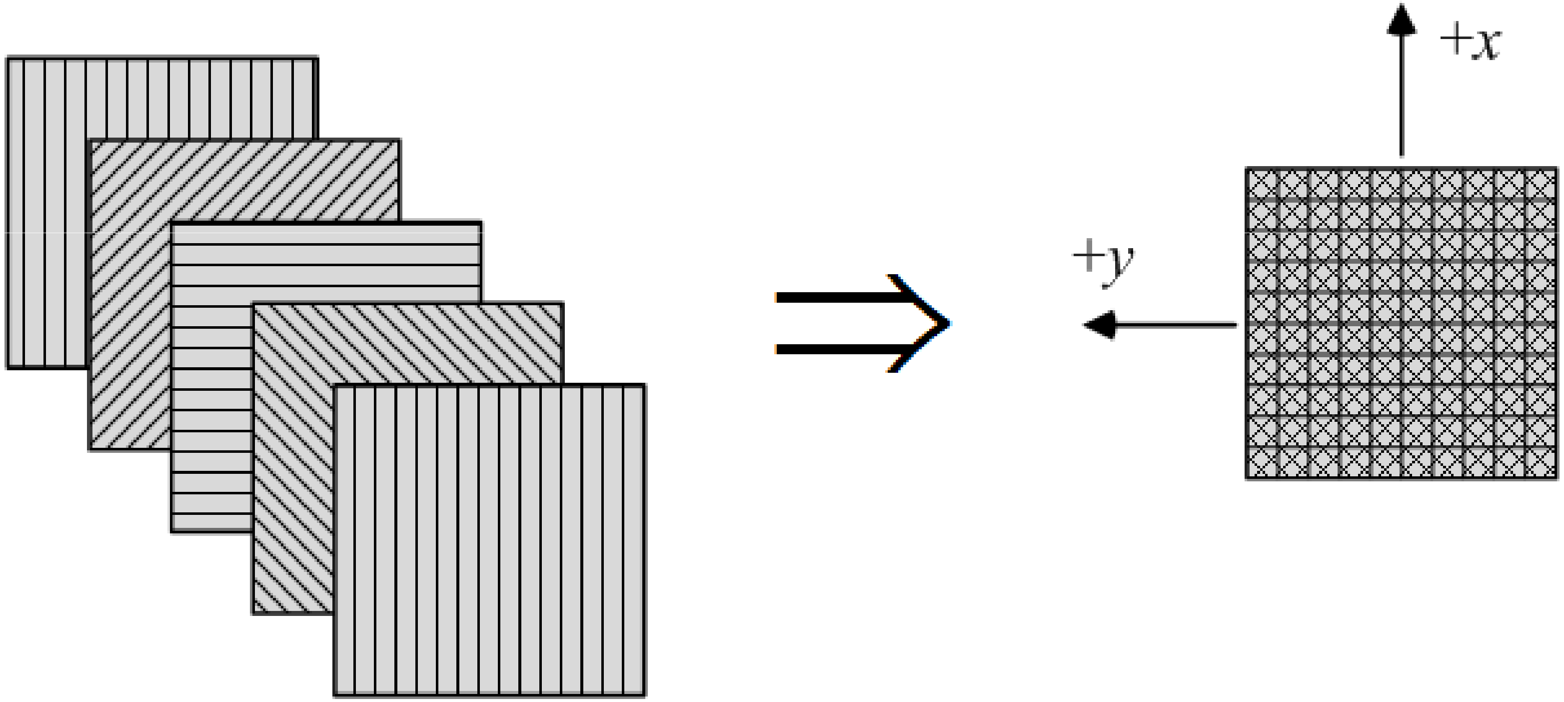
Macroscopic Failure Theories: Comparisons

Off-axis Gr/Ep specimen subjected to pure shear stress

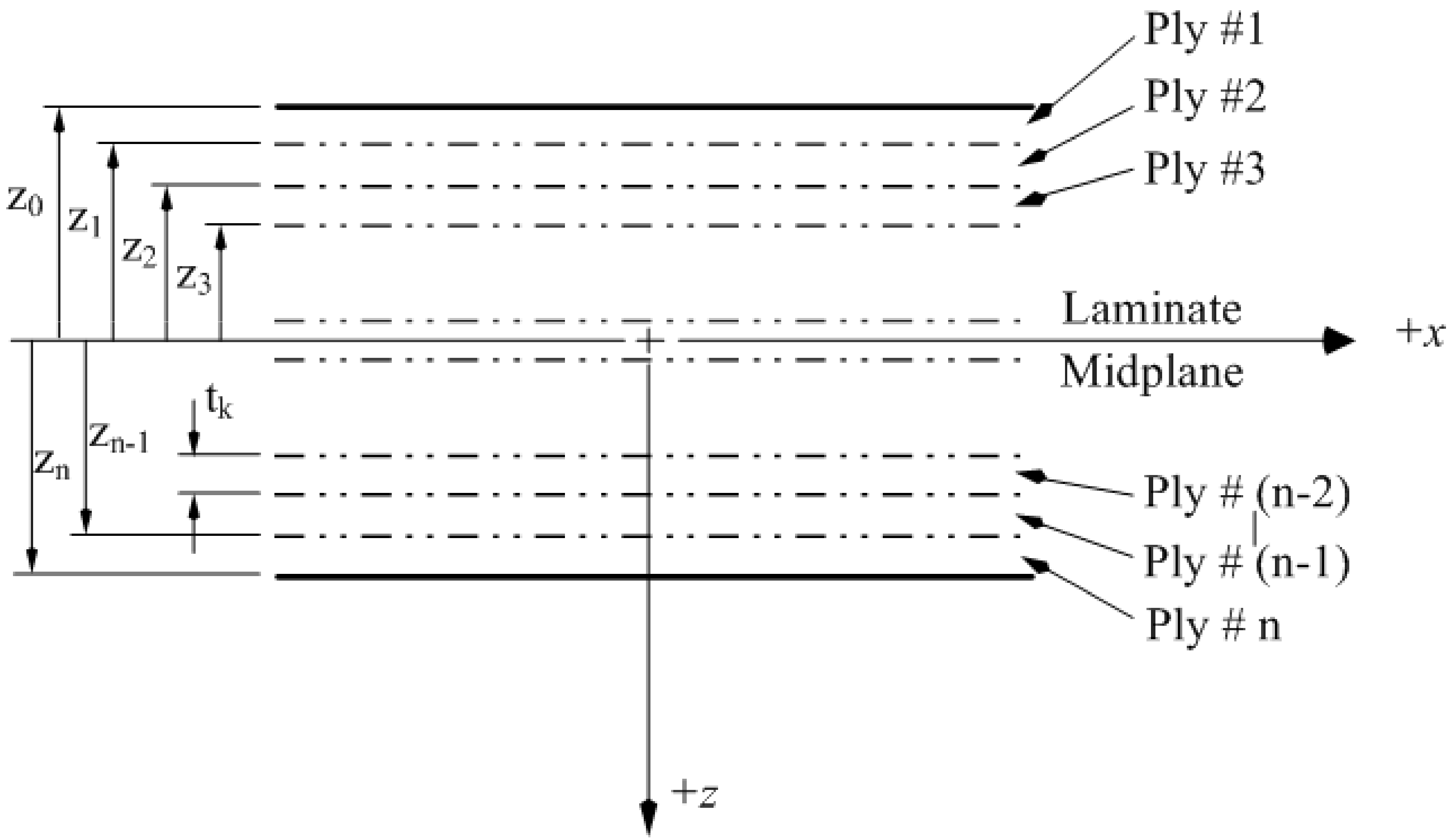




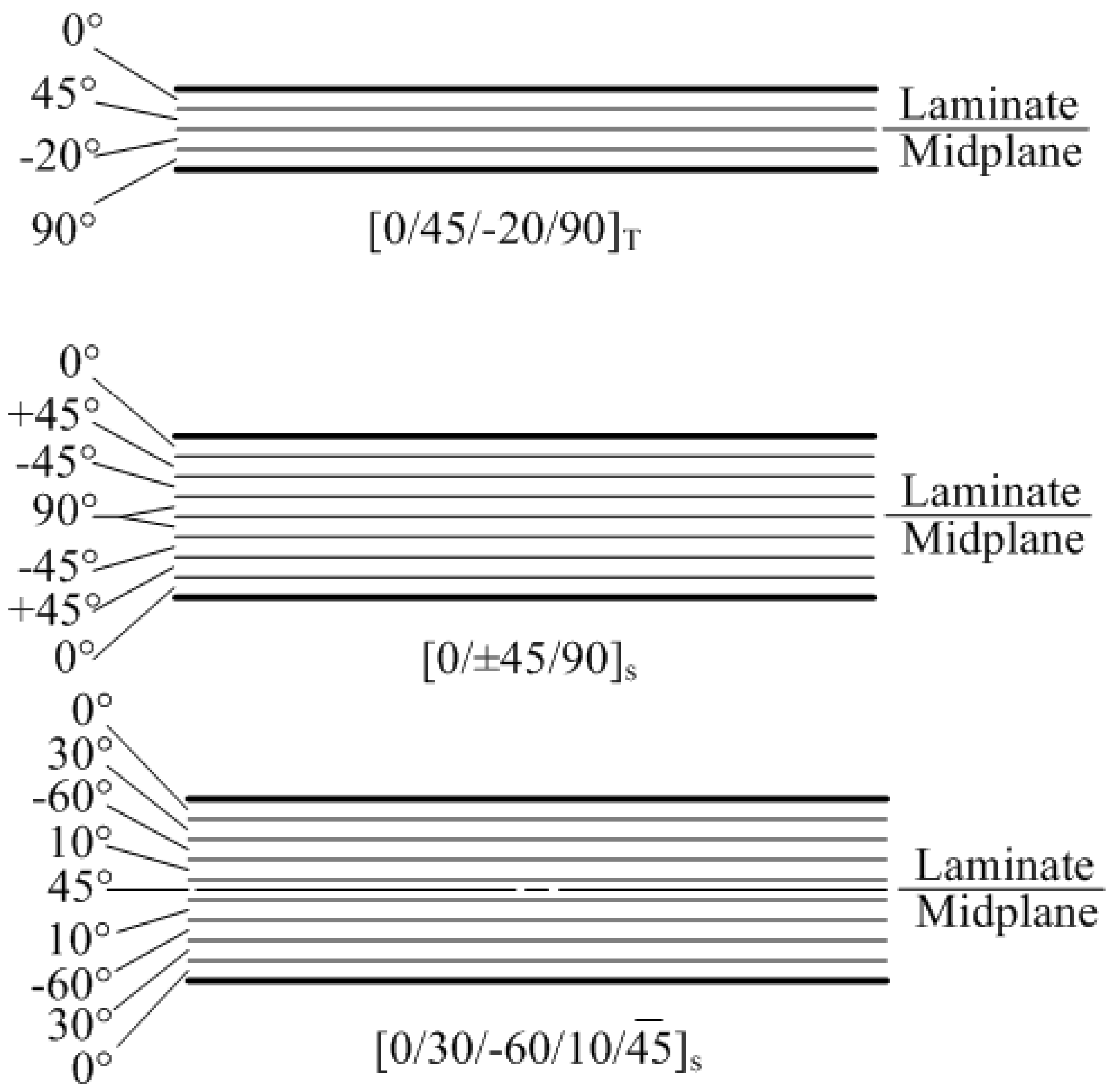
Multi-ply, Multi-directional Composite Laminates



Defining Ply Interface Positions



Describing Stacking Sequences



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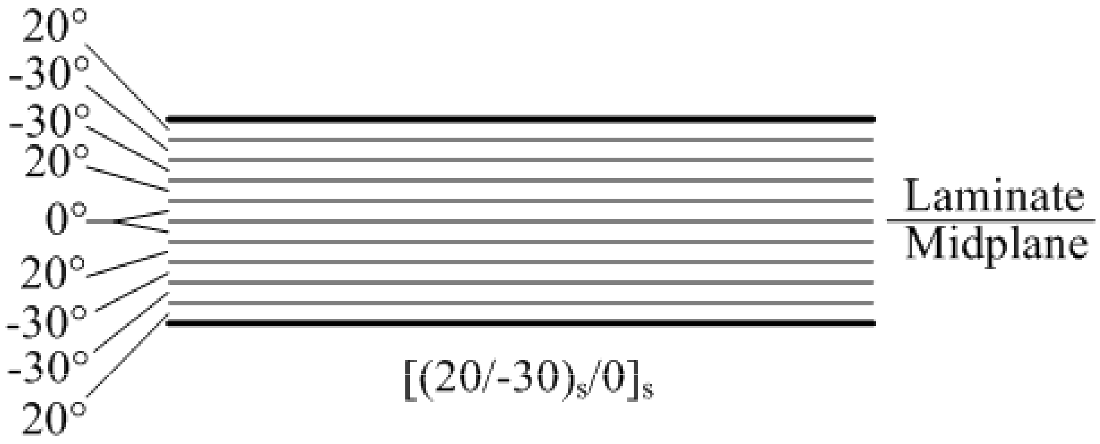
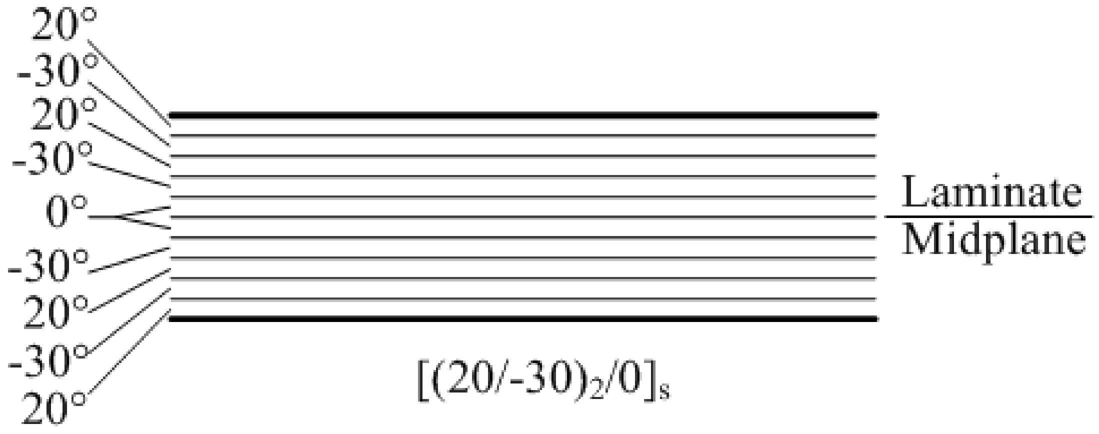




Kirchhoff Hypothesis

"a straight line which is initially perpendicular to the midplane of a thin plate remains straight and perpendicular to the midplane after deformation"

- Ultimately allows us to calculate the strain at any through-thickness position z:

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{xx}^{o} \\ \varepsilon_{yy}^{o} \\ \gamma_{xy}^{o} \end{Bmatrix} + z \begin{Bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{Bmatrix}$$

where:

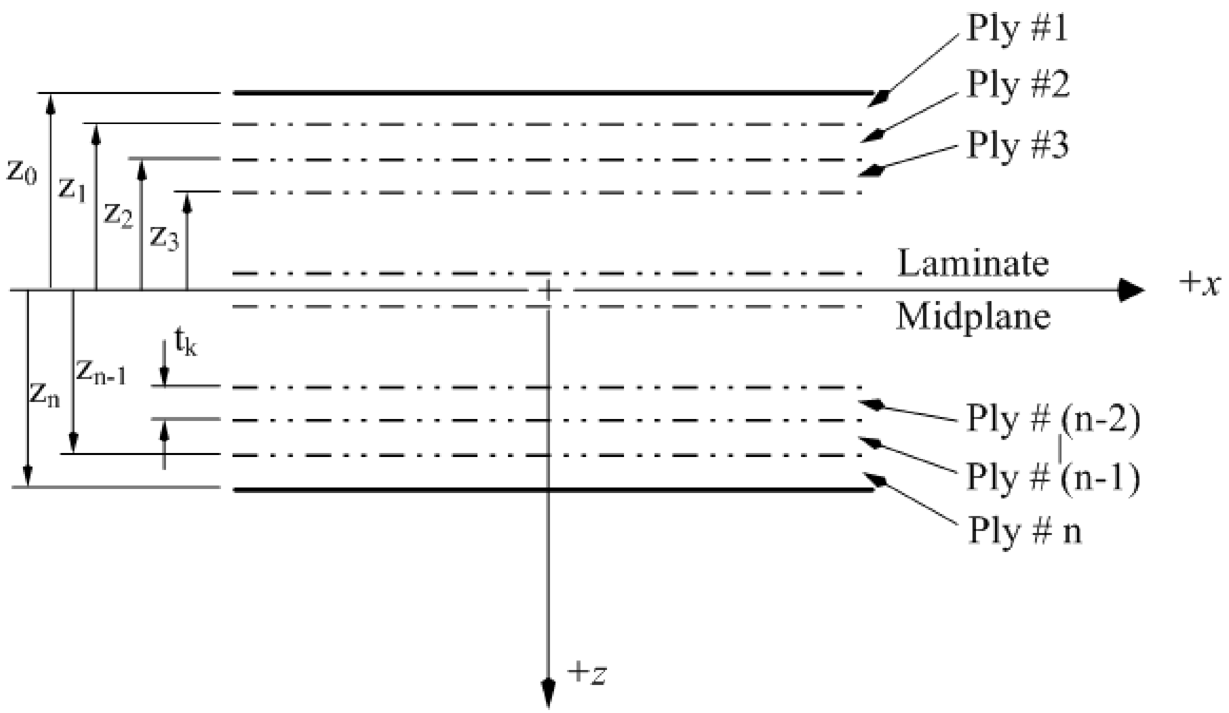
$$\varepsilon_{xx}^{o}, \varepsilon_{yy}^{o}, \gamma_{xy}^{o} = \text{midplane strains}$$

$$\kappa_{xx}, \kappa_{yy}, \kappa_{xy} = \text{midplane curvatures}$$

Ply Strains

- Strains at ply interfaces are usually of greatest interest:

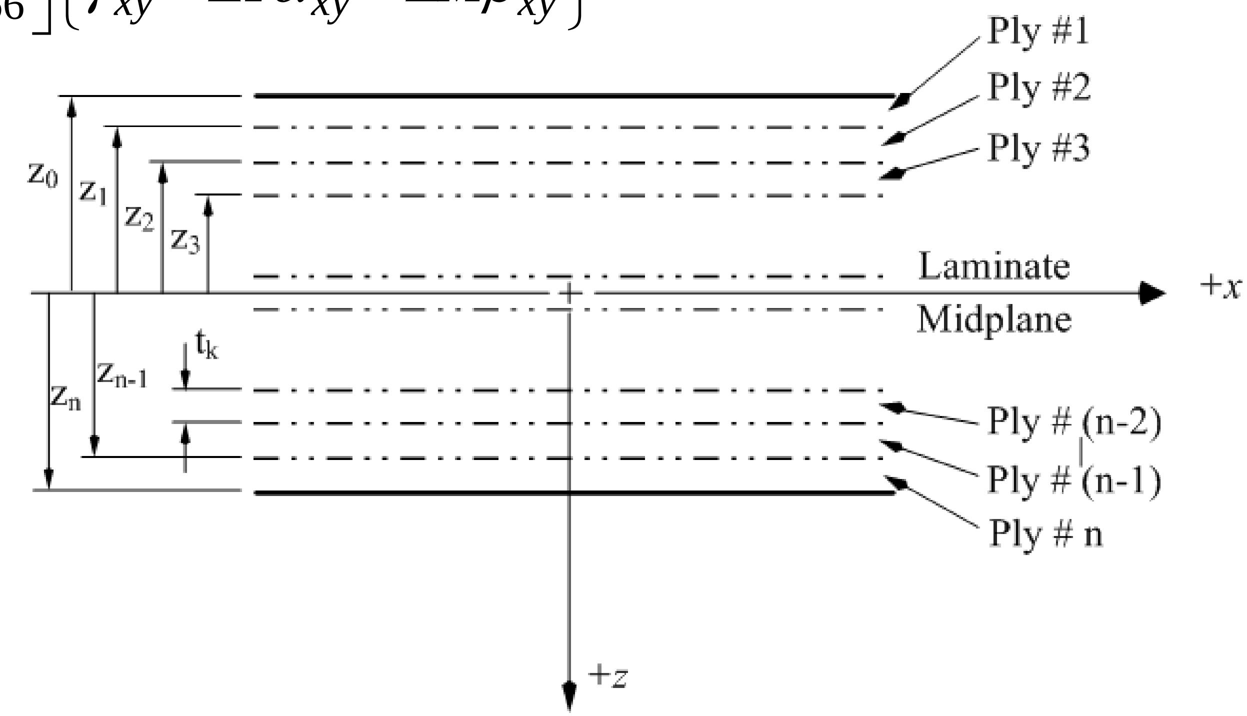
$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon^{o}_{xx} \\ \varepsilon^{o}_{yy} \\ \gamma^{o}_{xy} \end{Bmatrix} + z \begin{Bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{Bmatrix}$$



Ply Stresses

- Ply stresses can be calculated using Hooke's law:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} - \Delta T \alpha_{xx} - \Delta M \beta_{xx} \\ \varepsilon_{yy} - \Delta T \alpha_{yy} - \Delta M \beta_{yy} \\ \gamma_{xy} - \Delta T \alpha_{xy} - \Delta M \beta_{xy} \end{Bmatrix}$$



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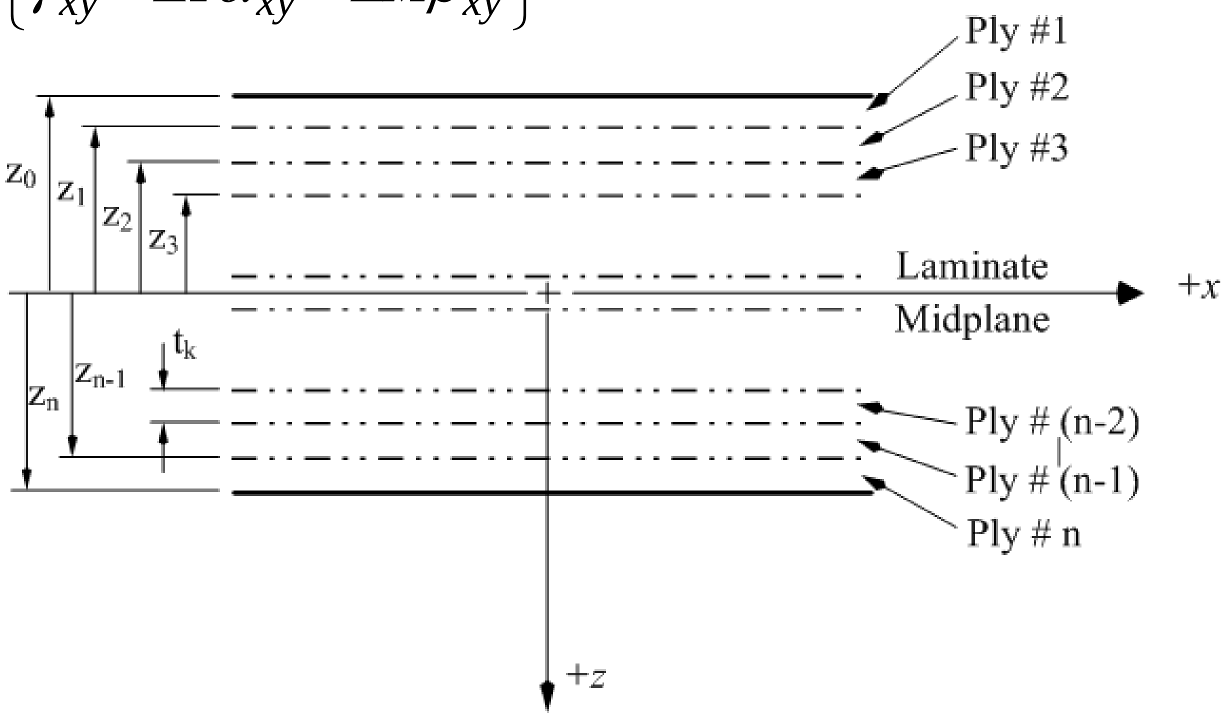
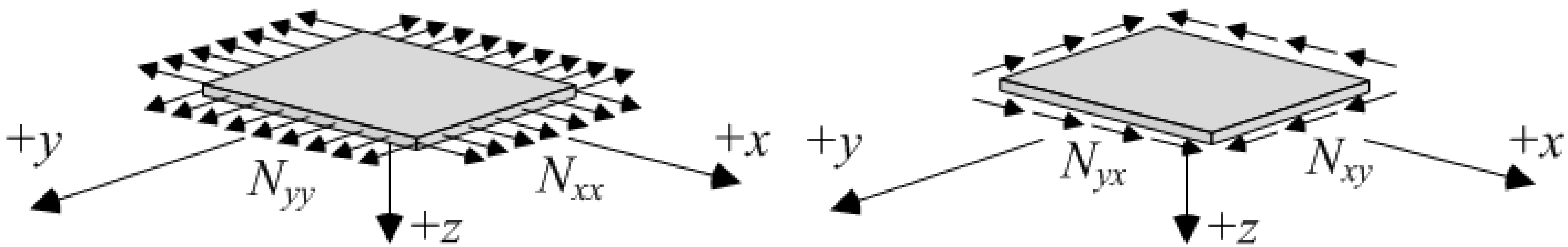


Laminate Loading

- Loads considered are restricted to those that lead to plane stress conditions…two types:
  - Stress resultants, N_{xx}, N_{yy} , and N_{xy}….units = force/length



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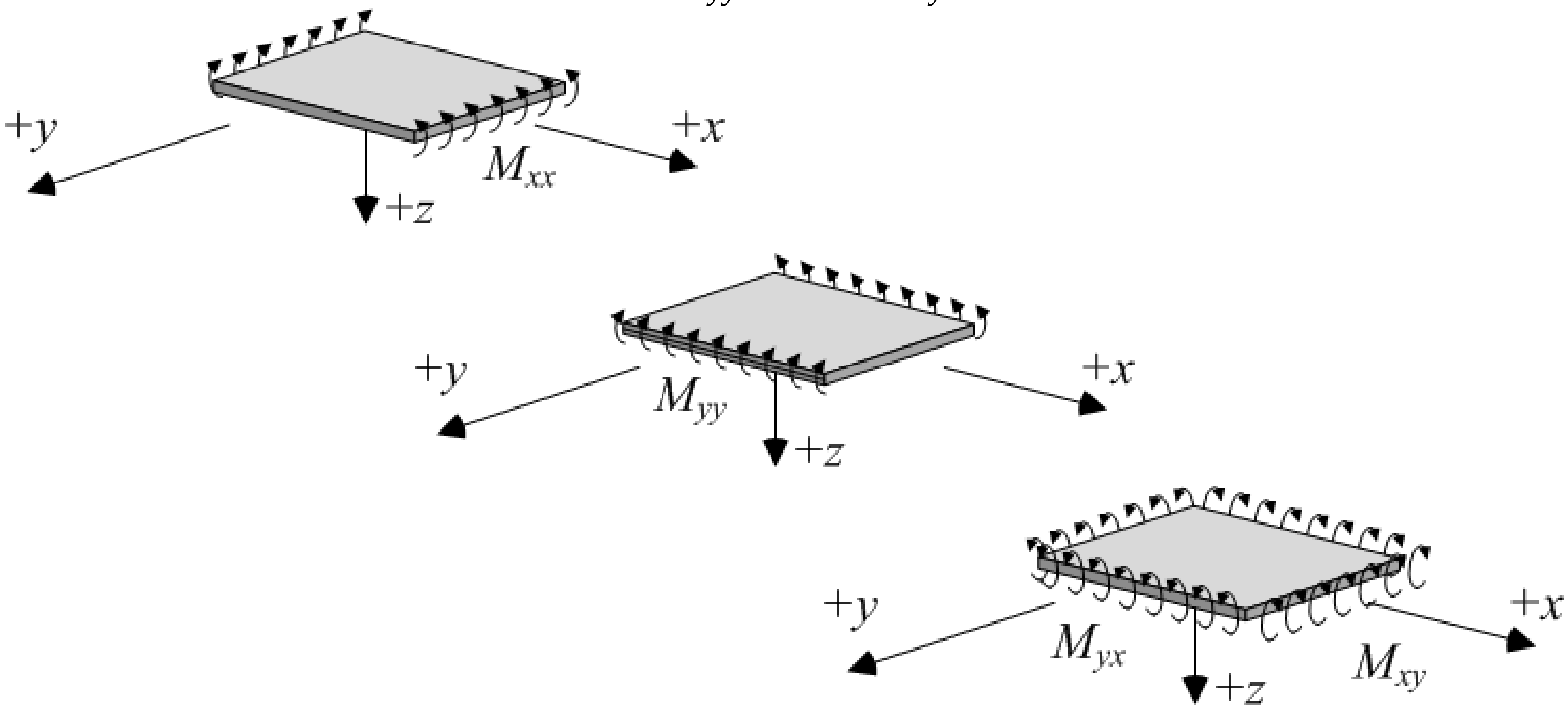

Laminate Loading

- It can be shown:

$$N_{xx} = \int_{-t/2}^{t/2} \sigma_{xx}\, dz$$

- Hooke's law:

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- Substituting for σ_{xx} and integrating in piece-wise fashion:

$$N_{xx} = A_{11}\varepsilon_{xx}^{o} + A_{12}\varepsilon_{yy}^{o} + A_{16}\gamma_{xy}^{o} + B_{11}\kappa_{xx} + B_{12}\kappa_{yy} + B_{16}\kappa_{xy} - N_{xx}^{T} - N_{xx}^{M}$$

Laminate Loading

- Where:

$$A_{11} = \left\{ \left(\overline{Q}_{11}\right)_1 [z_1 - z_0] + \left(\overline{Q}_{11}\right)_2 [z_2 - z_1] + \left(\overline{Q}_{11}\right)_3 [z_3 - z_2] + \ldots. + \left(\overline{Q}_{11}\right)_n [z_n - z_{n-1}] \right\}$$

$$A_{12} = \left\{ \left(\overline{Q}_{12}\right)_1 [z_1 - z_0] + \left(\overline{Q}_{12}\right)_2 [z_2 - z_1] + \left(\overline{Q}_{12}\right)_3 [z_3 - z_2] + \ldots. + \left(\overline{Q}_{12}\right)_n [z_n - z_{n-1}] \right\}$$

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$$B_{11} = \frac{1}{2} \left\{ \left(\overline{Q}_{11}\right)_1 [z_1^2 - z_0^2] + \left(\overline{Q}_{11}\right)_2 [z_2^2 - z_1^2] + \left(\overline{Q}_{11}\right)_3 [z_3^2 - z_2^2] + \ldots. + \left(\overline{Q}_{11}\right)_n [z_n^2 - z_{n-1}^2] \right\}$$

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$$N_{xx}^T \equiv \Delta T \sum_{k=1}^{n} \left\{ \left[\overline{Q}_{11}\alpha_{xx} + \overline{Q}_{12}\alpha_{yy} + \overline{Q}_{16}\alpha_{xy} \right]_k [z_k - z_{k-1}] \right\}$$

$$N_{xx}^M \equiv \Delta M \sum_{k=1}^{n} \left\{ \left[\overline{Q}_{11}\beta_{xx} + \overline{Q}_{12}\beta_{yy} + \overline{Q}_{16}\beta_{xy} \right]_k [z_k - z_{k-1}] \right\}$$

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Laminate Loading

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$$D_{11} = \frac{1}{3}\left\{\left(\overline{Q}_{11}\right)_1[z_1^3 - z_0^3] + \left(\overline{Q}_{11}\right)_2[z_2^3 - z_1^3] + \left(\overline{Q}_{11}\right)_3[z_3^3 - z_2^3] + \ldots + \left(\overline{Q}_{11}\right)_n[z_n^3 - z_{n-1}^3]\right\}$$

$$D_{12} = \frac{1}{3}\left\{\left(\overline{Q}_{12}\right)_1[z_1^3 - z_0^3] + \left(\overline{Q}_{12}\right)_2[z_2^3 - z_1^3] + \left(\overline{Q}_{12}\right)_3[z_3^3 - z_2^3] + \ldots + \left(\overline{Q}_{12}\right)_n[z_n^3 - z_{n-1}^3]\right\}$$

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$$M_{xx}^{T} \equiv \frac{\Delta T}{2}\sum_{k=1}^{n}\left\{\left[\overline{Q}_{11}\alpha_{xx} + \overline{Q}_{12}\alpha_{yy} + \overline{Q}_{16}\alpha_{xy}\right]_k \left[z_k^2 - z_{k-1}^2\right]\right\}$$

$$M_{xx}^{M} \equiv \frac{\Delta M}{2}\sum_{k=1}^{n}\left\{\left[\overline{Q}_{11}\beta_{xx} + \overline{Q}_{12}\beta_{yy} + \overline{Q}_{16}\beta_{xy}\right]_k \left[z_k^2 - z_{k-1}^2\right]\right\}$$

Laminate Loading

- Process repeated for all stress and moment resultants, finally resulting in:

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \\ M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{o} \\ \varepsilon_{yy}^{o} \\ \gamma_{xy}^{o} \\ \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{Bmatrix} - \begin{Bmatrix} N_{xx}^{T} \\ N_{yy}^{T} \\ N_{xy}^{T} \\ M_{xx}^{T} \\ M_{yy}^{T} \\ M_{xy}^{T} \end{Bmatrix} - \begin{Bmatrix} N_{xx}^{M} \\ N_{yy}^{M} \\ N_{xy}^{M} \\ M_{xx}^{M} \\ M_{yy}^{M} \\ M_{xy}^{M} \end{Bmatrix}$$

Laminate Loading

- Inverting:

$$\begin{Bmatrix} \varepsilon_{xx}^{o} \\ \varepsilon_{yy}^{o} \\ \gamma_{xy}^{o} \\ \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{16} & b_{11} & b_{12} & b_{16} \\ a_{12} & a_{22} & a_{26} & b_{21} & b_{22} & b_{26} \\ a_{16} & a_{26} & a_{66} & b_{61} & b_{62} & b_{66} \\ b_{11} & b_{21} & b_{61} & d_{11} & d_{12} & d_{16} \\ b_{12} & b_{22} & b_{62} & d_{12} & d_{22} & d_{26} \\ b_{16} & b_{26} & b_{66} & d_{16} & d_{26} & d_{66} \end{bmatrix} \begin{Bmatrix} N_{xx} + N_{xx}^{T} + N_{xx}^{M} \\ N_{yy} + N_{yy}^{T} + N_{yy}^{M} \\ N_{xy} + N_{xy}^{T} + N_{xy}^{M} \\ M_{xx} + M_{xx}^{T} + M_{xx}^{M} \\ M_{yy} + M_{yy}^{T} + M_{yy}^{M} \\ M_{xy} + M_{xy}^{T} + M_{xy}^{M} \end{Bmatrix}$$

Simplifications Due to Stacking Sequence

- Various terms within the $[ABD]$ and $[abd]$ matrices are always zero for certain stacking sequences (see Section 6.7)
- The most important simplification occurs for *symmetric* laminates…in this case:

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \\ M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & 0 & 0 & 0 \\ A_{12} & A_{22} & A_{26} & 0 & 0 & 0 \\ A_{16} & A_{26} & A_{66} & 0 & 0 & 0 \\ 0 & 0 & 0 & D_{11} & D_{12} & D_{16} \\ 0 & 0 & 0 & D_{12} & D_{22} & D_{26} \\ 0 & 0 & 0 & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon^o_{xx} \\ \varepsilon^o_{yy} \\ \gamma^o_{xy} \\ \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{Bmatrix} - \begin{Bmatrix} N^T_{xx} \\ N^T_{yy} \\ N^T_{xy} \\ 0 \\ 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} N^M_{xx} \\ N^M_{yy} \\ N^M_{xy} \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Simplifications Due to Stacking Sequence

- Various terms within the $[ABD]$ and $[abd]$ matrices are always zero for certain stacking sequences (see Section 6.7)
- The most important simplification occurs for *symmetric* laminates…in this case:

$$\begin{Bmatrix} \varepsilon_{xx}^{o} \\ \varepsilon_{yy}^{o} \\ \gamma_{xy}^{o} \\ \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{16} & 0 & 0 & 0 \\ a_{12} & a_{22} & a_{26} & 0 & 0 & 0 \\ a_{16} & a_{26} & a_{66} & 0 & 0 & 0 \\ 0 & 0 & 0 & d_{11} & d_{12} & d_{16} \\ 0 & 0 & 0 & d_{12} & d_{22} & d_{26} \\ 0 & 0 & 0 & d_{16} & d_{26} & d_{66} \end{bmatrix} \begin{Bmatrix} N_{xx} + N_{xx}^{T} + N_{xx}^{M} \\ N_{yy} + N_{yy}^{T} + N_{yy}^{M} \\ N_{xy} + N_{xy}^{T} + N_{xy}^{M} \\ M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix}$$

Effective Laminate Properties

- Effective elastic properties of a laminate can be determined using elements of the $[a_{ij}]$ an $[d_{ij}]$ matrices

- Extensional (in-plane):

$$\overline{E}^{ex}_{xx} = \frac{1}{ta_{11}} \qquad \overline{\nu}^{ex}_{xy} = \frac{-a_{12}}{a_{11}} \qquad \overline{\eta}^{ex}_{xx,xy} = \frac{a_{16}}{a_{11}}$$

$$\overline{E}^{ex}_{yy} = \frac{1}{ta_{22}} \qquad \overline{\nu}^{ex}_{yx} = \frac{-a_{12}}{a_{22}} \qquad \overline{\eta}^{ex}_{yy,xy} = \frac{a_{26}}{a_{22}}$$

$$\overline{G}_{xy} = \frac{1}{ta_{66}} \qquad \overline{\eta}_{xy,xx} = \frac{a_{16}}{a_{66}} \qquad \overline{\eta}_{xy,yy} = \frac{a_{26}}{a_{66}}$$

Effective Laminate Properties

- Effective elastic properties of a laminate can be determined using elements of the $[a_{ij}]$ an $[d_{ij}]$ matrices

- Flexural (bending):

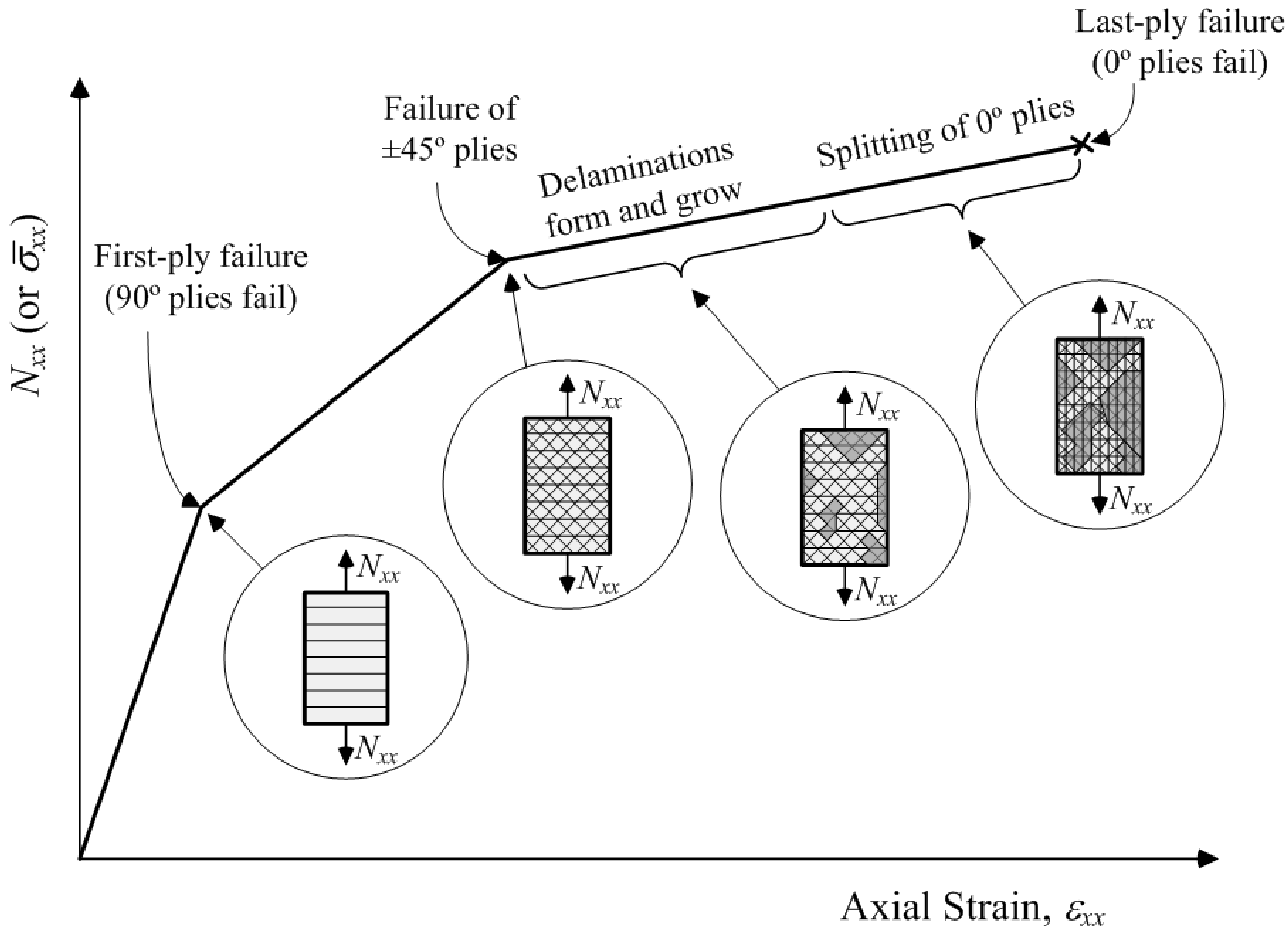
$$\overline{E}_{xx}^{fl} = \frac{12}{t^3 d_{11}} \qquad \overline{\nu}_{xy}^{fl} = \frac{-d_{12}}{d_{11}} \qquad \overline{\eta}_{xx,xy}^{fl} = \frac{d_{16}}{d_{11}}$$

$$\overline{E}_{yy}^{fl} = \frac{12}{t^3 d_{22}} \qquad \overline{\nu}_{yx}^{fl} = \frac{-d_{12}}{d_{22}} \qquad \overline{\eta}_{yy,xy}^{fl} = \frac{d_{26}}{d_{22}}$$

Program CLT

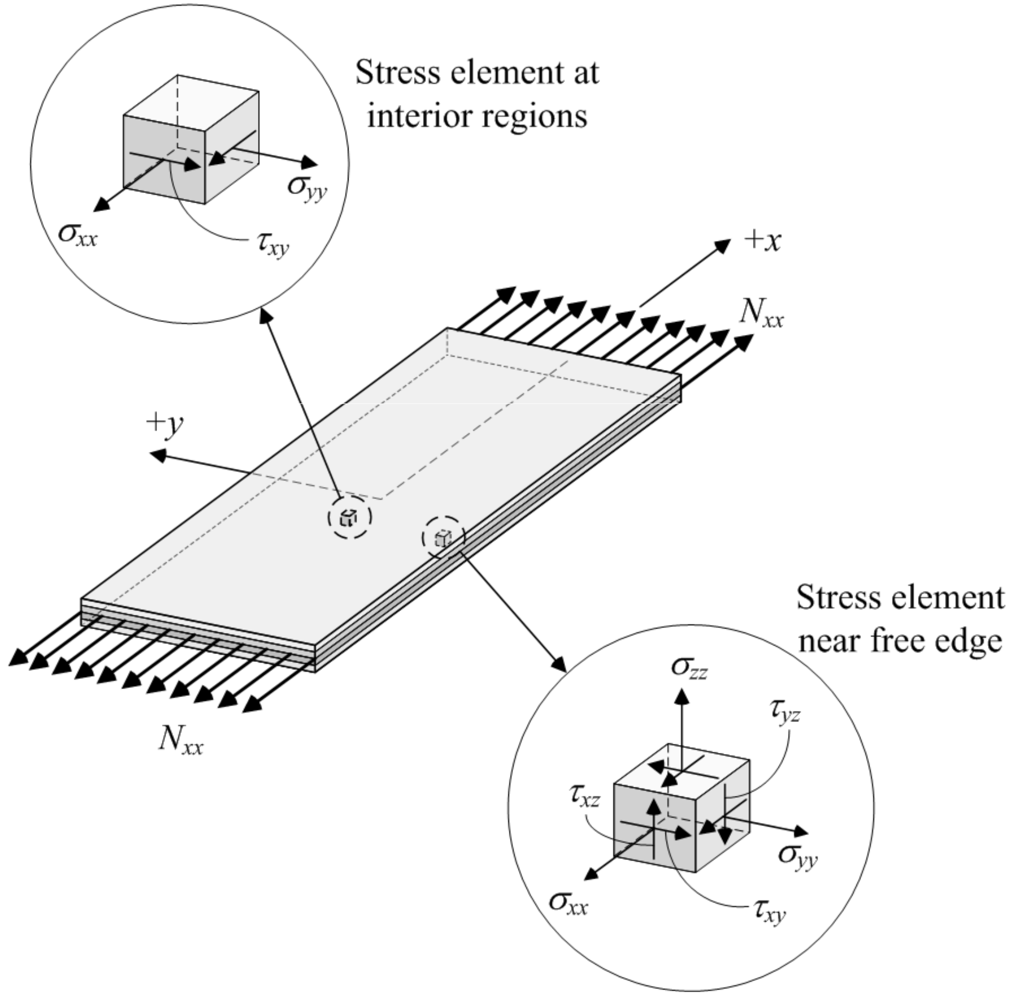
- Most of the topics included in this review are implemented in the program CLT (*C*lassical *L*amination *T*heory)

Laminate Damage Progression and Failure



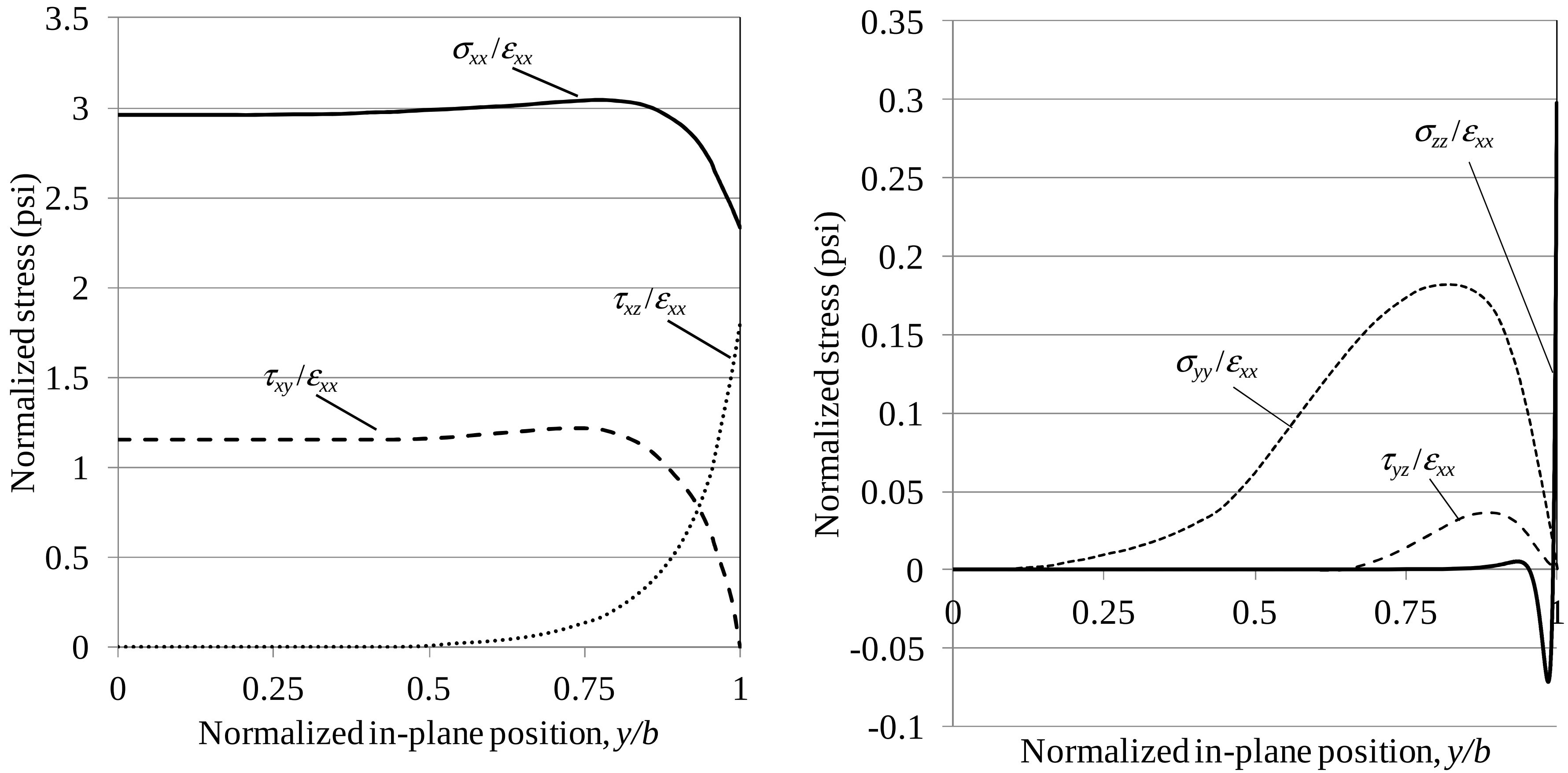
3-D Stress-State Exists Near a Free-edge and Complicate Failure Predictions

(see Section 6.13)

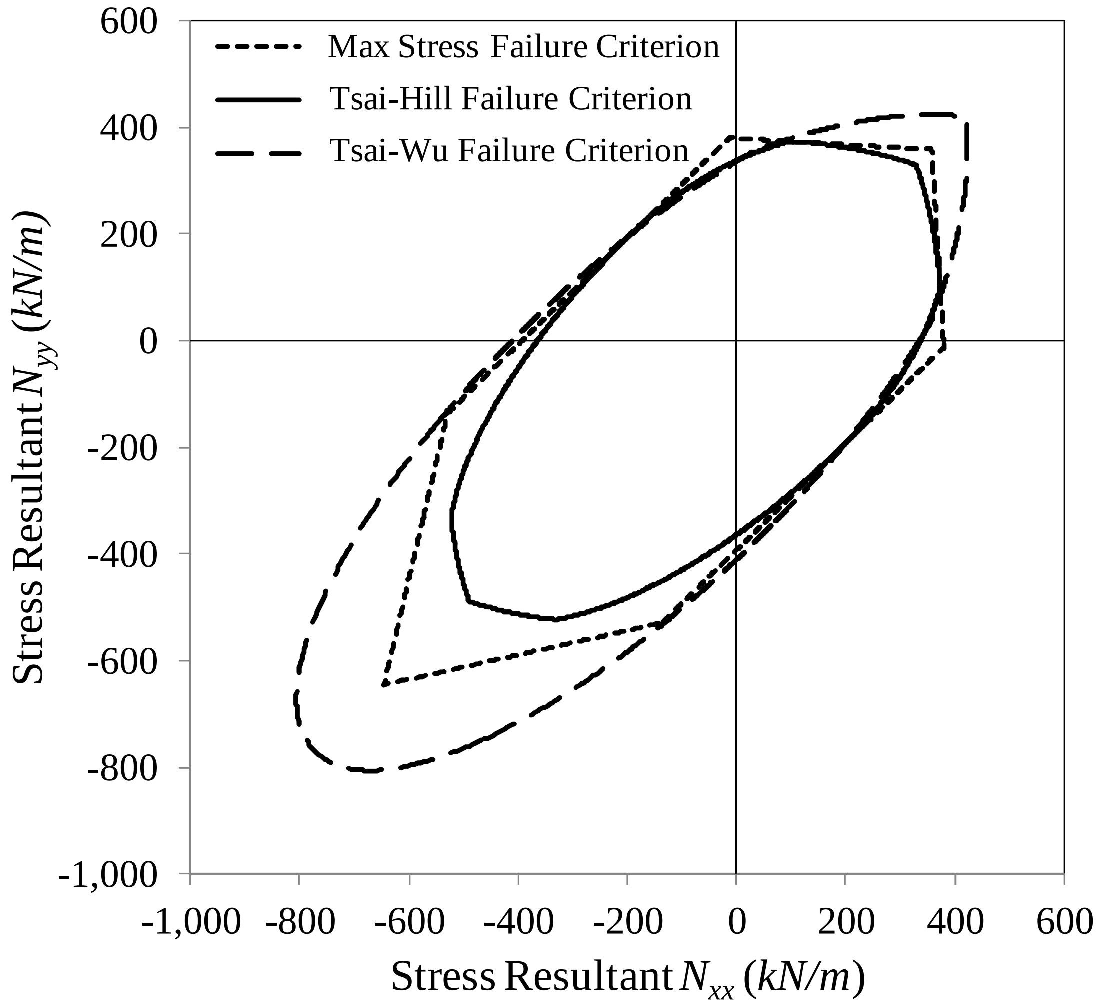


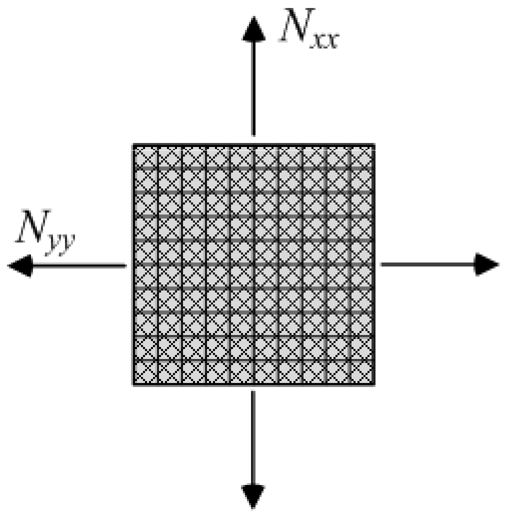
3-D Stress-State Exists Near a Free-edge and Complicate Failure Predictions *(see Section 6.13)*

(Typical results for a $[45/-45]_s$ laminate subject to uniaxial loading)

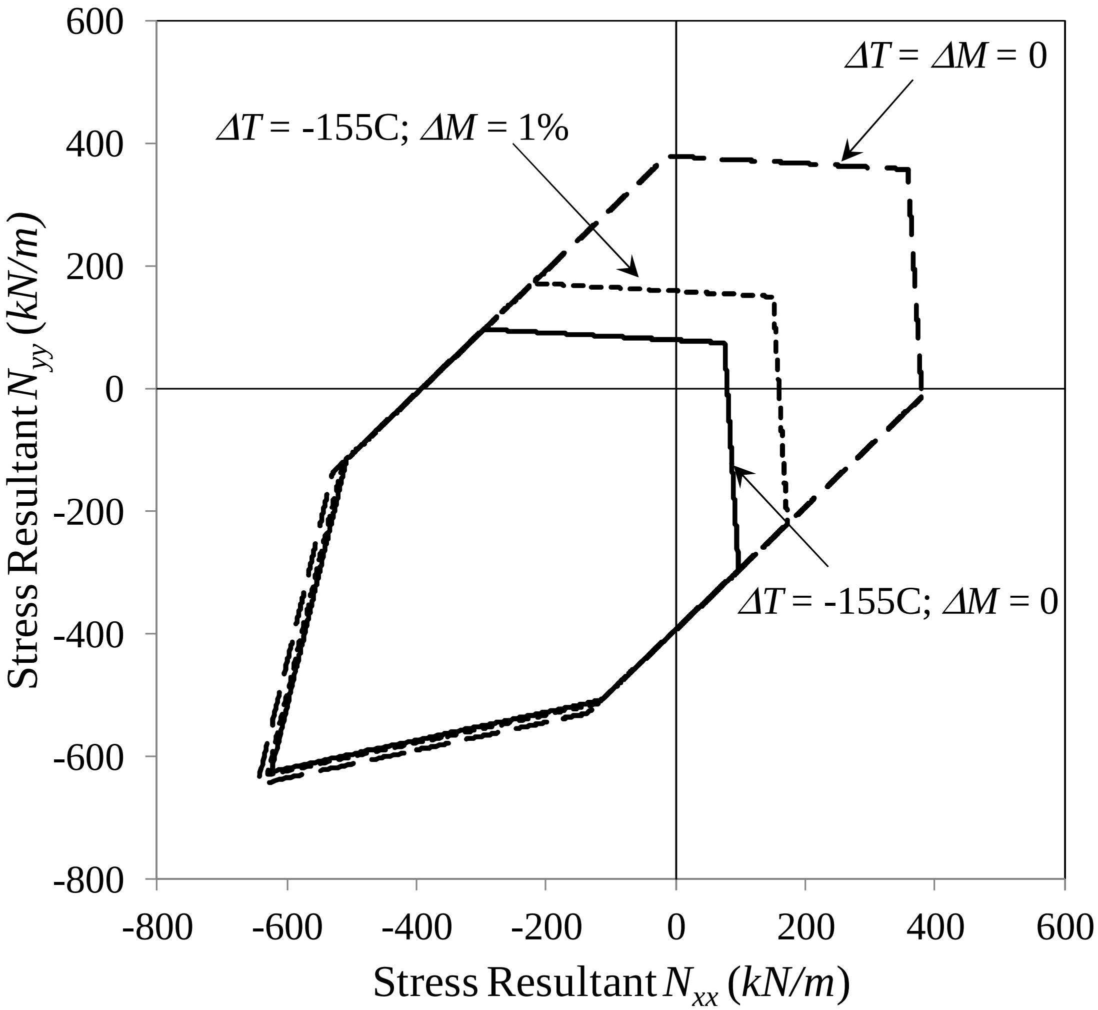


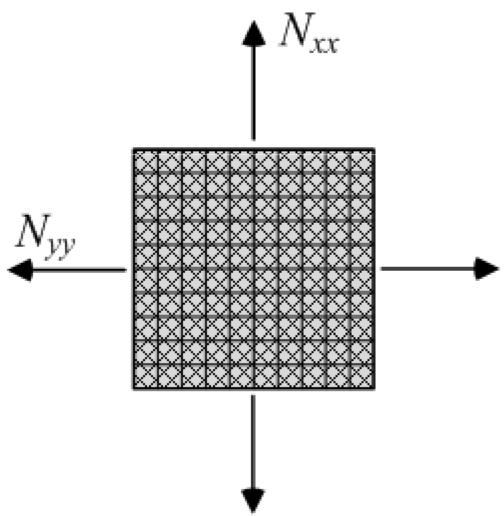
First-ply Failure Loads and/or First-ply Failure Envelopes can be Predicted Using Program LAMFAIL



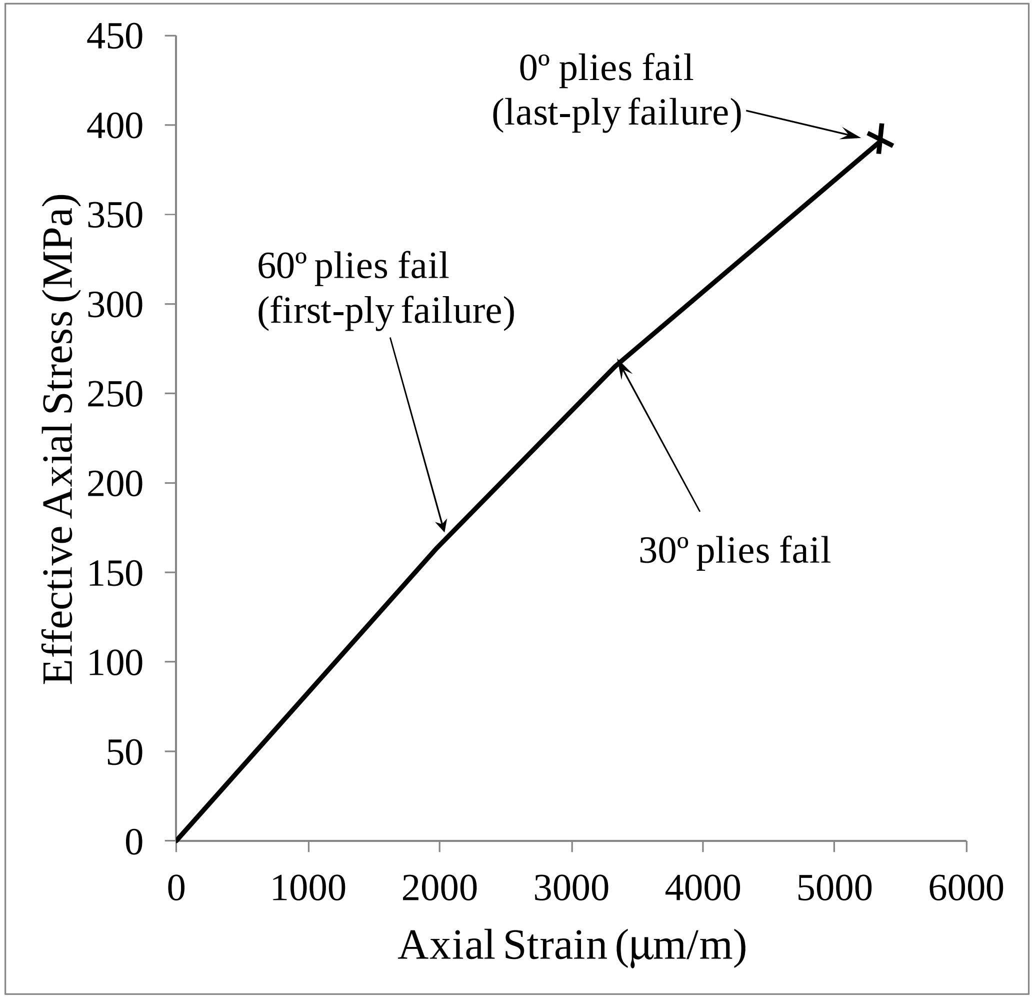


Environmental Effects Dramatically Effect Predicted First-ply Failure Loads





Last-Ply Failure Loads Can be Predicted Using the Ply-Discount Scheme (Program LAMFAIL)



Predicted stress-strain curve for a $[0/30/60]_s$ laminate