3 KINEMATICS OF CONTINUA

The man who cannot occasionally imagine events and conditions of existence that are contrary to the causal principle as he knows it will never enrich his science by the addition of a new idea.
—— Max Planck (1858–1947)

It is through science that we prove, but through intuition that we discover.
—— Henri Poincaré (1854–1912)

3.1 Introduction

Material or matter is composed of discrete molecules, which in turn are made up of atoms. An atom consists of negatively charged electrons, positively charged protons, and neutrons. Electrons form chemical bonds. The study of matter at molecular or atomistic levels is very useful for understanding a variety of phenomena, but studies at these scales are not useful to solve common engineering problems. *Continuum mechanics* is concerned with a study of various forms of matter at the macroscopic level. Central to this study is the assumption that the discrete nature of matter can be overlooked, provided the length scales of interest are large compared to the length scales of discrete molecular structures. Thus, matter at sufficiently large length scales can be treated as a *continuum*,[1] in which all physical quantities of interest, including density, are continuously differentiable almost everywhere.

Engineers and scientists undertake the study of continuous systems to understand their behavior under "working conditions," so that the systems can be designed to function properly and to be produced economically. For example, if we were to repair or replace a damaged artery in the human body, we must understand the function of the original artery and the conditions that led to its damage. An artery carries blood from the heart to different parts of the body. Conditions such as high blood pressure and increase in cholesterol levels in the blood may lead to deposition of particles in the arterial wall, as shown in Fig. 3.1.1. With time, accumulation of these particles in the arterial wall hardens and constricts the passage, leading to cardiovascular diseases. A possible remedy for such diseases is to repair or replace the damaged portion of the artery. This in turn requires an understanding of the deformations and stresses caused in the arterial wall by the blood flow. The understanding is then used to design a vascular prosthesis (that is, artificial artery).

[1]We mean a differentiable manifold with a boundary. Inherent in this assumption is that material particles that are neighbors will remain neighbors during the motion.

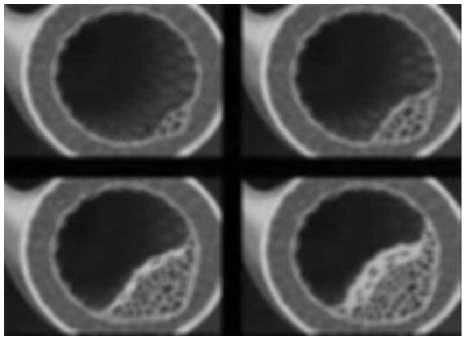


Fig. 3.1.1: Progressive damage of an artery due to deposition of particles in the arterial wall.

The present chapter is devoted to the study of geometric changes in a continuous medium (such as the artery) that is in equilibrium. The study of geometric changes in a continuum without regard to the forces causing the changes is known as *kinematics*. Sections or subsections that are considered to be too advanced for a first course can be skipped without loss of continuity (or returned to when needed).

3.2 Descriptions of Motion

3.2.1 Configurations of a Continuous Medium

Consider a body $\mathcal{B}$ of known geometry in a three-dimensional Euclidean space $\Re^3$; $\mathcal{B}$ may be viewed as a set of particles, each particle representing a large collection of molecules with a continuous distribution of matter in space and time. An example of a body $\mathcal{B}$ is a diving board. Under external stimuli, body $\mathcal{B}$ will undergo macroscopic geometric changes, which are termed *deformations*. The geometric changes are accompanied by stresses that are induced in the body. If the applied loads are time dependent, the deformation of the body will be a function of time; that is, the geometry of the body $\mathcal{B}$ will change with time. If the loads are applied slowly so that the deformation is dependent only on the loads, the body will occupy a sequence of geometrical regions. The region occupied by the continuum at a given time t is termed a *configuration* and denoted by κ. Thus, the positions occupied in space $\Re^3$ by all material points of the continuum $\mathcal{B}$ at different instants of time are called configurations.

Suppose that the continuum initially occupies a configuration κ_0, in which a particle X occupies position $\mathbf{X}$, referred to a *reference frame* of right-handed, rectangular Cartesian axes (X_1, X_2, X_3) at a fixed origin O with orthonormal basis vectors $\hat{\mathbf{E}}_i$, as shown in Fig. 3.2.1. Note that X (lightface roman letter) is the name of the particle that occupies location $\mathbf{X}$ (boldface letter) in configuration κ_0, and therefore (X_1, X_2, X_3) are called the *material coordinates*. After the application of some external stimuli (e.g., loads), the continuum changes its geometric shape and thus assumes a new configuration κ, called the *current* or *deformed configuration*. Particle X now occupies position $\mathbf{x}$ in the deformed configuration κ, as shown in Fig. 3.2.1. The mapping $\boldsymbol{\chi} : \mathcal{B}_{\kappa_0} \rightarrow \mathcal{B}_\kappa$ is called the *deformation mapping* of the body $\mathcal{B}$ from κ_0 to κ. The deformation mapping $\boldsymbol{\chi}(\mathbf{X}, t)$ takes the position vector $\mathbf{X}$ from the reference configuration and places the same point in the deformed configuration as $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$. The inverse mapping $\boldsymbol{\chi}^{-1} : \mathcal{B}_\kappa \rightarrow \mathcal{B}_{\kappa_0}$ takes the position vector $\mathbf{x}$ from the deformed configuration κ back to the reference configuration κ_0, $\mathbf{X} = \boldsymbol{\chi}^{-1}(\mathbf{x}, t)$. It is not always possible to construct the inverse mapping $\boldsymbol{\chi}^{-1}(\mathbf{x}, t)$ from a known deformation mapping $\boldsymbol{\chi}(\mathbf{X}, t)$. In the following discussion, we shall use $\boldsymbol{\chi}(\mathbf{X}, t)$ to denote the deformation mapping and $\mathbf{x}$ to denote the value of $\boldsymbol{\chi}(\mathbf{X}, t)$.

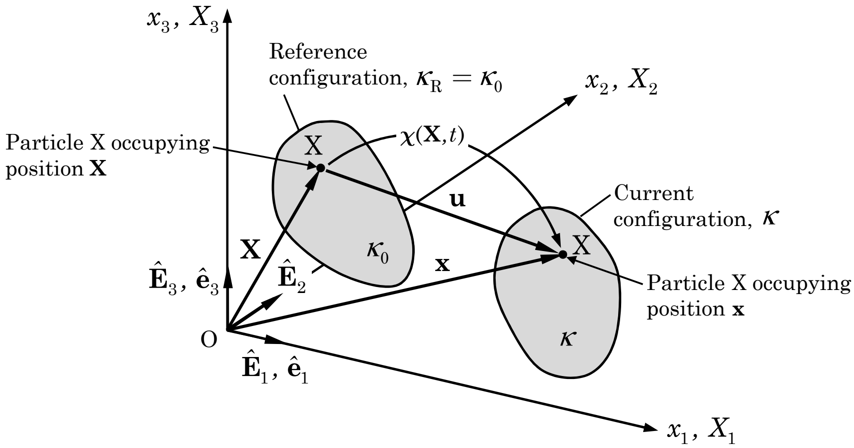


Fig. 3.2.1: Reference and deformed configurations of a body.

A nonrotating frame of reference is chosen, explicitly or implicitly, to describe the deformation. A frame of reference is a coordinate system with respect to which a configuration is described (or measured). We shall use the same coordinate system to describe reference and current configurations. The components X_i and x_i of vectors $\mathbf{X} = X_i \hat{\mathbf{E}}_i$ and $\mathbf{x} = x_i \hat{\mathbf{e}}_i$ are along the coordinates used, with the origins of the basis vectors $\hat{\mathbf{E}}_i$ and $\hat{\mathbf{e}}_i$ being the same.

The mathematical description of the deformation of a continuous body follows one of two approaches: (1) the material description or (2) the spatial description. The material description is also known as the *Lagrangian description*, and the spatial description is known as the *Eulerian description.* These descriptions are discussed next.

3.2.2 Material Description

In the material description, the motion of the body is referred to a reference configuration κ_R, which is often chosen to be the initial configuration[2], $\kappa_R = \kappa_0$, although any other known configuration can serve as a reference configuration. Thus, in the Lagrangian description, the current coordinates $\mathbf{x} \in \kappa$ are expressed in terms of the reference coordinates $\mathbf{X} \in \kappa_0$:

[2]Typically, the initial configuration is one without any stimuli and hence undeformed.

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t), \quad \mathbf{X} = \boldsymbol{\chi}(\mathbf{X}, 0), \tag{3.2.1}$$

and the variation of a typical variable ϕ over the body is described with respect to the material coordinates $\mathbf{X}$ and time t:

$$\phi = \phi(\mathbf{x}(\mathbf{X}), t) = \phi(\mathbf{X}, t). \tag{3.2.2}$$

For a fixed value of $\mathbf{X} \in \kappa_0$, $\phi(\mathbf{X}, t)$ gives the value of ϕ at time t associated with the fixed material particle X whose position in the reference configuration is $\mathbf{X}$, as shown in Fig. 3.2.2. Thus, a change in time t implies that the *same* material particle X, occupying position $\mathbf{X}$ in κ_0, has a different value ϕ. Figure 3.2.3 shows the deformation of a *fixed material volume* with time. Thus the attention is focused on the fixed material of the continuum.

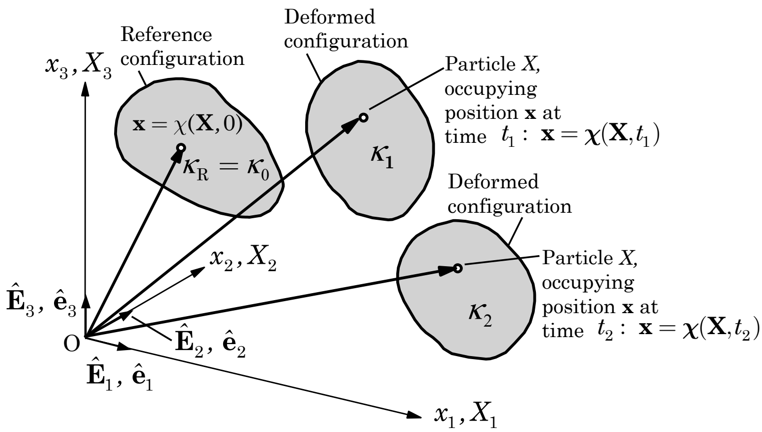


Fig. 3.2.2: Reference and deformed configurations in the material description.

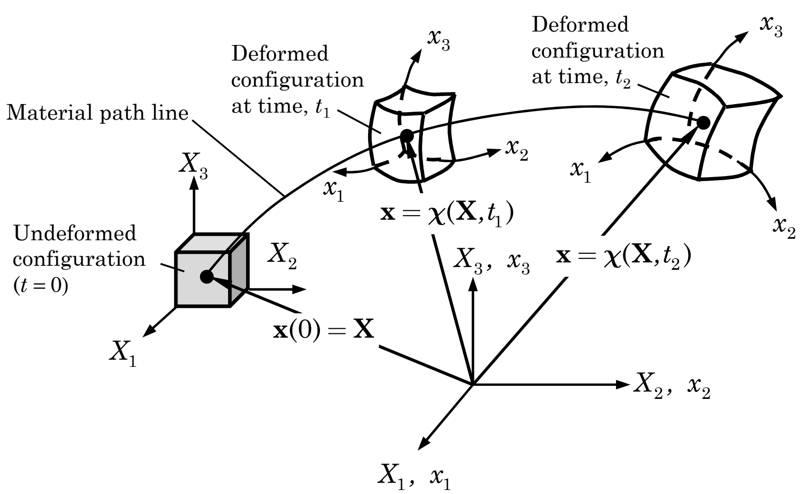


Fig. 3.2.3: Deformation of *fixed material volume* with time.

3.2.3 Spatial Description

In the spatial description, the motion is referred to the current configuration κ occupied by the body $\mathcal{B}$, and ϕ is described with respect to the current position $\mathbf{x} \in \kappa$ in space, currently occupied by material particle X:

$$\phi = \phi(\mathbf{x}, t), \quad \mathbf{X} = \mathbf{X}(\mathbf{x}, t) = \boldsymbol{\chi}^{-1}(\mathbf{x}, t). \tag{3.2.3}$$

The coordinates $\mathbf{x}$ are termed the *spatial coordinates*. For a fixed value of $\mathbf{x} \in \kappa$, $\phi(\mathbf{x}, t)$ gives the value of ϕ associated with a fixed point $\mathbf{x}$ in space, which will be the value of ϕ associated with different material points at different times, because different material points occupy position $\mathbf{x} \in \kappa$ at different times, as shown in Fig. 3.2.4. Thus, a change in time t implies that a different value ϕ is observed at the *same* spatial location $\mathbf{x} \in \kappa$, now probably occupied by a different material particle X. Hence, attention is focused on a spatial position $\mathbf{x} \in \kappa$. The notation $\mathbf{X} = \mathbf{X}(\mathbf{x}, t)$ is only symbolic because in all practical cases one is not interested (or does not know) where the material particle X comes from before it occupies the current position $\mathbf{x}$ and where it goes when it leaves the position. That is, material particles are of no interest in the spatial description.

When ϕ is known in the material description, $\phi = \phi(\mathbf{X}, t)$, its total time derivative, D/Dt, is simply the partial derivative with respect to time because the material coordinates $\mathbf{X}$ do not change with time:

$$\frac{D}{Dt}[\phi(\mathbf{X}, t)] \equiv \frac{\partial}{\partial t}[\phi(\mathbf{X}, t)]\bigg|_{\mathbf{X}\text{ fixed}} = \frac{\partial \phi}{\partial t}. \tag{3.2.4}$$

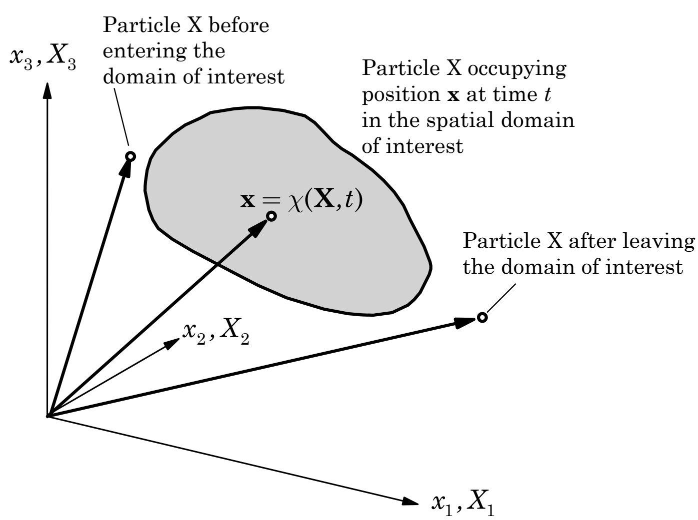


Fig. 3.2.4: Material points within and outside the spatial domain of interest in the spatial description.

However, when ϕ is known in the spatial description, $\phi = \phi(\mathbf{x}, t)$, its time derivative for a given particle, known as the *material derivative*,[3] is

$$\begin{aligned}\frac{D}{Dt}[\phi(\mathbf{x},t)] &= \frac{\partial}{\partial t}[\phi(\mathbf{x},t)] + \frac{Dx_i}{Dt}\frac{\partial}{\partial x_i}[\phi(\mathbf{x},t)]\\ &= \frac{\partial \phi}{\partial t} + v_i\frac{\partial \phi}{\partial x_i} = \frac{\partial \phi}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}\phi,\end{aligned} \tag{3.2.5}$$

where $\mathbf{v}$ is the velocity $\mathbf{v} = D\mathbf{x}/Dt = \dot{\mathbf{x}}$. Thus, if the velocity of a particle in the spatial description is $v(x,t)$, then the acceleration of a particle is

$$\mathbf{a} = \frac{D\mathbf{v}}{Dt} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v} \quad \left(a_i = \frac{\partial v_i}{\partial t} + v_j\frac{\partial v_i}{\partial x_j}\right). \tag{3.2.6}$$

Example 3.2.1 illustrates the determination of the inverse of a given mapping and computation of the material time derivative of a given function.

Example 3.2.1

Suppose that the motion of a continuous medium $\mathcal{B}$ is described by the mapping $\boldsymbol{\chi}: \kappa_0 \to \kappa$,

$$\boldsymbol{\chi}(\mathbf{X},t) = \mathbf{x} = (X_1 + AtX_2)\hat{\mathbf{e}}_1 + (X_2 - AtX_1)\hat{\mathbf{e}}_2 + X_3\,\hat{\mathbf{e}}_3,$$

and that the temperature T in the continuum in the spatial description is given by

$$T(\mathbf{x},t) = c_1(x_1 + c_2\,tx_2) = x_1 + tx_2,$$

where, in the interest of brevity, constants c_1 and c_2 are omitted; in SI units, $c_1 = 1$ K/m and $c_2 = 1$ /s. Determine (a) the inverse of the mapping $\boldsymbol{\chi}$, (b) the velocity components, and (c) the total time derivatives of T in the two descriptions.

Solution: A known deformation mapping $\boldsymbol{\chi}(\mathbf{X},t)$ relates the material coordinates (X_1, X_2, X_3) to the spatial coordinates (x_1, x_2, x_3) of a particle X. In the present case, we have

$$x_1 = X_1 + At\,X_2,\quad x_2 = X_2 - At\,X_1,\quad x_3 = X_3 \quad \text{or} \quad \begin{Bmatrix} x_1\\ x_2\\ x_3\end{Bmatrix} = \begin{bmatrix} 1 & At & 0\\ -At & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{Bmatrix} X_1\\ X_2\\ X_3\end{Bmatrix}. \tag{1}$$

Clearly, the relationships between x_i and X_i are linear (that is, the mapping is linear). Therefore, polygons are mapped into polygons. In particular, a unit square is mapped into a square that is rotated in a clockwise direction, as shown in Fig. 3.2.5. This can be verified by checking where the four corner points have moved in the "deformed" body:

(X_1, X_2, X_3)		(x_1, x_2, x_3)
$(0,0,0)$	$\to$	$(0,0,0)$
$(1,0,0)$	$\to$	$(1,-At,0)$
$(0,1,0)$	$\to$	$(At,1,0)$
$(1,1,0)$	$\to$	$(1+At, 1-At, 0)$

Note that, in general, the deformed square is not a unit square as the side now has a length of $1/\cos\alpha$, where $\alpha = \tan^{-1}(At)$. The reference configuration and deformed configurations at four different times, $t = 1, 2, 3$, and 4, for a value of $A = 0.25$, are shown in Fig. 3.2.6.

(a) The inverse mapping can be determined, when possible, by expressing (x_1, x_2, x_3) in terms of (X_1, X_2, X_3). In the present case, it is possible to invert the relations in Eq. (1) and obtain

$$\begin{Bmatrix} X_1\\ X_2\\ X_3\end{Bmatrix} = \frac{1}{(1+A^2t^2)}\begin{bmatrix} 1 & -At & 0\\ At & 1 & 0\\ 0 & 0 & 1+A^2t^2\end{bmatrix}\begin{Bmatrix} x_1\\ x_2\\ x_3\end{Bmatrix}. \tag{2}$$

[3] As opposed to d/dt, here we use Stokes's notation D/Dt for material derivative.

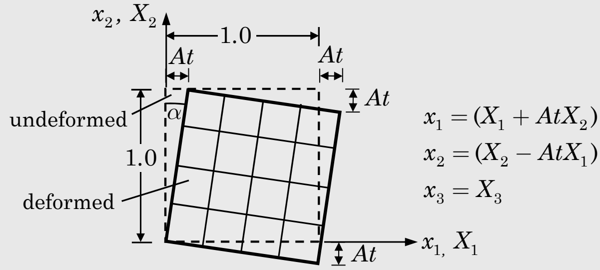


Fig. 3.2.5: A sketch of the mapping χ as applied to a unit square.

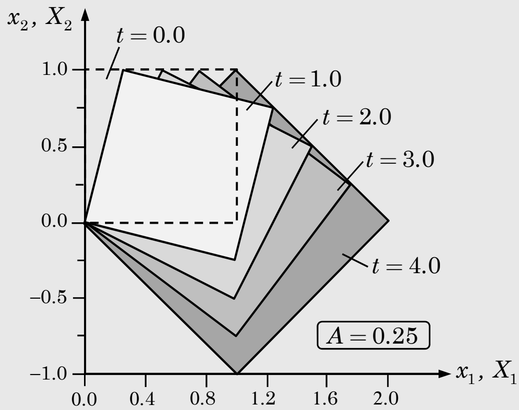


Fig. 3.2.6: Deformed configurations of the unit square at four different times ($A = 0.25$).

Therefore, we can write the inverse mapping as $\chi^{-1} : \kappa \rightarrow \kappa_0$ as

$$\chi^{-1}(\mathbf{x}, t) = \left(\frac{x_1 - Atx_2}{1 + A^2t^2}\right)\hat{\mathbf{E}}_1 + \left(\frac{x_2 + Atx_1}{1 + A^2t^2}\right)\hat{\mathbf{E}}_2 + x_3\,\hat{\mathbf{E}}_3. \tag{3}$$

(b) The velocity vector is given by $\mathbf{v} = v_1\hat{\mathbf{E}}_1 + v_2\hat{\mathbf{E}}_2$, with

$$v_1 = \frac{Dx_1}{Dt} = AX_2, \quad v_2 = \frac{Dx_2}{Dt} = -AX_1. \tag{4}$$

(c) The time rate of change of temperature of a material particle in $\mathcal{B}$ is simply

$$\frac{D}{Dt}[T(\mathbf{X}, t)] = \frac{\partial}{\partial t}[T(\mathbf{X}, t)]\bigg|_{\mathbf{x}\ \text{fixed}} = -2AtX_1 + (1 + A)X_2. \tag{5}$$

On the other hand, the time rate of change of temperature at point $\mathbf{x}$, which is now occupied by particle X, is

$$\begin{aligned}\frac{D}{Dt}[T(\mathbf{x}, t)] &= \frac{\partial T}{\partial t} + v_i\frac{\partial T}{\partial x_i} = x_2 + v_1 \cdot 1 + v_2 \cdot t \\ &= -2AtX_1 + (1 + A)X_2.\end{aligned} \tag{6}$$

In the study of solid bodies, the Eulerian description is less useful because the configuration κ is unknown. On the other hand, it is the preferred description for the study of motion of fluids because the configuration is known and remains unchanged, and we wish to determine the changes in the fluid velocities, pressure, density, and so on. Thus, in the Eulerian description, attention is focused on a given region of space instead of a given body of matter.

3.2.4 Displacement Field

The phrase "deformation of a continuum" refers to relative displacements and changes in the geometry experienced by the continuum $\mathcal{B}$ under the influence of a force system. The displacement of the particle X is defined, as shown in Fig. 3.2.7, by

$$\mathbf{u} = \mathbf{x} - \mathbf{X}. \tag{3.2.7}$$

In the Lagrangian description, the displacement vector $\mathbf{u}$ is expressed in terms of the material coordinates $\mathbf{X}$:

$$\mathbf{u}(\mathbf{X}, t) = \mathbf{x}(\mathbf{X}, t) - \mathbf{X}. \tag{3.2.8}$$

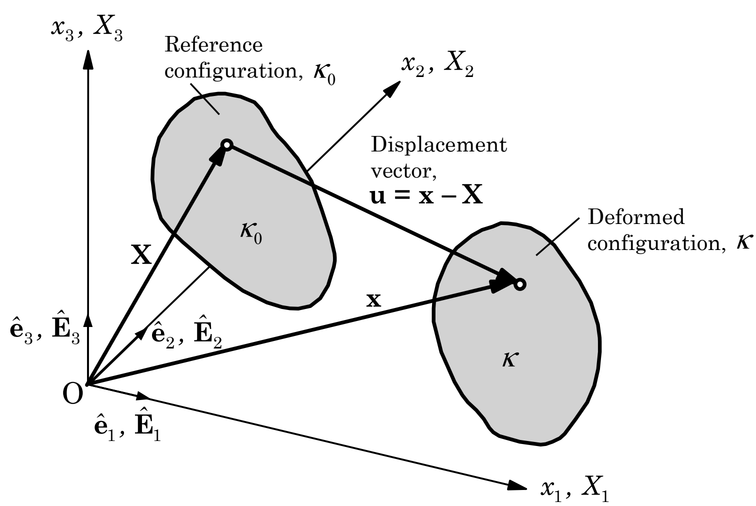


Fig. 3.2.7: Position vectors in the initial and current configurations and the displacement $\mathbf{u}$ of a particle X.

If the displacement of every particle in the body $\mathcal{B}$ is known, we can construct the current configuration κ from the reference configuration κ_0, $\boldsymbol{\chi}(\mathbf{X}, t) = \mathbf{x} = \mathbf{X} + \mathbf{u}(\mathbf{X}, t)$. On the other hand, in the Eulerian description the displacements are expressed in terms of the spatial coordinates (or current position) $\mathbf{x}$:

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{x} - \boldsymbol{\chi}^{-1}(\mathbf{x}, t) = \mathbf{x} - \mathbf{X}(\mathbf{x}, t). \tag{3.2.9}$$

To see the difference between the two descriptions further, consider the one-dimensional mapping $\chi(X, t) = x = X(1 + 0.5t)$ defining the motion of a rod of initial length two units. The inverse mapping is $\chi^{-1}(x, t) = X = x/(1 + 0.5t)$. The rod experiences a temperature distribution T given by the material description $T = 2t^2X$ or by the spatial description $T = 2t^2x/(1+0.5t)$, as shown in Fig. 3.2.8 [see Bonet and Wood (2008)].

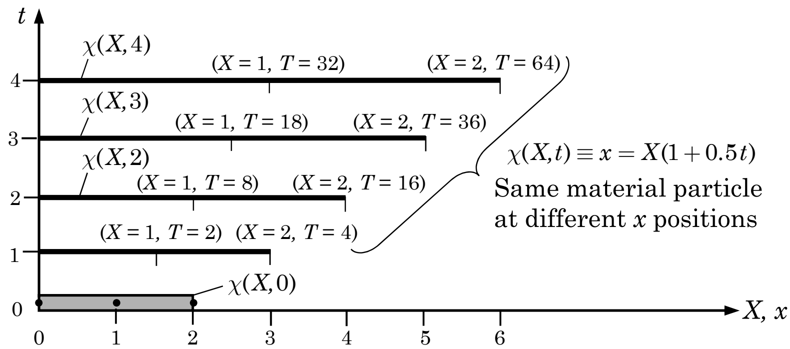


Fig. 3.2.8: Material and spatial descriptions of motion.

From Fig. 3.2.8, we see that the particle's material coordinate X remains associated with the particle while its spatial position x changes. The temperature at a given time can be found in one of the two ways: for example, at time $t = 3$ the temperature of the particle labeled 2 with material coordinate $X = 2$ is $T = 2 \times 2(3)^2 = 36$; alternatively, the temperature of the same particle, which at $t = 3$ is at a spatial position $x = 2(1 + 0.5 \times 3) = 5$, is $T = 2 \times 5(3)^2/(1 + 0.5 \times 3) = 36$. The displacement of a material point X occupying position X in κ_0 is

$$u(X,t) = x - X = X(1 + 0.5t) - X = 0.5Xt.$$

A *rigid body* is one in which the distance between any two material particles remains the same whereas a *deformable body* is one in which the material particles can move relative to each other under the action of external stimuli. A rigid-body motion is one in which *all* material particles of the body undergo the same displacement. Then the deformation of a continuum can be determined only by considering the change of distance between any two arbitrary but infinitesimally close points of the continuum.

3.3 Analysis of Deformation

3.3.1 Deformation Gradient

One of the key quantities in deformation analysis is the *deformation gradient* of κ relative to the reference configuration κ_0, denoted $\mathbf{F}_\kappa$, which provides the relationship between a material line $d\mathbf{X}$ before deformation and the line $d\mathbf{x}$, consisting of the same material as $d\mathbf{X}$ after deformation. It is defined as follows (in the interest of brevity, the subscript κ on $\mathbf{F}$ is dropped):

$$d\mathbf{x} = \mathbf{F} \cdot d\mathbf{X} = d\mathbf{X} \cdot \mathbf{F}^{\mathrm{T}}, \tag{3.3.1}$$

$$\mathbf{F} = \left(\frac{\partial \boldsymbol{\chi}}{\partial \mathbf{X}}\right)^{\mathrm{T}} = \left(\frac{\partial \mathbf{x}}{\partial \mathbf{X}}\right)^{\mathrm{T}} \equiv \mathbf{x}\overleftarrow{\nabla}_0 = (\boldsymbol{\nabla}_0 \mathbf{x})^{\mathrm{T}}. \tag{3.3.2}$$

Here $\overleftarrow{\nabla}$ denotes the backward gradient operator (see the note at the end of Section 2.4.5) and $\boldsymbol{\nabla}$ is the forward gradient operator with respect to $\mathbf{x}$. The subscript 0 on the del operator denotes that the differentiation is with respect to $\mathbf{X}$ (the material coordinates). Many authors [see, e.g., Gurtin (1981); Bonet and Wood (2008); and Gurtin, Fried, and Anand (2010)] use $\boldsymbol{\nabla}$ in defining the deformation gradient, but a close look reveals that they actually mean the backward gradient operator discussed in Section 2.4.5.

By definition, $\mathbf{F}$ is a function of both position $\mathbf{X}$ and time t; $\mathbf{F}$ is sometimes referred to as a *two-point tensor*[4] (or a linear transformation of points in the small neighborhood of $\mathbf{X}$ from κ_0 into the neighborhood of $\mathbf{x}$ in κ) because it describes the local deformation of a material line element at point $\mathbf{X}$ in the reference configuration κ_0 to the point $\mathbf{x}$ in the current configuration κ; $\mathbf{F}$ involves, in general, both stretch and rotation. For example, in the case of pure stretch followed by rotation, the deformation of $d\mathbf{X}$ into $d\mathbf{X}'$ involves only pure stretch and the deformation from $d\mathbf{X}'$ into $d\mathbf{x}$ involves only pure rotation (although, in reality, stretch and rotation occur simultaneously). Thus, we can write $d\mathbf{X}' = \mathbf{U} \cdot d\mathbf{X}$ and $d\mathbf{x} = \mathbf{R} \cdot d\mathbf{X}'$, where $\mathbf{U}$ is a stretch tensor and $\mathbf{R}$ is a *proper* orthogonal tensor, $\mathbf{R}^{\mathrm{T}} \cdot \mathbf{R} = \mathbf{I}$ and $|\mathbf{R}| = 1$. The tensors $\mathbf{U}$ and $\mathbf{R}$ can be interpreted as linear transformations, $\mathbf{U} : d\mathbf{X} \rightarrow d\mathbf{X}'$ and $\mathbf{R} : d\mathbf{X}' \rightarrow d\mathbf{x}$. In other words, the linear transformation $\mathbf{F} : d\mathbf{X} \rightarrow d\mathbf{x}$ is replaced by the composite transformation $\mathbf{F} = \mathbf{R} \cdot \mathbf{U}$. The requirement that $\mathbf{R}$ be a proper orthogonal transformation follows from the fact that a pure rotation should not change the length of the line element $d\mathbf{X}'$:

$$d\mathbf{x} \cdot d\mathbf{x} = (\mathbf{R} \cdot d\mathbf{X}') \cdot (\mathbf{R} \cdot d\mathbf{X}') = d\mathbf{X}' \cdot \mathbf{R}^{\mathrm{T}} \cdot \mathbf{R} \cdot d\mathbf{X}' = d\mathbf{X}' \cdot d\mathbf{X}'.$$

The multiplicative decomposition of $\mathbf{F}$ into pure stretch $\mathbf{U}$ and pure rotation $\mathbf{R}$ is discussed further in Section 3.9 on polar decomposition.

In index notation, Eq. (3.3.2) can be written as

$$\mathbf{F} = F_{iJ}\,\hat{\mathbf{e}}_i\,\hat{\mathbf{E}}_J, \quad F_{iJ} = \frac{\partial x_i}{\partial X_J}\,. \tag{3.3.3}$$

More explicitly, we have

$$[F] = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix}, \tag{3.3.4}$$

where the lowercase indices refer to the current (spatial) Cartesian coordinates, whereas uppercase indices refer to the reference (material) Cartesian coordinates. The determinant of $[F]$ is called the *Jacobian of the motion*, and it is denoted by $J = |F|$. The equation $\mathbf{F} \cdot d\mathbf{X} = \mathbf{0}$ for $d\mathbf{X} \neq \mathbf{0}$ implies that a material line in the reference configuration is reduced to zero by the deformation. Since this is physically not realistic, we conclude that $\mathbf{F} \cdot d\mathbf{X} \neq \mathbf{0}$ for $d\mathbf{X} \neq \mathbf{0}$. That

[4]Strictly speaking, $\mathbf{F}$ is not a tensor because its components do not transform like those of a tensor.

is, $\mathbf{F}$ is a nonsingular tensor, $J \neq 0$. Hence, $\mathbf{F}$ has an inverse $\mathbf{F}^{-1}$, and we can write

$$d\mathbf{X} = \mathbf{F}^{-1} \cdot d\mathbf{x} = d\mathbf{x} \cdot \mathbf{F}^{-\mathrm{T}}, \quad \text{where} \quad \mathbf{F}^{-\mathrm{T}} = \frac{\partial \mathbf{X}}{\partial \mathbf{x}} \equiv \boldsymbol{\nabla} \mathbf{X}, \tag{3.3.5}$$

and in index notation

$$\mathbf{F}^{-1} = F_{Ji}^{-1}\, \hat{\mathbf{E}}_J\, \hat{\mathbf{e}}_i, \quad F_{Ji}^{-1} = \frac{\partial X_J}{\partial x_i}. \tag{3.3.6}$$

In explicit form the matrix associated with $\mathbf{F}^{-1}$ is

$$[F]^{-1} = \begin{bmatrix} \frac{\partial X_1}{\partial x_1} & \frac{\partial X_1}{\partial x_2} & \frac{\partial X_1}{\partial x_3} \\ \frac{\partial X_2}{\partial x_1} & \frac{\partial X_2}{\partial x_2} & \frac{\partial X_2}{\partial x_3} \\ \frac{\partial X_3}{\partial x_1} & \frac{\partial X_3}{\partial x_2} & \frac{\partial X_3}{\partial x_3} \end{bmatrix}. \tag{3.3.7}$$

Note that $[F]$ stretches or compresses a material vector $\{X\}$ into the current vector $\{x\}$, while $\boldsymbol{\chi}$ gives the current position $\mathbf{x} \in \kappa$ of the material point X that occupied position $\mathbf{X} \in \kappa_0$. The deformation gradient and its inverse can be expressed in terms of the displacement vector as

$$\mathbf{F} = (\boldsymbol{\nabla}_0 \mathbf{x})^{\mathrm{T}} = (\boldsymbol{\nabla}_0 \mathbf{u} + \mathbf{I})^{\mathrm{T}} \quad \text{or} \quad \mathbf{F}^{-1} = (\boldsymbol{\nabla} \mathbf{X})^{\mathrm{T}} = (\mathbf{I} - \boldsymbol{\nabla} \mathbf{u})^{\mathrm{T}}. \tag{3.3.8}$$

Example 3.3.1 illustrates computation of the components of the deformation gradient and the displacement vector from known mapping of motion.

Example 3.3.1

Consider the uniform deformation of a square block of side 2 units and initially centered at $\mathbf{X} = (0, 0)$. If the deformation is defined by the mapping

$$\boldsymbol{\chi}(\mathbf{X}) = \big(3.5 + X_1 + 0.5X_2\big)\, \hat{\mathbf{e}}_1 + \big(4 + X_2\big)\, \hat{\mathbf{e}}_2 + X_3\, \hat{\mathbf{e}}_3,$$

(a) sketch the deformation, (b) determine the deformation gradient $\mathbf{F}$, and (c) compute the displacements.

Solution: (a) From the given mapping, we have $x_1 = 3.5 + X_1 + 0.5X_2$, $x_2 = 4 + X_2$, and $x_3 = X_3$; in matrix form, we have

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{bmatrix} 1.0 & 0.5 & 0.0 \\ 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} + \begin{Bmatrix} 3.5 \\ 4.0 \\ 0.0 \end{Bmatrix}. \tag{1}$$

These are linear relations and therefore the square is mapped, in general, into a parallelogram. To see where the corner points of the square are mapped to, apply the above equations to the corner points (no change in X_3):

(X_1, X_2)		(x_1, x_2)
$(-1, -1)$	$\to$	$(2, 3)$
$(\ \ 1, -1)$	$\to$	$(4, 3)$
$(\ \ 1, \ \ 1)$	$\to$	$(5, 5)$
$(-1, \ \ 1)$	$\to$	$(3, 5)$

Thus, under the mapping the square moved and became a parallelogram, centered at (3.5, 4), as shown in Fig. 3.3.1. The base and height of the parallelogram remained 2 units each.

$$x_1 = 3.5 + X_1 + 0.5X_2,$$
$$x_2 = 4 + X_2,$$
$$x_3 = X_3.$$

$$X_1 = -1.5 + x_1 - 0.5x_2,$$
$$X_2 = -4 + x_2,$$
$$X_3 = x_3.$$

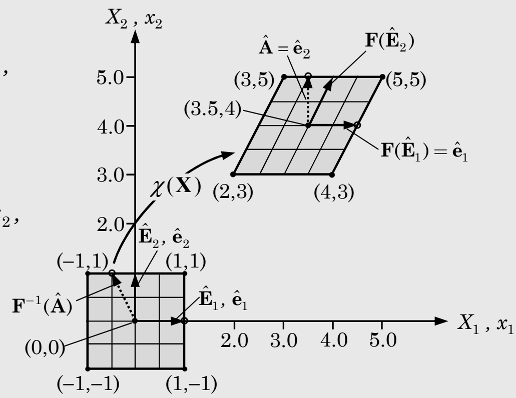


Fig. 3.3.1: Uniform deformation of a square block of material.

The linear relations in Eq. (1) can be inverted to obtain X_1, X_2, and X_3 in terms of x_1, x_2, and x_3:

$$\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{bmatrix} 1.0 & -0.5 & 0.0 \\ 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix} \left(\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} - \begin{Bmatrix} 3.5 \\ 4.0 \\ 0.0 \end{Bmatrix} \right), \tag{2}$$

or $X_1 = -1.5 + x_1 - 0.5x_2$, $X_2 = -4 + x_2$, and $X_3 = x_3$. Thus, the inverse mapping is

$$\boldsymbol{\chi}^{-1}(\mathbf{x}) = \big(-1.5 + x_1 - 0.5x_2\big)\,\hat{\mathbf{E}}_1 + \big(-4 + x_2\big)\,\hat{\mathbf{E}}_2 + x_3\,\hat{\mathbf{E}}_3, \tag{3}$$

which recovers the square shape from the parallelogram shape shown in Fig. 3.3.1. This type of deformation is known as *simple shear*, in which there exists a set of line elements, in the present case, lines parallel to the X_1-axis, whose orientation is such that they are unchanged in length and orientation by the deformation.

(b) The components of the deformation gradient and its inverse in matrix form are

$$[F] = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} = \begin{bmatrix} 1.0 & 0.5 & 0.0 \\ 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix}, \quad [F]^{-1} = \begin{bmatrix} \frac{\partial X_1}{\partial x_1} & \frac{\partial X_1}{\partial x_2} & \frac{\partial X_1}{\partial x_3} \\ \frac{\partial X_2}{\partial x_1} & \frac{\partial X_2}{\partial x_2} & \frac{\partial X_2}{\partial x_3} \\ \frac{\partial X_3}{\partial x_1} & \frac{\partial X_3}{\partial x_2} & \frac{\partial X_3}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 1.0 & -0.5 & 0.0 \\ 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix}.$$

The unit vectors $\hat{\mathbf{E}}_1$ and $\hat{\mathbf{E}}_2$ in the initial configuration deform to the lengths

$$\begin{bmatrix} 1.0 & 0.5 & 0.0 \\ 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}, \quad \begin{bmatrix} 1.0 & 0.5 & 0.0 \\ 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix} \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0.5 \\ 1.0 \\ 0.0 \end{Bmatrix}.$$

The unit vector $\hat{\mathbf{A}} = \hat{\mathbf{e}}_2$ in the current configuration is deformed from the vector

$$\begin{bmatrix} 1.0 & -0.5 & 0.0 \\ 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix} \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -0.5 \\ 1.0 \\ 0.0 \end{Bmatrix} \quad \Rightarrow \quad \mathbf{F}^{-1}(\hat{\mathbf{A}}) = -0.5\hat{\mathbf{E}}_1 + \hat{\mathbf{E}}_2.$$

(c) The displacement vector is given by

$$\mathbf{u} = \mathbf{x} - \mathbf{X} = \big(3.5 + 0.5X_2\big)\mathbf{e}_1 + 4\;\mathbf{e}_2,$$

which is independent of X_1. The displacement components are $u_1 = 3.5 + 0.5X_2$, $u_2 = 4$, and $u_3 = 0$. Thus, a $X_2 =$ constant line moved 4 units up and $3.5 + 0.5X_2$ units to the right.

3.3.2 Isochoric, Homogeneous, and Inhomogeneous Deformations

3.3.2.1 Isochoric deformation

If the Jacobian is unity $J = 1$, then the deformation is volume-preserving or the current and reference configurations coincide. If volume does not change locally during the deformation, the deformation is said to be *isochoric* at $\mathbf{X}$. If $J = 1$ everywhere in the body $\mathcal{B}$, then the deformation of the body is isochoric.

3.3.2.2 Homogeneous deformation

In general, the deformation gradient $\mathbf{F}$ is a function of $\mathbf{X}$. If $\mathbf{F} = \mathbf{I}$ everywhere in the body, then the body is not rotated but might be rigidly translated. If $\mathbf{F}$ has the same value at every material point in a body (that is, $\mathbf{F}$ is independent of $\mathbf{X}$), then the mapping $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$ is said to be a *homogeneous motion* of the body and the deformation is said to be homogeneous. In general, at any given time $t > 0$, a mapping $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$ is said to be a homogeneous motion if and only if it can be expressed as (so that $\mathbf{F}$ is a constant tensor)

$$\mathbf{x} = \mathbf{A} \cdot \mathbf{X} + \mathbf{c}, \tag{3.3.9}$$

where the second-order tensor $\mathbf{A}$ and vector $\mathbf{c}$ are functions of time t only; $\mathbf{c}$ represents a rigid-body translation. Note that for a homogeneous motion we have $\mathbf{F}$=$\mathbf{A}$. Clearly, the motion described by the mapping of Example 3.3.1 is homogeneous and isochoric. Next, we consider several simple forms of homogeneous deformations.

Pure dilatation. Consider a cube of material with edges of length L and ℓ in the reference and current configurations, respectively. If the deformation mapping has the form (see Fig. 3.3.2)

$$\boldsymbol{\chi}(\mathbf{X}) = \lambda X_1\, \hat{\mathbf{e}}_1 + \lambda X_2\, \hat{\mathbf{e}}_2 + \lambda X_3\, \hat{\mathbf{e}}_3, \quad \lambda = \frac{L}{\ell}\,, \tag{3.3.10}$$

then $\mathbf{F}$ has the matrix representation

$$[F] = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}. \tag{3.3.11}$$

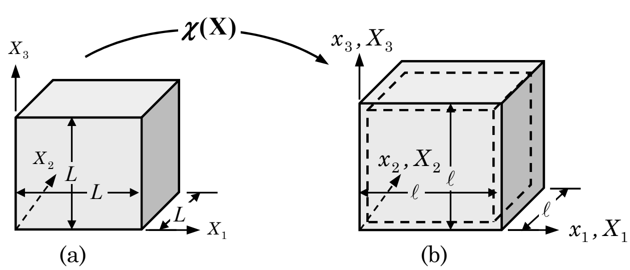


Fig. 3.3.2: A deformation mapping of pure dilatation.

This deformation is known as pure dilatation or pure stretch, and it is isochoric if and only if $\lambda = 1$ (λ is called the principal stretch), as shown in Fig. 3.3.2.

Simple extension. An example of homogeneous extension in the X_1-direction, as shown in Fig. 3.3.3, is provided by the deformation mapping

$$\boldsymbol{\chi}(\mathbf{X}) = (1+\alpha)X_1\,\hat{\mathbf{e}}_1 + X_2\,\hat{\mathbf{e}}_2 + X_3\,\hat{\mathbf{e}}_3. \tag{3.3.12}$$

The inverse mapping is [because $x_1 = (1+\alpha)X_1$, $x_2 = X_2$, and $x_3 = X_3$]

$$\boldsymbol{\chi}^{-1}(\mathbf{x}) = \frac{1}{(1+\alpha)}x_1\,\hat{\mathbf{e}}_1 + x_2\,\hat{\mathbf{e}}_2 + x_3\,\hat{\mathbf{e}}_3.$$

The matrices of the deformation gradient and its inverse are

$$[F] = \begin{bmatrix} 1+\alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad [F]^{-1} = \frac{1}{(1+\alpha)}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1+\alpha & 0 \\ 0 & 0 & 1+\alpha \end{bmatrix}. \tag{3.3.13}$$

For example, a line $X_2 = a + bX_1$ in the initial configuration κ_0 transforms under the mapping to the line

$$x_2 = X_2 = a + bX_1 = a + \frac{b}{1+\alpha}x_1$$

in the current configuration κ.

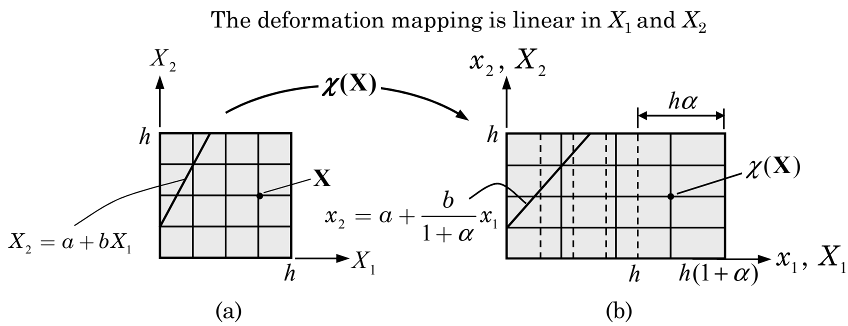


Fig. 3.3.3: A deformation mapping of simple extension. Typical material lines inside the body are also shown.

Simple shear. This deformation, also known as *uniform shear deformation*, as discussed in Example 3.3.1, is defined by a linear deformation mapping of the form (see Fig. 3.3.4)

$$\boldsymbol{\chi}(\mathbf{X}) = (X_1 + \gamma X_2)\hat{\mathbf{e}}_1 + X_2\,\hat{\mathbf{e}}_2 + X_3\,\hat{\mathbf{e}}_3, \tag{3.3.14}$$

where material planes X_2 = constant slide in the X_1-direction in linear proportion to X_2, the proportionality constant being γ, which is a measure of the amount of shear. The planes x_2 = constant are the shear planes and the direction along x_1 is the shear direction. The matrix representation of the deformation gradient in this case is

$$[F] = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.3.15}$$

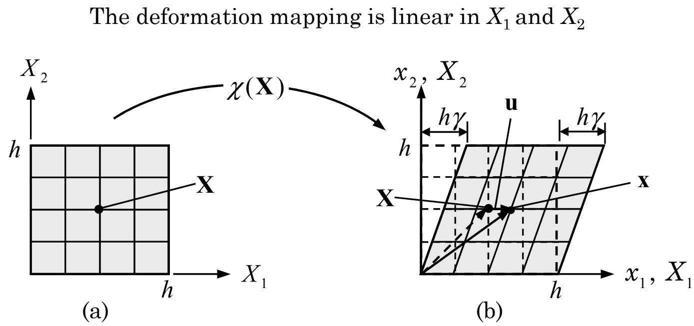


Fig. 3.3.4: A deformation mapping of simple shear. Typical material lines inside the body are also shown.

3.3.2.3 Nonhomogeneous deformation

A nonhomogeneous deformation is one in which the deformation gradient **F** is a function of **X**. An example of nonhomogeneous deformation mapping is provided, as shown in Fig. 3.3.5, by

$$\boldsymbol{\chi}(\mathbf{X}) = X_1(1+\gamma_1 X_2)\hat{\mathbf{e}}_1 + X_2(1+\gamma_2 X_1)\hat{\mathbf{e}}_2 + X_3\,\hat{\mathbf{e}}_3. \tag{3.3.16}$$

The matrix representation of the deformation gradient is

$$[F] = \begin{bmatrix} 1+\gamma_1 X_2 & \gamma_1 X_1 & 0 \\ \gamma_2 X_2 & 1+\gamma_2 X_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.3.17}$$

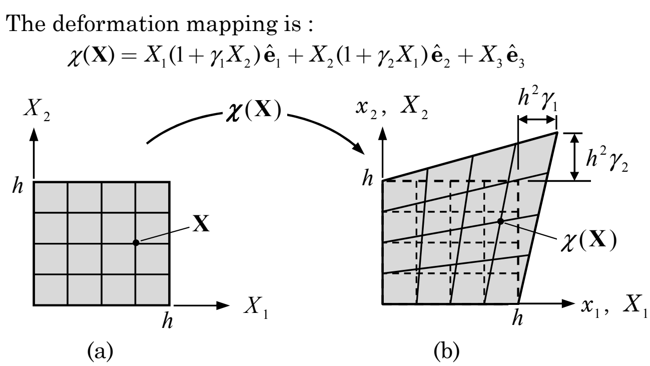


Fig. 3.3.5: A deformation mapping of combined shearing and extension. Typical material lines inside the body are also shown.

It is rather difficult to invert the mapping even for this simple nonhomogeneous deformation.

Figure 3.3.6 shows the deformed configuration for the values of $h = 1$ and $\gamma_2 = 3\gamma_1 = 3\gamma = 3$. Note that the straight line AC (that is, line $X_2 = X_1$) in the undeformed configuration becomes a curve in the deformed configuration, although the edges of the deformed configuration remain as straight lines.

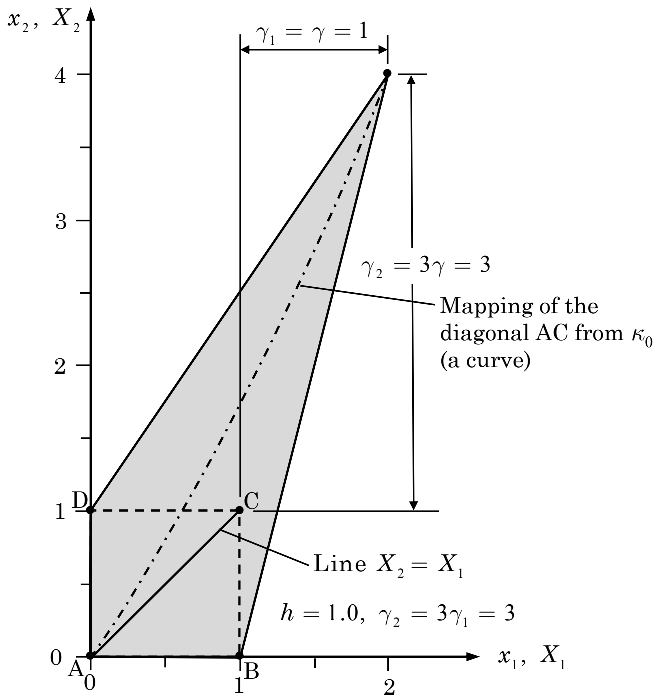


Fig. 3.3.6: The deformed configuration of a unit square under the deformation mapping in Eq. (3.3.16) for $\gamma_1 = 1$ and $\gamma_2 = 3$.

3.3.3 Change of Volume and Surface

Here we study how deformation mapping affects surface areas and volumes of a continuum. The motivation for this study comes from the need to write global equilibrium statements that involve integrals over areas and volumes.

3.3.3.1 Volume change

We can define volume and surface elements in the reference and deformed configurations. Consider three noncoplanar line elements $d\mathbf{X}^{(1)}$, $d\mathbf{X}^{(2)}$, and $d\mathbf{X}^{(3)}$ forming the edges of a parallelepiped at point P with position vector $\mathbf{X}$ in the reference body $\mathcal{B}$, as shown in Fig. 3.3.7, so that

$$d\mathbf{x}^{(i)} = \mathbf{F} \cdot d\mathbf{X}^{(i)}, \quad i = 1, 2, 3. \tag{3.3.18}$$

Note that the vectors $d\mathbf{x}^{(i)}$ are not necessarily parallel to or have the same length as the vectors $d\mathbf{X}^{(i)}$ due to shearing and stretching of the parallelepiped. We assume that the triad $(d\mathbf{X}^{(1)}, d\mathbf{X}^{(2)}, d\mathbf{X}^{(3)})$ is positively oriented in the sense that the triple scalar product $d\mathbf{X}^{(1)} \cdot d\mathbf{X}^{(2)} \times d\mathbf{X}^{(3)} > 0$. We denote the volume of the parallelepiped as

$$\begin{aligned} dV &= d\mathbf{X}^{(1)} \cdot d\mathbf{X}^{(2)} \times d\mathbf{X}^{(3)} = \left(\hat{\mathbf{N}}_1 \cdot \hat{\mathbf{N}}_2 \times \hat{\mathbf{N}}_3\right) dX^{(1)} dX^{(2)} dX^{(3)} \\ &= dX^{(1)} dX^{(2)} dX^{(3)}, \end{aligned} \tag{3.3.19}$$

where $\hat{\mathbf{N}}_i$ denotes the unit vector along $d\mathbf{X}^{(i)}$. The corresponding volume in the deformed configuration is given by

$$dv = d\mathbf{x}^{(1)} \cdot d\mathbf{x}^{(2)} \times d\mathbf{x}^{(3)} = \left(\mathbf{F} \cdot \hat{\mathbf{N}}_1\right) \cdot \left(\mathbf{F} \cdot \hat{\mathbf{N}}_2\right) \times \left(\mathbf{F} \cdot \hat{\mathbf{N}}_3\right) dX^{(1)} dX^{(2)} dX^{(3)},$$

or

$$dv = |F|\, dX^{(1)} dX^{(2)} dX^{(3)} = J\, dV. \tag{3.3.20}$$

We assume that the volume elements are positive so that the relative orientation of the line elements is preserved under the deformation, that is, $J > 0$. Thus, J has the physical meaning of being the local ratio of current to reference volume of a material volume element.

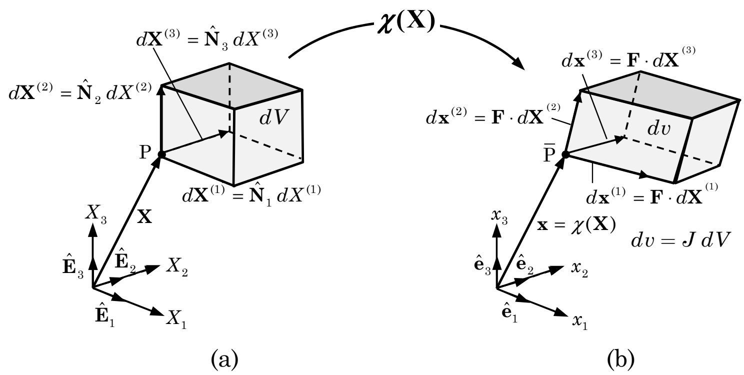


Fig. 3.3.7: Transformation of a volume element under a deformation mapping.

3.3.3.2 Area change

Next, consider an infinitesimal vector element of material surface $d\mathbf{A}$ in a neighborhood of the point $\mathbf{X}$ in the reference configuration, as shown in Fig. 3.3.8. The surface vector can be expressed as $d\mathbf{A} = dA\,\hat{\mathbf{N}}$, where $\hat{\mathbf{N}}$ is the positive unit normal to the surface in the reference configuration.

Suppose that $d\mathbf{A}$ from the reference configuration becomes $d\mathbf{a}$ in the current configuration, where $d\mathbf{a} = da\,\hat{\mathbf{n}}$, $\hat{\mathbf{n}}$ being the outward unit normal to the surface in the current configuration. The outward unit normals in the reference and current configurations can be expressed as

$$\hat{\mathbf{N}} = \frac{\hat{\mathbf{N}}_1 \times \hat{\mathbf{N}}_2}{|\hat{\mathbf{N}}_1 \times \hat{\mathbf{N}}_2|}, \quad \hat{\mathbf{n}} = \frac{\mathbf{F} \cdot \hat{\mathbf{n}}_1 \times \mathbf{F} \cdot \hat{\mathbf{n}}_2}{|\mathbf{F} \cdot \hat{\mathbf{n}}_1 \times \mathbf{F} \cdot \hat{\mathbf{n}}_2|}. \tag{3.3.21}$$

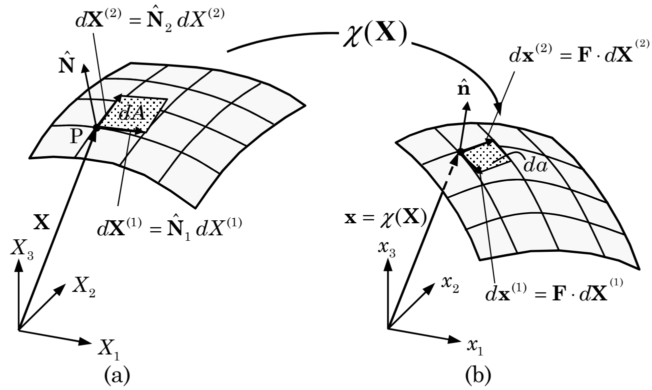


Fig. 3.3.8: Transformation of a surface element under a deformation mapping.

The areas of the parallelograms in the reference and current configurations are

$$dA \equiv |\hat{\mathbf{N}}_1 \times \hat{\mathbf{N}}_2|\; dX_1\, dX_2, \quad da \equiv |\mathbf{F}\cdot\hat{\mathbf{n}}_1 \times \mathbf{F}\cdot\hat{\mathbf{n}}_2|\, dx_1\, dx_2. \tag{3.3.22}$$

The area vectors are

$$\begin{aligned} d\mathbf{A} = \hat{\mathbf{N}}\, dA &= \frac{\hat{\mathbf{N}}_1 \times \hat{\mathbf{N}}_2}{|\hat{\mathbf{N}}_1 \times \hat{\mathbf{N}}_2|}\; |\hat{\mathbf{N}}_1 \times \hat{\mathbf{N}}_2|\; dX_1\, dX_2 \\ &= \left(\hat{\mathbf{N}}_1 \times \hat{\mathbf{N}}_2\right) dX_1\, dX_2 = \left(\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2\right) dX_1\, dX_2, \end{aligned} \tag{3.3.23}$$

$$\begin{aligned} d\mathbf{a} = \hat{\mathbf{n}}\, da &= \frac{\mathbf{F}\cdot\hat{\mathbf{n}}_1 \times \mathbf{F}\cdot\hat{\mathbf{n}}_2}{|\mathbf{F}\cdot\hat{\mathbf{n}}_1 \times \mathbf{F}\cdot\hat{\mathbf{n}}_2|}\; |\mathbf{F}\cdot\hat{\mathbf{n}}_1 \times \mathbf{F}\cdot\hat{\mathbf{n}}_2|\, dx_1\, dx_2 \\ &= \left(\mathbf{F}\cdot\hat{\mathbf{n}}_1 \times \mathbf{F}\cdot\hat{\mathbf{n}}_2\right) dx_1\, dx_2. \end{aligned} \tag{3.3.24}$$

Then it can be shown that (see the result of Problem **3.16**)

$$d\mathbf{a} = J\,\mathbf{F}^{-T}\cdot d\mathbf{A} \quad \text{or} \quad \hat{\mathbf{n}}\, da = J\,\mathbf{F}^{-\mathrm{T}}\cdot\hat{\mathbf{N}}\, dA. \tag{3.3.25}$$

3.4 Strain Measures

3.4.1 Cauchy–Green Deformation Tensors

The geometric changes that a continuous medium experiences can be measured in a number of ways. Here, we discuss a general measure of deformation of a continuous medium, independent of both translation and rotation.

Consider two material particles P and Q in the neighborhood of each other, separated by $d\mathbf{X}$ in the reference configuration, as shown in Fig. 3.4.1. In the current (deformed) configuration the material points P and Q occupy positions $\bar{\mathrm{P}}$ and $\bar{\mathrm{Q}}$, and they are separated by $d\mathbf{x}$. We wish to determine the change in the distance $d\mathbf{X}$ between the material points P and Q as the body deforms and the material points move to the new locations $\bar{\mathrm{P}}$ and $\bar{\mathrm{Q}}$.

The distances between points P and Q and points $\bar{\text{P}}$ and $\bar{\text{Q}}$ are given, respectively, by

$$(dS)^2 = d\mathbf{X} \cdot d\mathbf{X}, \tag{3.4.1}$$

$$(ds)^2 = d\mathbf{x} \cdot d\mathbf{x} = d\mathbf{X} \cdot (\mathbf{F}^{\mathrm{T}} \cdot \mathbf{F}) \cdot d\mathbf{X} \equiv d\mathbf{X} \cdot \mathbf{C} \cdot d\mathbf{X}, \tag{3.4.2}$$

where $\mathbf{C}$ is called the *right Cauchy–Green deformation tensor*

$$\mathbf{C} = \mathbf{F}^{\mathrm{T}} \cdot \mathbf{F}. \tag{3.4.3}$$

By definition, $\mathbf{C}$ is a symmetric second-order tensor. The *left Cauchy–Green deformation tensor* or *Finger tensor* is defined by

$$\mathbf{B} = \mathbf{F} \cdot \mathbf{F}^{\mathrm{T}}, \tag{3.4.4}$$

which is also a symmetric tensor.

Recall from Eq. (2.4.25) that the directional (or tangential) derivative of a field $\phi(\mathbf{X})$ is given by

$$\frac{d\phi}{dS} = \hat{\mathbf{N}} \cdot \boldsymbol{\nabla}_0 \phi, \quad \hat{\mathbf{N}} = \frac{d\mathbf{X}}{|d\mathbf{X}|} = \frac{d\mathbf{X}}{dS}\,, \tag{3.4.5}$$

where $\hat{\mathbf{N}}$ is the unit vector in the direction of $d\mathbf{X}$ at point $\mathbf{X}$. Therefore, a parameterized curve in the deformed configuration is determined by the deformation mapping $\mathbf{x}(S) = \boldsymbol{\chi}(\mathbf{X}(S))$, and we have ($\mathbf{F} = F_{iJ}\,\hat{\mathbf{e}}_i\,\hat{\mathbf{E}}_J$ and $\hat{\mathbf{N}} = N_K \hat{\mathbf{E}}_K$)

$$\frac{d\mathbf{x}}{dS} = \frac{d\mathbf{X}}{dS} \cdot \boldsymbol{\nabla}_0 \boldsymbol{\chi}(\mathbf{X}) = \mathbf{F} \cdot \frac{d\mathbf{X}}{dS} = \mathbf{F} \cdot \hat{\mathbf{N}} = F_{iJ} N_J\,\hat{\mathbf{e}}_i. \tag{3.4.6}$$

Note that $d\mathbf{x}/dS = F_{iJ} N_J\,\hat{\mathbf{e}}_i$ is a vector defined in the current (deformed) configuration because $\hat{\mathbf{e}}_i$ is a unit vector in the current configuration.

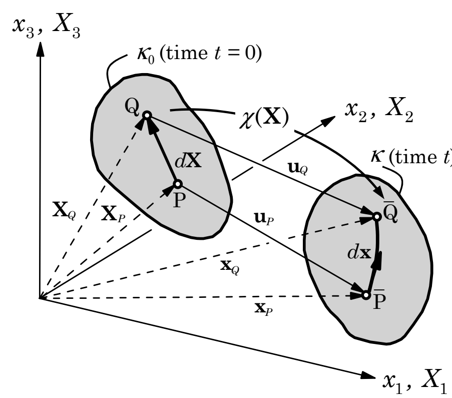


Fig. 3.4.1: Points P and Q separated by a distance $d\mathbf{X}$ in the reference configuration κ_0 take up positions $\bar{\text{P}}$ and $\bar{\text{Q}}$, respectively, in the deformed configuration κ, where they are separated by distance $d\mathbf{x}$.

The *stretch* of a curve at a point in the deformed configuration is defined as the ratio of the deformed length of the curve to its original length, $\lambda = ds/dS$. Let us consider an infinitesimal length dS of the curve in the neighborhood of the material point $\mathbf{X}$. Then the stretch λ of the curve is simply the length of the tangent vector $\mathbf{F}\cdot\hat{\mathbf{N}}$ in the deformed configuration

$$\lambda^2(S) = (\mathbf{F}\cdot\hat{\mathbf{N}})\cdot(\mathbf{F}\cdot\hat{\mathbf{N}}) \tag{3.4.7}$$

$$= \hat{\mathbf{N}}\cdot(\mathbf{F}^{\mathrm{T}}\cdot\mathbf{F})\cdot\hat{\mathbf{N}} = \hat{\mathbf{N}}\cdot\mathbf{C}\cdot\hat{\mathbf{N}}. \tag{3.4.8}$$

Equation (3.4.8) holds for any arbitrary curve with $d\mathbf{X} = dS\,\hat{\mathbf{N}}$, and thus allows us to compute the stretch in any direction at a given point. In particular, the square of the stretch in the direction of the unit base vector $\hat{\mathbf{E}}_I$ is given by

$$\lambda^2(\hat{\mathbf{E}}_I) = \hat{\mathbf{E}}_I\cdot\mathbf{C}\cdot\hat{\mathbf{E}}_I = C_{II} \quad \text{(no sum on } I). \tag{3.4.9}$$

That is, the diagonal terms of the right Cauchy–Green deformation tensor $\mathbf{C}$ represent the squares of the stretches in the direction of the coordinate axes (X_1, X_2, X_3). The off-diagonal elements of $\mathbf{C}$ give a measure of the angle of shearing between two base vectors $\hat{\mathbf{E}}_I$ and $\hat{\mathbf{E}}_J$, for $I \neq J$, under the deformation mapping $\boldsymbol{\chi}$. Further, the squares of the principal stretches at a point are equal to the eigenvalues of $\mathbf{C}$. We shall return to this aspect in Section 3.9 on the polar decomposition theorem.

3.4.2 Green–Lagrange Strain Tensor

The change in the squared lengths that occurs as a body deforms from the reference to the current configuration can be expressed relative to the original length as

$$(ds)^2 - (dS)^2 = 2\,d\mathbf{X}\cdot\mathbf{E}\cdot d\mathbf{X}, \tag{3.4.10}$$

where $\mathbf{E}$ is called the *Green–St. Venant (Lagrangian) strain tensor*, the *Green–Lagrange strain tensor*, or simply the *Green strain tensor*.[5] The Green–Lagrange strain tensor can be expressed, in view of Eqs. (3.4.1)–(3.4.3), as

$$\begin{aligned}\mathbf{E} &= \tfrac{1}{2}\left(\mathbf{F}^{\mathrm{T}}\cdot\mathbf{F} - \mathbf{I}\right) = \tfrac{1}{2}(\mathbf{C} - \mathbf{I})\\ &= \tfrac{1}{2}\left[(\mathbf{I} + \boldsymbol{\nabla}_0\mathbf{u})\cdot(\mathbf{I} + \boldsymbol{\nabla}_0\mathbf{u})^{\mathrm{T}} - \mathbf{I}\right]\\ &= \tfrac{1}{2}\left[\boldsymbol{\nabla}_0\mathbf{u} + (\boldsymbol{\nabla}_0\mathbf{u})^{\mathrm{T}} + (\boldsymbol{\nabla}_0\mathbf{u})\cdot(\boldsymbol{\nabla}_0\mathbf{u})^{\mathrm{T}}\right],\end{aligned} \tag{3.4.11}$$

where Eq. (3.3.8) is used in writing $\mathbf{F}$ in terms of $\boldsymbol{\nabla}_0\mathbf{u}$. By definition, the Green–Lagrange strain tensor is a symmetric second-order tensor. Also, the change in the squared lengths is zero if and only if $\mathbf{E} = \mathbf{0}$.

The vector form of the Green–Lagrange strain tensor in Eq. (3.4.11) allows us to express it in terms of its components in any coordinate system. In particular, in the rectangular Cartesian coordinate system (X_1, X_2, X_3), the components of $\mathbf{E}$ are given by

[5] Readers should not confuse the symbol $\mathbf{E}$ used for the Green–Lagrange strain tensor and $\hat{\mathbf{E}}_i$ used for the basis vectors in the reference configuration. One should always pay attention to different typefaces and subscripts used.

$$E_{IJ} = \frac{1}{2}\left(\frac{\partial u_I}{\partial X_J} + \frac{\partial u_J}{\partial X_I} + \frac{\partial u_K}{\partial X_I}\frac{\partial u_K}{\partial X_J}\right), \tag{3.4.12}$$

where summation on repeated index (K) is implied. Clearly, the last term in Eqs. (3.4.11) and (3.4.12) is nonlinear in the displacement gradients. In expanded notation, the Green–Lagrange strain tensor components are given by

$$\begin{aligned}
E_{11} &= \frac{\partial u_1}{\partial X_1} + \frac{1}{2}\left[\left(\frac{\partial u_1}{\partial X_1}\right)^2 + \left(\frac{\partial u_2}{\partial X_1}\right)^2 + \left(\frac{\partial u_3}{\partial X_1}\right)^2\right],\\
E_{22} &= \frac{\partial u_2}{\partial X_2} + \frac{1}{2}\left[\left(\frac{\partial u_1}{\partial X_2}\right)^2 + \left(\frac{\partial u_2}{\partial X_2}\right)^2 + \left(\frac{\partial u_3}{\partial X_2}\right)^2\right],\\
E_{33} &= \frac{\partial u_3}{\partial X_3} + \frac{1}{2}\left[\left(\frac{\partial u_1}{\partial X_3}\right)^2 + \left(\frac{\partial u_2}{\partial X_3}\right)^2 + \left(\frac{\partial u_3}{\partial X_3}\right)^2\right],\\
E_{12} &= \frac{1}{2}\left(\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1} + \frac{\partial u_1}{\partial X_1}\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1}\frac{\partial u_2}{\partial X_2} + \frac{\partial u_3}{\partial X_1}\frac{\partial u_3}{\partial X_2}\right),\\
E_{13} &= \frac{1}{2}\left(\frac{\partial u_1}{\partial X_3} + \frac{\partial u_3}{\partial X_1} + \frac{\partial u_1}{\partial X_1}\frac{\partial u_1}{\partial X_3} + \frac{\partial u_2}{\partial X_1}\frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_1}\frac{\partial u_3}{\partial X_3}\right),\\
E_{23} &= \frac{1}{2}\left(\frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2} + \frac{\partial u_1}{\partial X_2}\frac{\partial u_1}{\partial X_3} + \frac{\partial u_2}{\partial X_2}\frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2}\frac{\partial u_3}{\partial X_3}\right).
\end{aligned} \tag{3.4.13}$$

The components E_{11}, E_{22}, and E_{33} are termed normal strains and E_{12}, E_{23}, and E_{13} are called shear strains, as shown in Fig. 3.4.2.

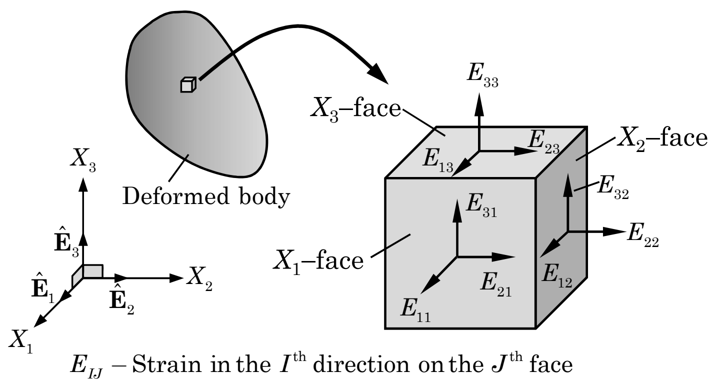


Fig. 3.4.2: Green–Lagrange strain tensor components in rectangular Cartesian coordinates.

3.4.3 Physical Interpretation of Green–Lagrange Strain Tensor Components

To see the physical meaning of the normal strain component E_{11}, consider a line element initially parallel to the X_1-axis, that is, $d\mathbf{X} = dX_1\hat{\mathbf{E}}_1$ in the reference configuration of the body, as shown in Fig. 3.4.3. Then

$$(ds)^2 - (dS)^2 = 2E_{IJ}\,dX_I\,dX_J = 2E_{11}\,dX_1\,dX_1 = 2E_{11}\,(dS)^2.$$

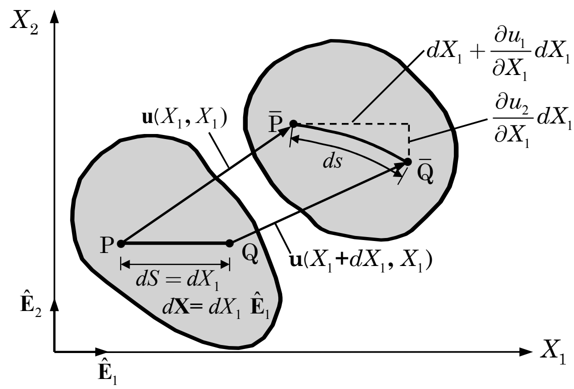


Fig. 3.4.3: Physical interpretation of normal strain component E_{11}.

Solving for E_{11}, we obtain

$$E_{11} = \tfrac{1}{2}\frac{(ds)^2 - (dS)^2}{(dS)^2} = \tfrac{1}{2}\left[\left(\frac{ds}{dS}\right)^2 - 1\right],$$

where the initial and final lengths are given by (approximating the curve as a straight line)

$$(dS)^2 = (dX_1)^2, \quad (ds)^2 = \left(dX_1 + \frac{\partial u_1}{\partial X_1} dX_1\right)^2 + \left(\frac{\partial u_2}{\partial X_1} dX_1\right)^2. \qquad (3.4.14)$$

Thus, we have

$$E_{11} = \tfrac{1}{2}\left[2\frac{\partial u_1}{\partial X_1} + \left(\frac{\partial u_1}{\partial X_1}\right)^2 + \left(\frac{\partial u_2}{\partial X_1}\right)^2\right].$$

We can also write

$$E_{11} = \tfrac{1}{2}\frac{(ds)^2 - (dS)^2}{(dS)^2} = \tfrac{1}{2}\left[\left(\frac{ds}{dS}\right)^2 - 1\right] = \tfrac{1}{2}\left(\lambda^2 - 1\right), \qquad (3.4.15)$$

where λ is the stretch of the line element $d\mathbf{X}$:

$$\lambda = \frac{ds}{dS} = \sqrt{1 + 2E_{11}}. \qquad (3.4.16)$$

The shear strain components E_{IJ}, $I \neq J$, can be interpreted as a measure of the change in the angle between line elements that were perpendicular to each other in the reference configuration. To see this, consider line elements $d\mathbf{X}^{(1)} = dX_1\hat{\mathbf{E}}_1$ and $d\mathbf{X}^{(2)} = dX_2\hat{\mathbf{E}}_2$ in the reference configuration of the body, which are perpendicular to each other, as shown in Fig. 3.4.4. The material line elements $d\mathbf{X}^{(1)}$ and $d\mathbf{X}^{(2)}$ occupy positions $d\mathbf{x}^{(1)}$ and $d\mathbf{x}^{(2)}$, respectively, in the deformed body. Then the cosine of the angle between the line elements $\bar{\mathrm{O}}\bar{\mathrm{Q}}$ and $\bar{\mathrm{O}}\bar{\mathrm{P}}$ in the deformed body is given by [see Eq. (3.3.1)]

$$\cos\theta_{12} = \hat{\mathbf{n}}_1 \cdot \hat{\mathbf{n}}_2 = \frac{d\mathbf{x}^{(1)} \cdot d\mathbf{x}^{(2)}}{|d\mathbf{x}^{(1)}|\,|d\mathbf{x}^{(2)}|} = \frac{[d\mathbf{X}^{(1)} \cdot \mathbf{F}^{\mathrm{T}}] \cdot [\mathbf{F} \cdot d\mathbf{X}^{(2)}]}{\sqrt{d\mathbf{X}^{(1)} \cdot \mathbf{C} \cdot d\mathbf{X}^{(1)}}\sqrt{d\mathbf{X}^{(2)} \cdot \mathbf{C} \cdot d\mathbf{X}^{(2)}}}. \tag{3.4.17}$$

In view of the relations

$$\mathbf{C} = \mathbf{F}^{\mathrm{T}} \cdot \mathbf{F}, \quad \hat{\mathbf{N}}_1 = \hat{\mathbf{E}}_1, \quad \hat{\mathbf{N}}_2 = \hat{\mathbf{E}}_2, \tag{3.4.18}$$

we have

$$\cos\theta_{12} = \frac{\hat{\mathbf{N}}_1 \cdot \mathbf{C} \cdot \mathbf{N}_2}{\sqrt{\hat{\mathbf{N}}_1 \cdot \mathbf{C} \cdot \mathbf{N}_1}\,\sqrt{\hat{\mathbf{N}}_2 \cdot \mathbf{C} \cdot \mathbf{N}_2}} = \frac{C_{12}}{\sqrt{C_{11}}\sqrt{C_{22}}}\,,$$

or

$$\cos\theta_{12} = \frac{C_{12}}{\lambda_1\lambda_2} = \frac{2E_{12}}{\sqrt{(1+2E_{11})}\sqrt{(1+2E_{22})}}\,. \tag{3.4.19}$$

Thus, $2E_{12}$ is equal to the cosine of the angle between the line elements, θ_{12}, multiplied by the product of extension ratios λ_1 and λ_2. Clearly, the finite strain E_{12} depends not only on the angle θ_{12} but also on the stretches of elements involved. When the unit extensions and the angle changes are small compared to unity, we find that $2E_{12}$ is the *decrease* from $\pi/2$:

$$\tfrac{\pi}{2} - \theta_{12} \approx \sin(\tfrac{\pi}{2} - \theta_{12}) = \cos\theta_{12} \approx 2E_{12}. \tag{3.4.20}$$

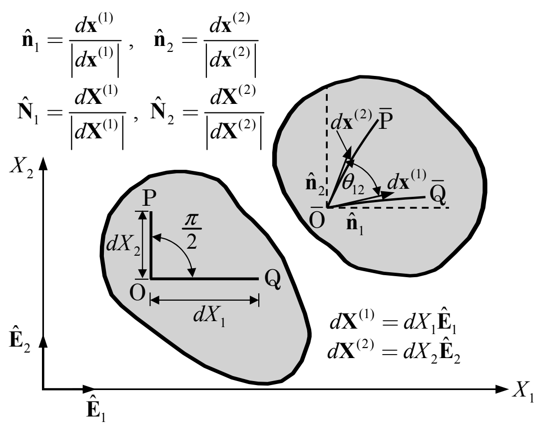


Fig. 3.4.4: Physical interpretation of shear strain component E_{12}.

3.4.4 Cauchy and Euler Strain Tensors

Returning to the strain measures, the change in the squared lengths that occurs as the body deforms from the initial to the current configuration can be expressed relative to the current length. First, we express dS in terms of $d\mathbf{x}$ as

$$(dS)^2 = d\mathbf{X} \cdot d\mathbf{X} = d\mathbf{x} \cdot (\mathbf{F}^{-\mathrm{T}} \cdot \mathbf{F}^{-1}) \cdot d\mathbf{x} \equiv d\mathbf{x} \cdot \tilde{\mathbf{B}} \cdot d\mathbf{x}, \tag{3.4.21}$$

where $\tilde{\mathbf{B}}$ is called the *Cauchy strain tensor*

$$\tilde{\mathbf{B}} = \mathbf{F}^{-\mathrm{T}} \cdot \mathbf{F}^{-1}, \quad \tilde{\mathbf{B}}^{-1} \equiv \mathbf{B} = \mathbf{F} \cdot \mathbf{F}^{\mathrm{T}}, \tag{3.4.22}$$

and $\mathbf{B}$ is the left Cauchy–Green tensor or Finger tensor. We can write

$$(ds)^2 - (dS)^2 = 2\, d\mathbf{x} \cdot \mathbf{e} \cdot d\mathbf{x}, \tag{3.4.23}$$

where $\mathbf{e}$, called the *Almansi–Hamel (Eulerian) strain tensor* or simply the *Euler strain tensor*, is defined as

$$\mathbf{e} = \tfrac{1}{2}\left(\mathbf{I} - \mathbf{F}^{-\mathrm{T}} \cdot \mathbf{F}^{-1}\right) = \tfrac{1}{2}\left(\mathbf{I} - \tilde{\mathbf{B}}\right) \tag{3.4.24}$$

$$= \tfrac{1}{2}\left[\mathbf{I} - (\mathbf{I} - \boldsymbol{\nabla}\mathbf{u}) \cdot (\mathbf{I} - \boldsymbol{\nabla}\mathbf{u})^{\mathrm{T}}\right]$$

$$= \tfrac{1}{2}\left[\boldsymbol{\nabla}\mathbf{u} + (\boldsymbol{\nabla}\mathbf{u})^{\mathrm{T}} - (\boldsymbol{\nabla}\mathbf{u}) \cdot (\boldsymbol{\nabla}\mathbf{u})^{\mathrm{T}}\right]. \tag{3.4.25}$$

The rectangular Cartesian components of $\mathbf{C}$, $\tilde{\mathbf{B}}$, and $\mathbf{e}$ are given by

$$C_{IJ} = \frac{\partial x_k}{\partial X_I}\frac{\partial x_k}{\partial X_J}, \quad \tilde{B}_{ij} = \frac{\partial X_K}{\partial x_i}\frac{\partial X_K}{\partial x_j}, \tag{3.4.26}$$

$$e_{ij} = \tfrac{1}{2}\left(\delta_{ij} - \frac{\partial X_K}{\partial x_i}\frac{\partial X_K}{\partial x_j}\right) = \tfrac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right). \tag{3.4.27}$$

Examples 3.4.1 and 3.4.2 illustrate the calculation of various measures of strain.

Example 3.4.1

For the deformation given in Example 3.3.1, determine the Cartesian components of the right Cauchy–Green deformation tensor $\mathbf{C}$, the Cauchy strain tensor $\tilde{\mathbf{B}}$, and the Green–Lagrange and Almansi strain tensors, $\mathbf{E}$ and $\mathbf{e}$.

Solution: The components of the right Cauchy–Green deformation tensor and the Cauchy strain tensor are

$$[C] = [F]^{\mathrm{T}}[F] = \begin{bmatrix} 1.0 & 0.0 & 0.0 \\ 0.5 & 1.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix}\begin{bmatrix} 1.0 & 0.5 & 0.0 \\ 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix} = \begin{bmatrix} 1.0 & 0.50 & 0.0 \\ 0.5 & 1.25 & 0.0 \\ 0.0 & 0.00 & 1.0 \end{bmatrix},$$

$$[\tilde{B}] = [F]^{-\mathrm{T}}[F]^{-1} = \begin{bmatrix} 1.0 & 0.0 & 0.0 \\ -0.5 & 1.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix}\begin{bmatrix} 1.0 & -0.5 & 0.0 \\ 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix} = \begin{bmatrix} 1.0 & -0.50 & 0.0 \\ -0.5 & 1.25 & 0.0 \\ 0.0 & 0.00 & 1.0 \end{bmatrix}.$$

The Green–Lagrange and Almansi strain tensor components in matrix form are given by

$$[E] = \tfrac{1}{2}\left([C] - [I]\right) = \tfrac{1}{2}\begin{bmatrix} 0.0 & 0.50 & 0.0 \\ 0.5 & 0.25 & 0.0 \\ 0.0 & 0.00 & 0.0 \end{bmatrix}; \quad [e] = \tfrac{1}{2}\left([I] - [\tilde{B}]\right) = \tfrac{1}{2}\begin{bmatrix} 0.0 & 0.50 & 0.0 \\ 0.5 & -0.25 & 0.0 \\ 0.0 & 0.00 & 0.0 \end{bmatrix}.$$

Example 3.4.2

Consider the uniform deformation of a square block $\mathcal{B}$ of side length 2 units, initially centered at $\mathbf{X} = (0, 0)$, as shown in Fig. 3.4.5. The deformation is defined by the mapping

$$\boldsymbol{\chi}(\mathbf{X}) = \tfrac{1}{4}(18 + 4X_1 + 6X_2)\hat{\mathbf{e}}_1 + \tfrac{1}{4}(14 + 6X_2)\hat{\mathbf{e}}_2 + X_3\hat{\mathbf{e}}_3.$$

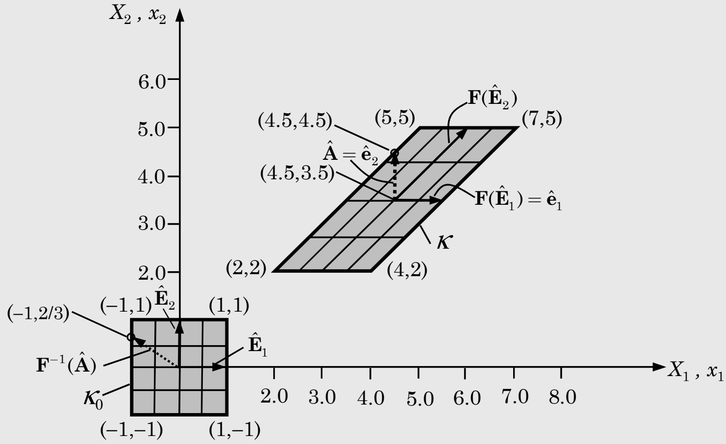


Fig. 3.4.5: Undeformed (κ_0) and deformed (κ) configurations of a rectangular block.

(a) Sketch the deformed configuration κ of the body $\mathcal{B}$.

(b) Compute the components of the deformation gradient $\mathbf{F}$ and its inverse.

(c) Compute the components of the right Cauchy–Green deformation tensor $\mathbf{C}$ and Cauchy strain tensor $\tilde{\mathbf{B}}$.

(d) Compute Green's and Almansi's strain tensor components (E_{IJ} and e_{ij}).

Solution:
(a) A sketch of the deformed configuration of the body is shown in Fig. 3.4.5.

(b) Note that the inverse transformation is given by ($X_3 = x_3$)

$$\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \frac{1}{6}\begin{bmatrix} 6 & -6 \\ 0 & 4 \end{bmatrix}\left(\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} - \frac{1}{4}\begin{Bmatrix} 18 \\ 14 \end{Bmatrix}\right) = -\frac{1}{3}\begin{Bmatrix} 3 \\ 7 \end{Bmatrix} + \frac{1}{3}\begin{bmatrix} 3 & -3 \\ 0 & 2 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix},$$

or

$$\boldsymbol{\chi}^{-1}(\mathbf{x}) = \big(-1 + x_1 - x_2\big)\hat{\mathbf{E}}_1 + \tfrac{1}{3}\big(-7 + 2x_2\big)\hat{\mathbf{E}}_2 + x_3\,\hat{\mathbf{E}}_3.$$

The matrix form of the deformation gradient and its inverse are

$$[F] = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 3 \\ 0 & 3 \end{bmatrix}; \quad [F]^{-1} = \begin{bmatrix} \frac{\partial X_1}{\partial x_1} & \frac{\partial X_1}{\partial x_2} \\ \frac{\partial X_2}{\partial x_1} & \frac{\partial X_2}{\partial x_2} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 3 & -3 \\ 0 & 2 \end{bmatrix}.$$

(c) The right Cauchy–Green deformation tensor and Cauchy strain tensor are, respectively,

$$[C] = [F]^{\mathrm{T}}[F] = \frac{1}{2}\begin{bmatrix} 2 & 3 \\ 3 & 9 \end{bmatrix}, \quad [B] = [F][F]^{\mathrm{T}} = \frac{1}{4}\begin{bmatrix} 13 & 9 \\ 9 & 9 \end{bmatrix}.$$

(d) The Green and Almansi strain tensor components in matrix form are, respectively,

$$[E] = \frac{1}{2}\Big([F]^{\mathrm{T}}[F] - [I]\Big) = \frac{1}{4}\begin{bmatrix} 0 & 3 \\ 3 & 7 \end{bmatrix}, \quad [e] = \frac{1}{2}\Big([I] - [F]^{-\mathrm{T}}[F]^{-1}\Big) = \frac{1}{18}\begin{bmatrix} 0 & 9 \\ 9 & -4 \end{bmatrix}.$$

3.4.5 Transformation of Strain Components

The tensors **E** and **e** can be expressed in any coordinate system much like any second-order tensor. For example, in a rectangular Cartesian system, we have

$$\mathbf{E} = E_{IJ}\,\hat{\mathbf{E}}_I\hat{\mathbf{E}}_J, \qquad \mathbf{e} = e_{ij}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j. \tag{3.4.28}$$

Further, the components of **E** and **e** transform according to Eq. (2.5.21):

$$\bar{E}_{IJ} = \ell_{IK}\,\ell_{JL}\,E_{KL}, \quad \bar{e}_{ij} = \ell_{ik}\,\ell_{j\ell}\,e_{k\ell}, \tag{3.4.29}$$

where ℓ_{IJ} (ℓ_{ij}) denotes the direction cosines [see Eq. (2.2.71)].

Example 3.4.3

Derive the transformation equations between the strain components E_{IJ} referred to (X_1, X_2, X_3) and $\bar{E}_{IJ}$ in the new coordinate system $(\bar{X}_1, \bar{X}_2, \bar{X}_3)$, which is obtained by rotating the former about the X_3-axis counterclockwise by the angle θ.

Solution: The two coordinate systems are related by [see Eq. (2.2.70)]

$$\begin{Bmatrix} \bar{X}_1 \\ \bar{X}_2 \\ \bar{X}_3 \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} \equiv [L] \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}. \tag{3.4.30}$$

Transformation of strain tensor components follows those of a second-order tensor [see Eq. (2.5.21), and note that $[L]^{-1} = [L]^{\mathrm{T}}$]

$$[\bar{E}] = [L][E][L]^{\mathrm{T}}; \quad [E] = [L]^{\mathrm{T}}[\bar{E}][L]. \tag{3.4.31}$$

Carrying out the indicated matrix multiplications and expressing the result in single-column format, we have

$$\begin{Bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ 2E_{23} \\ 2E_{13} \\ 2E_{12} \end{Bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 0 & 0 & 0 & -\frac{1}{2}\sin 2\theta \\ \sin^2\theta & \cos^2\theta & 0 & 0 & 0 & \frac{1}{2}\sin 2\theta \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta & \sin\theta & 0 \\ 0 & 0 & 0 & -\sin\theta & \cos\theta & 0 \\ \sin 2\theta & -\sin 2\theta & 0 & 0 & 0 & \cos 2\theta \end{bmatrix} \begin{Bmatrix} \bar{E}_{11} \\ \bar{E}_{22} \\ \bar{E}_{33} \\ 2\bar{E}_{23} \\ 2\bar{E}_{13} \\ 2\bar{E}_{12} \end{Bmatrix}. \tag{3.4.32}$$

The inverse relations are

$$\begin{Bmatrix} \bar{E}_{11} \\ \bar{E}_{22} \\ \bar{E}_{33} \\ 2\bar{E}_{23} \\ 2\bar{E}_{13} \\ 2\bar{E}_{12} \end{Bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 0 & 0 & 0 & \frac{1}{2}\sin 2\theta \\ \sin^2\theta & \cos^2\theta & 0 & 0 & 0 & -\frac{1}{2}\sin 2\theta \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 & \sin\theta & \cos\theta & 0 \\ -\sin 2\theta & \sin 2\theta & 0 & 0 & 0 & \cos 2\theta \end{bmatrix} \begin{Bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ 2E_{23} \\ 2E_{13} \\ 2E_{12} \end{Bmatrix}. \tag{3.4.33}$$

Example 3.4.4

Consider a rectangular block ($\mathcal{B}$) ABCD of dimensions $a \times b \times h$, where h is the thickness (very small compared to a and b). Suppose that block $\mathcal{B}$ is deformed into the diamond shape $\bar{\mathrm{A}}\bar{\mathrm{B}}\bar{\mathrm{C}}\bar{\mathrm{D}}$ shown in Fig. 3.4.6(a), without a change in its thickness. Determine the deformation mapping, displacements, and strains in the body. Assume that the mapping is a linear polynomial in X_1 and X_2. A *complete* linear polynomial in X_1 and X_2 is of the form $p(X_1, X2) = a_0 + a_1X_1 + a_2X_2 + a_3X_1X_2$.

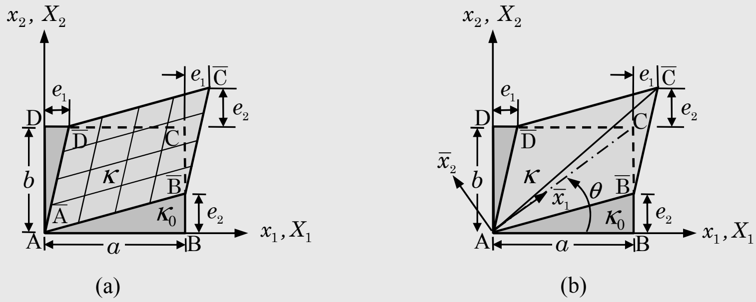


Fig. 3.4.6: Undeformed (κ_0) and deformed (κ) configurations of a rectangular block. Typical material lines inside the body are also shown.

Solution: Let (X_1, X_2, X_3) denote the coordinates of a material point in the reference configuration, κ_0. The X_3-axis is taken out of the plane of the page and not shown in the figure. By assumption, the geometry of the deformed body can be described by the mapping $\boldsymbol{\chi}(\mathbf{x}) = x_1\,\hat{\mathbf{e}}_1 + x_2\,\hat{\mathbf{e}}_2 + x_3\,\hat{\mathbf{e}}_3$, where

$$x_1 = a_0 + a_1X_1 + a_2X_2 + a_3X_1X_2,$$
$$x_2 = b_0 + b_1X_1 + b_2X_2 + b_3X_1X_2,$$
$$x_3 = X_3,$$

and a_0, a_1, a_2, a_3, b_0, b_1, b_2, and b_3 are constants to be determined using the values of (X_1, X_2) from the undeformed configuration and the corresponding values of (x_1, x_2) from the deformed configuration shown in Fig. 3.4.6(a). The eight constants are determined using the 8 conditions provided by the coordinate values at points A, B, C, and D. Since point A is at the origin of the coordinate system, we immediately obtain $a_0 = b_0 = 0$. Next, we have

$$(X_1, X_2) = (a, 0), \quad (x_1, x_2) = (a, e_2) \;\rightarrow\; a_1 = 1, \quad b_1 = \frac{e_2}{a},$$
$$(X_1, X_2) = (0, b), \quad (x_1, x_2) = (e_1, b) \;\rightarrow\; a_2 = \frac{e_1}{b}, \quad b_2 = 1,$$
$$(X_1, X_2) = (a, b), \quad (x_1, x_2) = (a + e_1, b + e_2) \;\rightarrow\; a_3 = 0, \quad b_3 = 0.$$

Thus, the deformation is defined by the transformation

$$\boldsymbol{\chi}(\mathbf{x}) = \big(X_1 + k_1X_2\big)\hat{\mathbf{e}}_1 + \big(X_2 + k_2X_1\big)\hat{\mathbf{e}}_2 + X_3\,\hat{\mathbf{e}}_3, \tag{1}$$

where $k_1 = e_1/b$ and $k_2 = e_2/a$. The inverse mapping is given by

$$\boldsymbol{\chi}^{-1}(\mathbf{X}) = \tfrac{1}{1-k_1k_2}\,(x_1 - k_1x_2)\,\hat{\mathbf{E}}_1 + \tfrac{1}{1-k_1k_2}\,(-k_2x_1 + x_2)\,\hat{\mathbf{E}}_2 + x_3\,\hat{\mathbf{E}}_3. \tag{2}$$

Thus, the displacement vector of a material point in the Lagrangian description is

$$\mathbf{u} = k_1X_2\,\hat{\mathbf{E}}_1 + k_2X_1\,\hat{\mathbf{E}}_2. \tag{3}$$

The only nonzero Green strain tensor components are given by

$$E_{11} = \tfrac{1}{2}k_2^2, \quad 2E_{12} = k_1 + k_2, \quad E_{22} = \tfrac{1}{2}k_1^2. \tag{4}$$

For the infinitesimal case (that is, k_1 and k_2 are small), we only have the shear strain $2\varepsilon_{12} = k_1 + k_2$. The components of the deformation gradient are

$$[F] = \begin{bmatrix} 1 & k_1 & 0 \\ k_2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The case in which $k_2 = 0$ is known as the *simple shear*. The Green's deformation tensor $\mathbf{C}$ is

$$\mathbf{C} = \mathbf{F}^{\mathrm{T}} \cdot \mathbf{F} \quad\rightarrow\quad [C] = [F]^{\mathrm{T}}[F] = \begin{bmatrix} 1 + k_1^2 & k_1 + k_2 & 0 \\ k_1 + k_2 & 1 + k_2^2 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and $2\mathbf{E} = \mathbf{C} - \mathbf{I}$ yields the results given in Eq. (4).

The displacements in the spatial description are

$$\begin{aligned} u_1 &= x_1 - X_1 = k_1 X_2 = \tfrac{k_1}{1-k_1k_2}\left(-k_2 x_1 + x_2\right), \\ u_2 &= x_2 - X_2 = k_2 X_1 = \tfrac{k_2}{1-k_1k_2}\left(x_1 - k_1 x_2\right), \\ u_3 &= x_3 - X_3 = 0. \end{aligned} \tag{5}$$

The Almansi strain tensor components are

$$\begin{aligned} e_{11} &= -\frac{k_1k_2}{1-k_1k_2} - \frac{1}{2}\left[\left(\frac{k_1k_2}{1-k_1k_2}\right)^2 + \left(\frac{k_2}{1-k_1k_2}\right)^2\right], \\ 2e_{12} &= \frac{k_1+k_2}{1-k_1k_2} + \frac{k_1k_2(k_1+k_2)}{(1-k_1k_2)^2}, \\ e_{22} &= -\frac{k_1k_2}{1-k_1k_2} - \frac{1}{2}\left[\left(\frac{k_1k_2}{1-k_1k_2}\right)^2 + \left(\frac{k_1}{1-k_1k_2}\right)^2\right]. \end{aligned} \tag{6}$$

Alternatively, the same results can be obtained using the elementary mechanics of materials approach, where the strains are defined as the ratio of the difference between the final length and original length to the original length. A line element AB in the initial (undeformed) configuration κ_0 of the body $\mathcal{B}$ moves to position $\bar{\mathrm{A}}\bar{\mathrm{B}}$ (point $\bar{\mathrm{A}}$ is the same as point A), as shown in Fig. 3.4.6(a). Then the Green strain in line AB is given by

$$\begin{aligned} E_{11} = E_{\mathrm{AB}} &= \frac{\bar{\mathrm{A}}\bar{\mathrm{B}} - \mathrm{AB}}{\mathrm{AB}} = \tfrac{1}{a}\sqrt{a^2+e_2^2} - 1 = \sqrt{1+\left(\frac{e_2}{a}\right)^2} - 1 \\ &= \left[1 + \tfrac{1}{2}\left(\frac{e_2}{a}\right)^2 + \cdots\right] - 1 \approx \tfrac{1}{2}\left(\frac{e_2}{a}\right)^2 = \tfrac{1}{2}k_2^2, \end{aligned}$$

where cubic and higher powers of e_2/a are considered to be smaller than e_2/a and e_2^2/a^2 and thus neglected. Similarly,

$$E_{22} = \left[1 + \tfrac{1}{2}\left(\frac{e_1}{b}\right)^2 + \cdots\right] - 1 \approx \tfrac{1}{2}\left(\frac{e_1}{b}\right)^2 = \tfrac{1}{2}k_1^2.$$

The shear strain $2E_{12}$ is equal to the change in the angle between two line elements that were originally at 90°, that is, change in the angle DAB. The change is equal to, as can be seen from Fig. 3.4.6(b)

$$2E_{12} = \angle\mathrm{DAB} - \angle\bar{\mathrm{D}}\bar{\mathrm{A}}\bar{\mathrm{B}} = \frac{e_1}{b} + \frac{e_2}{a} = k_1 + k_2.$$

Thus, the strains computed using mechanics of materials approach, when terms of order higher than e_1^2/b^2 and e_2^2/a^2 are neglected, yield the same as those in Eq. (4). On the other hand, if we define E_{11} and E_{22} as (consistent with the definition of the Green–Lagrange strain tensor),

$$\begin{aligned} 2E_{11} &= \frac{(\bar{\mathrm{A}}\bar{\mathrm{B}})^2 - (\mathrm{AB})^2}{(\mathrm{AB})^2} = \frac{a^2+e_2^2}{a^2} - 1 = \frac{e_2^2}{a^2} = k_2^2, \\ 2E_{22} &= \frac{(\bar{\mathrm{A}}\bar{\mathrm{D}})^2 - (\mathrm{AD})^2}{(\mathrm{AD})^2} = \frac{b^2+e_1^2}{b^2} - 1 = \frac{e_1^2}{b^2} = k_1^2, \end{aligned}$$

we obtain the results in Eq. (4) directly, without making any order of magnitude assumption. Thus, in general, the engineering strains defined in mechanics of materials and the Green–Lagrange strains are not the same.

The axial strain in line element AC is [see Fig. 3.4.6(b)]

$$\begin{aligned} E_{\mathrm{AC}} &= \frac{\bar{\mathrm{A}}\bar{\mathrm{C}} - \mathrm{AC}}{\mathrm{AC}} = \tfrac{1}{\sqrt{a^2+b^2}}\sqrt{(a+e_1)^2 + (b+e_2)^2} - 1 \\ &= \tfrac{1}{\sqrt{a^2+b^2}}\sqrt{a^2+b^2+e_1^2+e_2^2+2ae_1+2be_2} - 1 \\ &= \left[1 + \frac{e_1^2+e_2^2+2ae_1+2be_2}{a^2+b^2}\right]^{\frac{1}{2}} - 1 \approx \tfrac{1}{2}\,\frac{e_1^2+e_2^2+2ae_1+2be_2}{a^2+b^2} \\ &= \tfrac{1}{2(a^2+b^2)}\left[a^2k_2^2 + 2ab(k_1+k_2) + b^2k_1^2\right]. \end{aligned}$$

The axial strain E_{AC} can also be computed using the strain transformation equations (3.4.29). The line AC is oriented at $\theta = \tan^{-1}(b/a)$. Hence, we have

$$\beta_{11} = \cos\theta = \frac{a}{\sqrt{a^2+b^2}}, \quad \beta_{12} = \sin\theta = \frac{b}{\sqrt{a^2+b^2}},$$
$$\beta_{21} = -\sin\theta = -\frac{b}{\sqrt{a^2+b^2}}, \quad \beta_{22} = \cos\theta = \frac{a}{\sqrt{a^2+b^2}},$$

and

$$\begin{aligned} E_{AC} \equiv \bar{E}_{11} &= \beta_{1i}\beta_{1j}E_{ij} = \beta_{11}\beta_{11}E_{11} + 2\beta_{11}\beta_{12}E_{12} + \beta_{12}\beta_{12}E_{22} \\ &= \tfrac{1}{2(a^2+b^2)}\left[a^2k_2^2 + 2ab(k_1+k_2) + b^2k_1^2\right], \end{aligned}$$

which is the same as that computed previously.

3.4.6 Invariants and Principal Values of Strains

The principal invariants of the Green–Lagrange strain tensor $\mathbf{E}$ are [see Eqs. (2.5.16) and (2.5.17)]

$$J_1 = \operatorname{tr}\mathbf{E}, \quad J_2 = \frac{1}{2}\left[(\operatorname{tr}\mathbf{E})^2 - \operatorname{tr}\left(\mathbf{E}^2\right)\right], \quad J_3 = |\mathbf{E}|, \tag{3.4.34}$$

where the trace of $\mathbf{E}$, tr $\mathbf{E}$, is defined as the double-dot product of $\mathbf{E}$ with the identity tensor [see Eq. (2.5.15)]

$$\operatorname{tr}\mathbf{E} = \mathbf{E} : \mathbf{I}. \tag{3.4.35}$$

In terms of the rectangular Cartesian components, the three principal invariants of $\mathbf{E}$ have the form

$$J_1 = E_{II}, \quad J_2 = \tfrac{1}{2}\left(E_{II}E_{JJ} - E_{IJ}E_{JI}\right), \quad J_3 = |\mathbf{E}|. \tag{3.4.36}$$

It is of considerable interest (e.g., in the design of structures) to know the maximum and minimum values of the strain at a point. The eigenvalues of the matrix of the strain tensor (see Section 2.5.6), when ordered from large to small, characterize the maximum and minimum normal strains, and the eigenvectors represent the planes on which they occur. The maximum shear strains can be determined once the maximum normal strains are determined. The eigenvalues of a strain tensor are called the *principal values of strain*, and the corresponding eigenvectors are called the *principal directions of strain*.

The eigenvalue problem associated with the strain tensor $\mathbf{E}$ is to find μ and $\mathbf{X}$ such that

$$\mathbf{E}\mathbf{X} = \mu\mathbf{X} \ \text{ for all } \mathbf{X} \neq \mathbf{0}; \quad |[E] - \mu[I]| = 0, \tag{3.4.37}$$

where μ are the principal values and $\{X\}$ are the principal directions. The characteristic equation is of the form

$$-\mu^3 + J_1\mu^2 - J_2\mu + J_3 = 0. \tag{3.4.38}$$

Three eigenvalues μ_1, μ_2, and μ_3 provide the three principal values (one of them is the maximum and one of them is the minimum) of normal strain.

The maximum shear strains E_{ns} are computed from (for additional discussion see Section 4.3.3)

$$\begin{aligned} E_{ns}^2 &= \tfrac{1}{4}\big(\mu_1 - \mu_2\big)^2 \quad \text{for} \quad \hat{\mathbf{n}} = \tfrac{1}{\sqrt{2}}\big(\hat{\mathbf{e}}_1 \pm \hat{\mathbf{e}}_2\big), \\ E_{ns}^2 &= \tfrac{1}{4}\big(\mu_1 - \mu_3\big)^2 \quad \text{for} \quad \hat{\mathbf{n}} = \tfrac{1}{\sqrt{2}}\big(\hat{\mathbf{e}}_1 \pm \hat{\mathbf{e}}_3\big), \\ E_{ns}^2 &= \tfrac{1}{4}\big(\mu_2 - \mu_3\big)^2 \quad \text{for} \quad \hat{\mathbf{n}} = \tfrac{1}{\sqrt{2}}\big(\hat{\mathbf{e}}_2 \pm \hat{\mathbf{e}}_3\big). \end{aligned} \tag{3.4.39}$$

The largest shear strain is given by

$$(E_{ns})_{\max} = \tfrac{1}{2}\big(\mu_{\max} - \mu_{\min}\big), \tag{3.4.40}$$

where $\mu_{\max}$ and $\mu_{\min}$ are the maximum and minimum principal values of strain, respectively. The plane of the maximum shear strain lies between the planes of the maximum and minimum principal strain (that is, oriented at $\pm 45°$ to both planes).

Example 3.4.5 deals with the computation of principal strains and their directions.

Example 3.4.5

The state of strain at a point in an elastic body is given by (10^{-3} in./in.)

$$[E] = \begin{bmatrix} 4 & -4 & 0 \\ -4 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Determine the principal strains and principal directions of the strain.

Solution: In this case, we know from the given matrix that $\mu = 3$ is a root with eigenvector $\mathbf{X}^{(3)} = \pm\hat{\mathbf{E}}_3$. The principal invariants of $[E]$ are

$$J_1 = 4 + 0 + 3 = 7, \quad J_2 = \tfrac{1}{2}\left[7^2 - 4^2 - 3^2 - 2 \times (-4)^2\right] = -4, \quad J_3 = |E| = -48.$$

Hence, the characteristic equation is

$$-\mu^3 + 7\mu^2 + 4\mu - 48 = 0 \quad \rightarrow \quad (-\mu^2 + 4\mu + 16)(\mu - 3) = 0.$$

Then the roots (the principal strains) of the characteristic equation are (10^{-3} in/in.)

$$\mu_1 = 3, \quad \mu_2 = 2(1 + \sqrt{5}), \quad \mu_3 = 2(1 - \sqrt{5}).$$

The eigenvector components $X_I^{(1)}$ associated with $E_1 = \mu_1 = 3$ are calculated from

$$\begin{bmatrix} 4-3 & -4 & 0 \\ -4 & 0-3 & 0 \\ 0 & 0 & 3-3 \end{bmatrix} \begin{Bmatrix} X_1^{(1)} \\ X_2^{(1)} \\ X_3^{(1)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix},$$

which gives $X_1^{(1)} - 4X_2^{(1)} = 0$ and $-4X_1^{(1)} - 3X_2^{(1)} = 0$, or $X_1^{(1)} = X_2^{(1)} = 0$. Using the normalization $(X_1^{(1)})^2 + (X_2^{(1)})^2 + (X_3^{(1)})^2 = 1$, we obtain $X_3^{(1)} = 1$. Thus, the principal direction associated with the principal strain $E_1 = 3$ is $\{\hat{X}^{(1)}\}^{\mathrm{T}} = \pm\{0, 0, 1\}$ or $\hat{\mathbf{X}}^{(1)} = \pm\hat{\mathbf{E}}_3$.

The eigenvector components associated with principal strain $E_2 = \mu_2 = 2(1+\sqrt{5})$ are calculated from

$$\begin{bmatrix} 4-\mu_2 & -4 & 0 \\ -4 & -\mu_2 & 0 \\ 0 & 0 & 3-\mu_2 \end{bmatrix} \begin{Bmatrix} X_1^{(2)} \\ X_2^{(2)} \\ X_3^{(2)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix},$$

which gives

$$X_1^{(2)} = -\tfrac{2+2\sqrt{5}}{4} X_2^{(2)} = -1.618\, X_2^{(2)}, \quad X_3^{(2)} = 0, \quad \Rightarrow \quad \{\hat{X}^{(2)}\} = \pm \begin{Bmatrix} -0.851 \\ 0.526 \\ 0.000 \end{Bmatrix}.$$

Similarly, the eigenvector components associated with principal strain $E_3 = \mu_3 = 2(1-\sqrt{5})$ are obtained as

$$X_1^{(3)} = \tfrac{2+2\sqrt{5}}{4} X_2^{(3)} = 1.618\, X_2^{(3)}, \quad X_3^{(3)} = 0, \quad \Rightarrow \quad \{\hat{X}^{(3)}\} = \pm \begin{Bmatrix} 0.526 \\ 0.851 \\ 0.000 \end{Bmatrix}.$$

Note that the eigenvectors $\hat{\mathbf{X}}^{(1)}$, $\hat{\mathbf{X}}^{(2)}$, and $\hat{\mathbf{X}}^{(3)}$ are mutually orthogonal, as expected.

3.5 Infinitesimal Strain Tensor and Rotation Tensor

3.5.1 Infinitesimal Strain Tensor

When all displacement gradients are small, that is, $|\boldsymbol{\nabla}_0 \mathbf{u}| << 1$, we may neglect the nonlinear terms in the definition of the Green–Lagrange strain tensor $\mathbf{E}$ and obtain the linearized strain tensor $\boldsymbol{\varepsilon}$, called the *infinitesimal strain tensor*. To derive $\boldsymbol{\varepsilon}$ from $\mathbf{E}$, we must linearize $\mathbf{E}$ by using a measure of smallness.

We introduce the nonnegative function

$$\epsilon(t) = \|\boldsymbol{\nabla}_0 \mathbf{u}\|_\infty = \sup_{\mathbf{X}\in\kappa} |\boldsymbol{\nabla}_0 \mathbf{u}|, \tag{3.5.1}$$

where "sup" stands for supremum or the least upper bound of the set of all absolute values of $\boldsymbol{\nabla}_0 \mathbf{u}$ defined for all $\mathbf{X} \in \kappa$. If $f(\boldsymbol{\nabla}_0 \mathbf{u})$ is a scalar-, vector-, or tensor-valued function in the neighborhood of $\boldsymbol{\nabla}_0 \mathbf{u} = \mathbf{0}$ so that there exists a constant c such that

$$\|f(\boldsymbol{\nabla}_0 \mathbf{u})\|_\infty < c\,\epsilon^n,$$

we say that f is of the order ϵ^n, as $\epsilon \to 0$, and write $f = \mathrm{O}(\epsilon^n)$.

If $\mathbf{E}$ is of the order $\mathrm{O}(\epsilon)$ in $\boldsymbol{\nabla}_0 \mathbf{u}$, then we mean

$$\frac{\partial u_I}{\partial X_J} = \mathrm{O}(\epsilon) \quad \text{as} \quad \epsilon \to 0.$$

If terms of the order $\mathrm{O}(\epsilon^2)$, as $\epsilon \to 0$, can be omitted in Eq. (3.4.12), then

$$E_{IJ} = \tfrac{1}{2}\left(\frac{\partial u_I}{\partial X_J} + \frac{\partial u_J}{\partial X_I} + \frac{\partial u_K}{\partial X_I}\frac{\partial u_K}{\partial X_J}\right)$$

can be approximated as

$$E_{IJ} \approx \tfrac{1}{2}\left(\frac{\partial u_I}{\partial X_J} + \frac{\partial u_J}{\partial X_I}\right) = \mathrm{O}(\epsilon) \text{ as } \epsilon \to 0. \tag{3.5.2}$$

Next consider

$$\begin{aligned}\frac{\partial u_I}{\partial x_j} &= \frac{\partial u_I}{\partial X_K}\frac{\partial X_K}{\partial x_j} = \frac{\partial u_I}{\partial X_K}\left(\frac{\partial x_K}{\partial x_j} - \frac{\partial u_K}{\partial x_j}\right)\\ &= \frac{\partial u_I}{\partial X_K}\delta_{Kj} + \mathrm{O}(\epsilon^2) \text{ as } \epsilon \to 0,\\ \frac{\partial u_i}{\partial X_J} &= \frac{\partial u_i}{\partial x_k}\frac{\partial x_k}{\partial X_J} = \frac{\partial u_i}{\partial x_k}\left(\frac{\partial u_k}{\partial X_J} + \frac{\partial X_k}{\partial X_J}\right)\\ &= \frac{\partial u_i}{\partial x_k}\delta_{kJ} + \mathrm{O}(\epsilon^2) \text{ as } \epsilon \to 0.\end{aligned}$$

Thus, when terms of the order $\mathrm{O}(\epsilon^2)$, as $\epsilon \to 0$, are neglected, it is immaterial whether the partial derivative of the displacement field $\mathbf{u}$ is taken with respect to x_j or X_j so that $\frac{\partial u_i}{\partial x_j} = \frac{\partial u_i}{\partial X_j}$; that is, $|\boldsymbol{\nabla}\mathbf{u}| \approx |\boldsymbol{\nabla}_0\mathbf{u}| = \mathrm{O}(\epsilon)$. In other words, in the case of infinitesimal strains, no distinction is made between the material coordinates $\mathbf{X}$ and the spatial coordinates $\mathbf{x}$, and it is not necessary to distinguish between the Green–Lagrange strain tensor $\mathbf{E}$ and the Eulerian strain tensor $\mathbf{e}$. The infinitesimal strain tensor $\boldsymbol{\varepsilon}$ is defined as [see Eq. (3.5.2)]

$$\mathbf{E} \approx \boldsymbol{\varepsilon} = \tfrac{1}{2}\left[\boldsymbol{\nabla}_0\mathbf{u} + (\boldsymbol{\nabla}_0\mathbf{u})^{\mathrm{T}}\right]. \tag{3.5.3}$$

The rectangular Cartesian components of the infinitesimal strain tensor are

$$\varepsilon_{IJ} = \tfrac{1}{2}\left(\frac{\partial u_I}{\partial X_J} + \frac{\partial u_J}{\partial X_I}\right), \tag{3.5.4}$$

or, in expanded form,

$$\begin{aligned}\varepsilon_{11} &= \frac{\partial u_1}{\partial X_1}; & \varepsilon_{22} &= \frac{\partial u_2}{\partial X_2};\\ \varepsilon_{33} &= \frac{\partial u_3}{\partial X_3}; & \varepsilon_{12} &= \tfrac{1}{2}\left(\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1}\right);\\ \varepsilon_{13} &= \tfrac{1}{2}\left(\frac{\partial u_1}{\partial X_3} + \frac{\partial u_3}{\partial X_1}\right); & \varepsilon_{23} &= \tfrac{1}{2}\left(\frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2}\right).\end{aligned} \tag{3.5.5}$$

The strain components ε_{11}, ε_{22}, and ε_{33} are the infinitesimal normal strains and ε_{12}, ε_{13}, and ε_{23} are the infinitesimal shear strains. The shear strains $\gamma_{12} = 2\varepsilon_{12}$, $\gamma_{13} = 2\varepsilon_{13}$, and $\gamma_{23} = 2\varepsilon_{23}$ are called the *engineering shear strains.*

3.5.2 Physical Interpretation of Infinitesimal Strain Tensor Components

To gain insight into the physical meaning of the infinitesimal strain components, we write Eq. (3.4.10) in the form

$$(ds)^2 - (dS)^2 = 2\, d\mathbf{X}\cdot\boldsymbol{\varepsilon}\cdot d\mathbf{X} = 2\,\varepsilon_{ij}\, dX_i\, dX_j,$$

and dividing throughout by $(dS)^2$, we obtain

$$\frac{(ds)^2 - (dS)^2}{(dS)^2} = 2\,\varepsilon_{ij}\,\frac{dX_i}{dS}\,\frac{dX_j}{dS}.$$

Let $d\mathbf{X}/dS = \hat{\mathbf{N}}$, the unit vector in the direction of $d\mathbf{X}$. For small deformations, we have $ds + dS \approx 2dS$, and therefore we have

$$\frac{ds - dS}{dS} = \hat{\mathbf{N}} \cdot \boldsymbol{\varepsilon} \cdot \hat{\mathbf{N}} = \varepsilon_{ij}\, N_i\, N_j. \tag{3.5.6}$$

The left side of Eq. (3.5.6) is the ratio of change in length per unit original length for a line element in the direction of $\hat{\mathbf{N}}$. For example, consider $\hat{\mathbf{N}}$ along the X_1-direction. Then we have

$$\frac{ds - dS}{dS} = \varepsilon_{11}.$$

Thus, the normal strain ε_{11} is the ratio of change in length of a line element that was parallel to the X_1-axis in the undeformed body to its original length. Similarly, for a line element along X_2 direction, $(ds - dS)/dS$ is the normal strain ε_{22}, and for a line element along the X_3 direction, $(ds - dS)/dS$ denotes the normal strain ε_{33}.

To understand the meaning of shear components of infinitesimal strain tensor, consider line elements $d\mathbf{X}^{(1)}$ and $d\mathbf{X}^{(2)}$ at a point in the body, which deform into line elements $d\mathbf{x}^{(1)}$ and $d\mathbf{x}^{(2)}$, respectively. Then we have [see Eqs. (3.3.1) and (3.4.3), and the first line of Eq. (3.4.11)]:

$$\begin{aligned} d\mathbf{x}^{(1)} \cdot d\mathbf{x}^{(2)} &= d\mathbf{X}^{(1)} \cdot \mathbf{F}^{\mathrm{T}} \cdot \mathbf{F} \cdot d\mathbf{X}^{(2)} = d\mathbf{X}^{(1)} \cdot \mathbf{C} \cdot d\mathbf{X}^{(2)} \\ &= d\mathbf{X}^{(1)} \cdot (\mathbf{I} + 2\mathbf{E}) \cdot d\mathbf{X}^{(2)} \\ &= d\mathbf{X}^{(1)} \cdot d\mathbf{X}^{(2)} + 2d\mathbf{X}^{(1)} \cdot \mathbf{E} \cdot d\mathbf{X}^{(2)}. \end{aligned} \tag{3.5.7}$$

Now suppose that the line elements $d\mathbf{X}^{(1)}$ and $d\mathbf{X}^{(2)}$ are orthogonal to each other. Then

$$d\mathbf{x}^{(1)} \cdot d\mathbf{x}^{(2)} = 2d\mathbf{X}^{(1)} \cdot \mathbf{E} \cdot d\mathbf{X}^{(2)},$$

or

$$\begin{aligned} 2d\mathbf{X}^{(1)} \cdot \mathbf{E} \cdot d\mathbf{X}^{(2)} &= dx^{(1)} dx^{(2)} \cos\theta = dx^{(1)} dx^{(2)} \cos(\tfrac{\pi}{2} - \gamma_1 - \gamma_2) \\ &= dx^{(1)} dx^{(2)} \sin(\gamma_1 + \gamma_2) = dx^{(1)} dx^{(2)} \sin\gamma, \end{aligned} \tag{3.5.8}$$

where θ is the angle between the deformed line elements $dx^{(1)}$ and $dx^{(2)}$ and $\gamma = \gamma_1 + \gamma_2$ is the change in the angle from 90°. For small deformations, we take $\sin\gamma \approx \gamma$, and obtain

$$\gamma = 2\frac{d\mathbf{X}^{(1)}}{dx^{(1)}} \cdot \mathbf{E} \cdot \frac{d\mathbf{X}^{(2)}}{dx^{(2)}} = 2\hat{\mathbf{N}}^{(1)} \cdot \boldsymbol{\varepsilon} \cdot \hat{\mathbf{N}}^{(2)}, \tag{3.5.9}$$

where $\hat{\mathbf{N}}^{(1)} = d\mathbf{X}^{(1)}/dx^{(1)}$ and $\hat{\mathbf{N}}^{(2)} = d\mathbf{X}^{(2)}/dx^{(2)}$ are the unit vectors along the line elements $d\mathbf{X}^{(1)}$ and $d\mathbf{X}^{(2)}$, respectively. If the line elements $d\mathbf{X}^{(1)}$ and $d\mathbf{X}^{(2)}$ are taken along the X_1 and X_2 coordinates, respectively, then we have $2\varepsilon_{12} = \gamma$. Thus, the engineering shear strain $\gamma_{12} = 2\varepsilon_{12}$ represents the change in angle between line elements that were perpendicular to each other in the undeformed body.

3.5.3 Infinitesimal Rotation Tensor

The displacement gradient tensor can be expressed as the sum of a symmetric tensor and a skew symmetric tensor. We have

$$(\nabla \mathbf{u})^{\mathrm{T}} = \tfrac{1}{2}\left[(\nabla \mathbf{u})^{\mathrm{T}} + \nabla \mathbf{u}\right] + \tfrac{1}{2}\left[(\nabla \mathbf{u})^{\mathrm{T}} - \nabla \mathbf{u}\right] \equiv \tilde{\boldsymbol{\varepsilon}} + \boldsymbol{\Omega}, \tag{3.5.10}$$

where the symmetric part is similar to the infinitesimal strain tensor (and $\tilde{\boldsymbol{\varepsilon}} \approx \boldsymbol{\varepsilon}$ when $|\nabla \mathbf{u}| \approx |\nabla_0 \mathbf{u}| << 1$), and the skew symmetric part is known as the *infinitesimal rotation tensor*

$$\boldsymbol{\Omega} = \tfrac{1}{2}\left[(\nabla \mathbf{u})^{\mathrm{T}} - \nabla \mathbf{u}\right]. \tag{3.5.11}$$

We note that there is no restriction placed on the magnitude of $\nabla \mathbf{u}$ in writing Eq. (3.5.10); $\tilde{\varepsilon}$ and Ω do not have the meaning of infinitesimal strain and infinitesimal rotation tensors unless the deformation is infinitesimal (that is, $|\nabla_0 \mathbf{u}| \approx |\nabla \mathbf{u}|$).

From the definition, it follows that $\boldsymbol{\Omega}$ is a skew symmetric tensor, that is, $\boldsymbol{\Omega}^{\mathrm{T}} = -\boldsymbol{\Omega}$. In Cartesian component form we have

$$\Omega_{ij} = \tfrac{1}{2}\left(u_{i,j} - u_{j,i}\right), \qquad \Omega_{ij} = -\Omega_{ji}. \tag{3.5.12}$$

Thus, there are only three independent components of Ω:

$$[\Omega] = \begin{bmatrix} 0 & \Omega_{12} & \Omega_{13} \\ -\Omega_{12} & 0 & \Omega_{23} \\ -\Omega_{13} & -\Omega_{23} & 0 \end{bmatrix}. \tag{3.5.13}$$

The three components can be used to define the components of a vector $\boldsymbol{\omega}$,

$$\begin{aligned} \boldsymbol{\Omega} &= -\mathcal{E} \cdot \boldsymbol{\omega} \quad \text{or} \quad \boldsymbol{\omega} = -\tfrac{1}{2}\,\mathcal{E} : \boldsymbol{\Omega}, \\ \Omega_{ij} &= -e_{ijk}\,\omega_k \quad \text{or} \quad \omega_i = -\tfrac{1}{2}\,e_{ijk}\,\Omega_{jk}, \end{aligned} \tag{3.5.14}$$

where $\mathcal{E}$ is the permutation (alternating) tensor, $\mathcal{E} = e_{ijk}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j\hat{\mathbf{e}}_k$. In view of Eqs. (3.5.11) and (3.5.14), it follows that

$$\omega_i = \tfrac{1}{2}\,e_{ijk}\,\frac{\partial u_k}{\partial x_j} \quad \text{or} \quad \boldsymbol{\omega} = \tfrac{1}{2}\,\nabla \times \mathbf{u}. \tag{3.5.15}$$

Infinitesimal displacements of the form $d\mathbf{u} = \boldsymbol{\Omega} \cdot d\mathbf{x}$, where $\boldsymbol{\Omega}$ is independent of the position $\mathbf{x}$, are rigid-body rotations because

$$du_i = \Omega_{ij}\,dx_j = -e_{ijk}\,\omega_k\,dx_j = -(d\mathbf{x} \times \boldsymbol{\omega})_i = (\boldsymbol{\omega} \times d\mathbf{x})_i \quad \text{or} \quad d\mathbf{u} = \boldsymbol{\omega} \times d\mathbf{x}.$$

Thus, $\boldsymbol{\omega}$ represents the *infinitesimal rotation vector*; its magnitude is the angle of rotation and its direction gives the axis of rotation. We also note that $\nabla \cdot \boldsymbol{\omega} = 0$. Such vectors are called *solenoidal*. A *rigid-body motion* is one in which the relative distance between points is preserved.

Certain motions do not produce infinitesimal strains but they may produce finite strains. For example, consider the following deformation mapping:

$$\begin{aligned} \boldsymbol{\chi}(\mathbf{X}) = &(b_1 + X_1 + c_2X_3 - c_3X_2)\,\hat{\mathbf{e}}_1 + (b_2 + X_2 + c_3X_1 - c_1X_3)\,\hat{\mathbf{e}}_2 \\ &+ (b_3 + X_3 + c_1X_2 - c_2X_1)\,\hat{\mathbf{e}}_3, \end{aligned} \tag{3.5.16}$$

where b_i and c_i $(i = 1, 2, 3)$ are arbitrary constants. The displacement vector is

$$\begin{aligned}\mathbf{u}(\mathbf{X}) = (b_1 + c_2X_3 - c_3X_2)\,\hat{\mathbf{e}}_1 + (b_2 + c_3X_1 - c_1X_3)\,\hat{\mathbf{e}}_2 \\ + (b_3 + c_1X_2 - c_2X_1)\,\hat{\mathbf{e}}_3.\end{aligned} \tag{3.5.17}$$

Therefore, we have

$$\begin{aligned}
&\frac{\partial u_1}{\partial X_1} = 0, \quad &&\frac{\partial u_1}{\partial X_2} = -c_3, \quad &&\frac{\partial u_1}{\partial X_3} = c_2,\\
&\frac{\partial u_2}{\partial X_1} = c_3, \quad &&\frac{\partial u_2}{\partial X_2} = 0, \quad &&\frac{\partial u_2}{\partial X_3} = -c_1,\\
&\frac{\partial u_3}{\partial X_1} = -c_2, \quad &&\frac{\partial u_3}{\partial X_2} = c_1, \quad &&\frac{\partial u_3}{\partial X_3} = 0,
\end{aligned}$$

Then the components of the deformation gradient **F** and left Cauchy–Green deformation tensor **C** associated with the mapping are

$$[F] = \begin{bmatrix} 1 & -c_3 & c_2 \\ c_3 & 1 & -c_1 \\ -c_2 & c_1 & 1 \end{bmatrix}, \quad [C] = \begin{bmatrix} 1 + c_2^2 + c_3^2 & -c_1c_2 & -c_1c_3 \\ -c_1c_2 & 1 + c_1^2 + c_3^2 & -c_2c_3 \\ -c_1c_3 & -c_2c_3 & 1 + c_1^2 + c_2^2 \end{bmatrix},$$

and the matrix of Green–Lagrange strain tensor components is

$$[E] = \frac{1}{2}\begin{bmatrix} c_2^2 + c_3^2 & -c_1c_2 & -c_1c_3 \\ -c_1c_2 & c_1^2 + c_3^2 & -c_2c_3 \\ -c_1c_3 & -c_2c_3 & c_1^2 + c_2^2 \end{bmatrix}. \tag{3.5.18}$$

Note that the linearized strains are all zero. Thus, for nonzero values of the constants c_i, the mapping produces nonzero finite strains. When all of the constants c_i are either zero or negligibly small (so that their products and squares are very small compared to unity), then $[F] = [I]$ and $[C] = [I]$, implying that the mapping $\mathbf{F} = \mathbf{R}$ represents a rigid-body rotation [that is, $\mathbf{U} = \mathbf{I}$; see Section 3.3.1). Figure 3.5.1 depicts the deformation for the two-dimensional case, with $b_1 = 2$, $b_2 = 3$, $c_3 = 1$, and all other constants zero. Thus, the finite strain tensor and deformation gradient give true measures of the deformation. The question of smallness of c_i in a given engineering application must be carefully examined before using linearized strains.

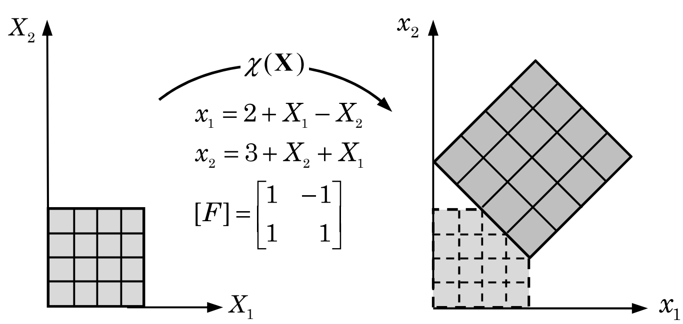


Fig. 3.5.1: A mapping that produces zero infinitesimal strains but nonzero finite strains.

Next, consider the mapping

$$\boldsymbol{\chi}(\mathbf{X}) = (u_1 + X_1\cos\theta - X_2\sin\theta)\,\hat{\mathbf{e}}_1 + (u_2 + X_1\sin\theta + X_2\cos\theta)\,\hat{\mathbf{e}}_2 + X_3\,\hat{\mathbf{e}}_3, \tag{3.5.19}$$

where u_1 and u_2 denote the horizontal and vertical displacements of the point $(0,0,0)$, as shown in Fig. 3.5.2.

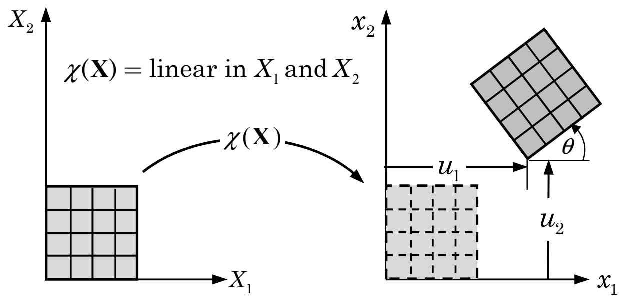


Fig. 3.5.2: A mapping that produces nonzero infinitesimal strains but zero finite strains.

The components of the deformation gradient **F** and left Cauchy–Green deformation tensor **C** are

$$[F] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad [C] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.5.20}$$

Since $\mathbf{C} = \mathbf{I}$, we have $\mathbf{E} = \mathbf{0}$, indicating that the body does not experience stretching or shearing. The mapping is a rigid-body motion (both rigid-body translation and rigid-body rotation).

If we linearize the deformation mapping by making the approximations $\cos\theta \approx 1$ and $\sin\theta \approx 0$, we obtain

$$[F] = \begin{bmatrix} 1 & -\theta & 0 \\ \theta & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad [C] = \begin{bmatrix} 1+\theta^2 & 0 & 0 \\ 0 & 1+\theta^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus, the Green strain tensor components are no longer zero. The principal stretches $\lambda_1 = \lambda_2 = 1 + \theta^2$ are not equal to 1, as required by the definition of rigid-body motion. Owing to the artificial stretch induced by the linearization of the mapping, the stretches get larger and larger as the block rotates.

3.5.4 Infinitesimal Strains in Cylindrical and Spherical Coordinate Systems

The strains defined by Eq. (3.5.3) are valid in any coordinate system. Hence, they can be expressed in component form in any given coordinate system by expanding the strain tensors in the dyadic form and the operator $\boldsymbol{\nabla}_0 = \boldsymbol{\nabla}$ in that coordinate system, as given in Table 2.4.2 (see also Fig. 2.4.5).

3.5.4.1 Cylindrical coordinate system

In the cylindrical coordinate system we have

$$\mathbf{u} = u_r\hat{\mathbf{e}}_r + u_\theta\hat{\mathbf{e}}_\theta + u_z\hat{\mathbf{e}}_z, \tag{3.5.21}$$

$$\boldsymbol{\nabla}_0 = \hat{\mathbf{e}}_r\frac{\partial}{\partial r} + \frac{1}{r}\hat{\mathbf{e}}_\theta\frac{\partial}{\partial \theta} + \hat{\mathbf{e}}_z\frac{\partial}{\partial z}, \tag{3.5.22}$$

$$\frac{\partial\hat{\mathbf{e}}_r}{\partial\theta} = \hat{\mathbf{e}}_\theta, \qquad \frac{\partial\hat{\mathbf{e}}_\theta}{\partial\theta} = -\hat{\mathbf{e}}_r\,. \tag{3.5.23}$$

Using Eqs. (3.5.21)–(3.5.23), we obtain [see Eq. (2.5.27)]

$$\begin{aligned}\boldsymbol{\nabla}_0\mathbf{u} = {} & \hat{\mathbf{e}}_r\hat{\mathbf{e}}_r\frac{\partial u_r}{\partial r} + \hat{\mathbf{e}}_r\hat{\mathbf{e}}_\theta\frac{\partial u_\theta}{\partial r} + \frac{1}{r}\hat{\mathbf{e}}_\theta\hat{\mathbf{e}}_r\left(\frac{\partial u_r}{\partial\theta} - u_\theta\right)\\ & + \hat{\mathbf{e}}_r\hat{\mathbf{e}}_z\frac{\partial u_z}{\partial r} + \hat{\mathbf{e}}_z\hat{\mathbf{e}}_r\frac{\partial u_r}{\partial z} + \frac{1}{r}\left(u_r + \frac{\partial u_\theta}{\partial\theta}\right)\hat{\mathbf{e}}_\theta\hat{\mathbf{e}}_\theta\\ & + \frac{1}{r}\hat{\mathbf{e}}_\theta\hat{\mathbf{e}}_z\frac{\partial u_z}{\partial\theta} + \hat{\mathbf{e}}_z\hat{\mathbf{e}}_\theta\frac{\partial u_\theta}{\partial z} + \hat{\mathbf{e}}_z\hat{\mathbf{e}}_z\frac{\partial u_z}{\partial z}\,,\end{aligned} \tag{3.5.24}$$

$$\begin{aligned}(\boldsymbol{\nabla}_0\mathbf{u})^{\mathrm{T}} = {} & \hat{\mathbf{e}}_r\hat{\mathbf{e}}_r\frac{\partial u_r}{\partial r} + \hat{\mathbf{e}}_\theta\hat{\mathbf{e}}_r\frac{\partial u_\theta}{\partial r} + \frac{1}{r}\hat{\mathbf{e}}_r\hat{\mathbf{e}}_\theta\left(\frac{\partial u_r}{\partial\theta} - u_\theta\right)\\ & + \hat{\mathbf{e}}_z\hat{\mathbf{e}}_r\frac{\partial u_z}{\partial r} + \hat{\mathbf{e}}_r\hat{\mathbf{e}}_z\frac{\partial u_r}{\partial z} + \frac{1}{r}\hat{\mathbf{e}}_\theta\hat{\mathbf{e}}_\theta\left(u_r + \frac{\partial u_\theta}{\partial\theta}\right)\\ & + \frac{1}{r}\hat{\mathbf{e}}_z\hat{\mathbf{e}}_\theta\frac{\partial u_z}{\partial\theta} + \hat{\mathbf{e}}_\theta\hat{\mathbf{e}}_z\frac{\partial u_\theta}{\partial z} + \hat{\mathbf{e}}_z\hat{\mathbf{e}}_z\frac{\partial u_z}{\partial z}\,.\end{aligned} \tag{3.5.25}$$

Substituting the above expressions into Eq. (3.5.3) and collecting the coefficients of various dyadics (that is, coefficients of $\hat{\mathbf{e}}_r\hat{\mathbf{e}}_r$, $\hat{\mathbf{e}}_r\hat{\mathbf{e}}_\theta$, and so on) we obtain the infinitesimal strain tensor components

$$\begin{aligned} \varepsilon_{rr} &= \frac{\partial u_r}{\partial r}, & \varepsilon_{r\theta} &= \tfrac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right),\\ \varepsilon_{rz} &= \tfrac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right), & \varepsilon_{\theta\theta} &= \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta},\\ \varepsilon_{z\theta} &= \tfrac{1}{2}\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial\theta}\right), & \varepsilon_{zz} &= \frac{\partial u_z}{\partial z}\,. \end{aligned} \tag{3.5.26}$$

3.5.4.2 Spherical coordinate system

In the spherical coordinate system, we have

$$\mathbf{u} = u_R\,\hat{\mathbf{e}}_R + u_\phi\,\hat{\mathbf{e}}_\phi + u_\theta\,\hat{\mathbf{e}}_\theta, \tag{3.5.27}$$

$$\boldsymbol{\nabla}_0 = \hat{\mathbf{e}}_R\,\frac{\partial}{\partial R} + \frac{1}{R}\,\hat{\mathbf{e}}_\phi\frac{\partial}{\partial\phi} + \frac{1}{R\sin\phi}\hat{\mathbf{e}}_\theta\,\frac{\partial}{\partial\theta}\,, \tag{3.5.28}$$

$$\begin{aligned} &\frac{\partial\hat{\mathbf{e}}_R}{\partial\phi} = \hat{\mathbf{e}}_\phi, \quad \frac{\partial\hat{\mathbf{e}}_R}{\partial\theta} = \sin\phi\,\hat{\mathbf{e}}_\theta, \quad \frac{\partial\hat{\mathbf{e}}_\phi}{\partial\phi} = -\hat{\mathbf{e}}_R,\\ &\frac{\partial\hat{\mathbf{e}}_\phi}{\partial\theta} = \cos\phi\,\hat{\mathbf{e}}_\theta, \quad \frac{\partial\hat{\mathbf{e}}_\theta}{\partial\theta} = -\sin\phi\,\hat{\mathbf{e}}_R - \cos\phi\,\hat{\mathbf{e}}_\phi\,. \end{aligned} \tag{3.5.29}$$

Using Eqs. (3.5.27)–(3.5.29), we obtain [see Eq. (2.5.29)]

$$\begin{aligned}
\boldsymbol{\nabla}_0\mathbf{u} = & \frac{\partial u_R}{\partial R}\hat{\mathbf{e}}_R\,\hat{\mathbf{e}}_R + \frac{\partial u_\phi}{\partial R}\hat{\mathbf{e}}_R\,\hat{\mathbf{e}}_\phi + \frac{\partial u_\theta}{\partial R}\hat{\mathbf{e}}_R\,\hat{\mathbf{e}}_\theta \\
& + \frac{1}{R}\left(\frac{\partial u_R}{\partial \phi} - u_\phi\right)\hat{\mathbf{e}}_\phi\,\hat{\mathbf{e}}_R + \frac{1}{R}\left(\frac{\partial u_\phi}{\partial \phi} + u_R\right)\hat{\mathbf{e}}_\phi\,\hat{\mathbf{e}}_\phi + \frac{1}{R}\frac{\partial u_\theta}{\partial \phi}\hat{\mathbf{e}}_\phi\,\hat{\mathbf{e}}_\theta \\
& + \frac{1}{R\sin\phi}\left(\frac{\partial u_R}{\partial \theta} - u_\theta\sin\phi\right)\hat{\mathbf{e}}_\theta\,\hat{\mathbf{e}}_R + \frac{1}{R\sin\phi}\left(\frac{\partial u_\phi}{\partial \theta} - u_\theta\cos\phi\right)\hat{\mathbf{e}}_\theta\,\hat{\mathbf{e}}_\phi \\
& + \frac{1}{R\sin\phi}\left(\frac{\partial u_\theta}{\partial \theta} + u_R\sin\phi + u_\phi\cos\phi\right)\hat{\mathbf{e}}_\theta\,\hat{\mathbf{e}}_\theta, \qquad (3.5.30)
\end{aligned}$$

$$\begin{aligned}
(\boldsymbol{\nabla}_0\mathbf{u})^{\mathrm{T}} = & \frac{\partial u_R}{\partial R}\hat{\mathbf{e}}_R\,\hat{\mathbf{e}}_R + \frac{\partial u_\phi}{\partial R}\hat{\mathbf{e}}_\phi\,\hat{\mathbf{e}}_R + \frac{\partial u_\theta}{\partial R}\hat{\mathbf{e}}_\theta\,\hat{\mathbf{e}}_R \\
& + \frac{1}{R}\left(\frac{\partial u_R}{\partial \phi} - u_\phi\right)\hat{\mathbf{e}}_R\,\hat{\mathbf{e}}_\phi + \frac{1}{R}\left(\frac{\partial u_\phi}{\partial \phi} + u_R\right)\hat{\mathbf{e}}_\phi\,\hat{\mathbf{e}}_\phi + \frac{1}{R}\frac{\partial u_\theta}{\partial \phi}\hat{\mathbf{e}}_\theta\,\hat{\mathbf{e}}_\phi \\
& + \frac{1}{R\sin\phi}\left(\frac{\partial u_R}{\partial \theta} - u_\theta\sin\phi\right)\hat{\mathbf{e}}_R\,\hat{\mathbf{e}}_\theta + \frac{1}{R\sin\phi}\left(\frac{\partial u_\phi}{\partial \theta} - u_\theta\cos\phi\right)\hat{\mathbf{e}}_\phi\,\hat{\mathbf{e}}_\theta \\
& + \frac{1}{R\sin\phi}\left(\frac{\partial u_\theta}{\partial \theta} + u_R\sin\phi + u_\phi\cos\phi\right)\hat{\mathbf{e}}_\theta\,\hat{\mathbf{e}}_\theta. \qquad (3.5.31)
\end{aligned}$$

Substituting the above expressions into Eq. (3.5.3) and collecting the coefficients of various dyadics, we obtain the following infinitesimal strain tensor components in the spherical coordinate system:

$$\begin{aligned}
\varepsilon_{RR} &= \frac{\partial u_R}{\partial R}, \quad \varepsilon_{\phi\phi} = \frac{1}{R}\left(\frac{\partial u_\phi}{\partial \phi} + u_R\right), \\
\varepsilon_{R\phi} &= \tfrac{1}{2}\left(\frac{1}{R}\frac{\partial u_R}{\partial \phi} + \frac{\partial u_\phi}{\partial R} - \frac{u_\phi}{R}\right), \\
\varepsilon_{R\theta} &= \tfrac{1}{2}\left(\frac{1}{R\sin\phi}\frac{\partial u_R}{\partial \theta} + \frac{\partial u_\theta}{\partial R} - \frac{u_\theta}{R}\right), \qquad (3.5.32) \\
\varepsilon_{\phi\theta} &= \tfrac{1}{2}\frac{1}{R}\left(\frac{1}{\sin\phi}\frac{\partial u_\phi}{\partial \theta} + \frac{\partial u_\theta}{\partial \phi} - u_\theta\cot\phi\right), \\
\varepsilon_{\theta\theta} &= \frac{1}{R\sin\phi}\left(\frac{\partial u_\theta}{\partial \theta} + u_R\sin\phi + u_\phi\cos\phi\right).
\end{aligned}$$

3.6 Velocity Gradient and Vorticity Tensors

3.6.1 Definitions

In fluid mechanics, the velocity vector $\mathbf{v}(\mathbf{x}, t)$ is the variable of interest. Similar to the displacement gradient tensor [see Eq. (3.5.10)], we can write the *velocity gradient tensor* $\mathbf{L}$ as the sum of symmetric $\mathbf{D}$ and skew-symmetric $\mathbf{W}$ tensors:

$$\mathbf{L} \equiv (\boldsymbol{\nabla}\mathbf{v})^{\mathrm{T}} = \tfrac{1}{2}\left[(\boldsymbol{\nabla}\mathbf{v})^{\mathrm{T}} + \boldsymbol{\nabla}\mathbf{v}\right] + \tfrac{1}{2}\left[(\boldsymbol{\nabla}\mathbf{v})^{\mathrm{T}} - \boldsymbol{\nabla}\mathbf{v}\right] \equiv \mathbf{D} + \mathbf{W}, \qquad (3.6.1)$$

where $\mathbf{D}$ is called the *rate of deformation tensor* (or *rate of strain tensor*) and $\mathbf{W}$ is called the *vorticity tensor* or *spin tensor*:

$$\mathbf{D} = \tfrac{1}{2}\left[(\boldsymbol{\nabla}\mathbf{v})^{\mathrm{T}} + \boldsymbol{\nabla}\mathbf{v}\right], \quad \mathbf{W} = \tfrac{1}{2}\left[(\boldsymbol{\nabla}\mathbf{v})^{\mathrm{T}} - \boldsymbol{\nabla}\mathbf{v}\right]. \qquad (3.6.2)$$

It follows that

$$\mathbf{D} = \tfrac{1}{2}\left(\mathbf{L} + \mathbf{L}^{\mathrm{T}}\right), \quad \mathbf{W} = \tfrac{1}{2}\left(\mathbf{L} - \mathbf{L}^{\mathrm{T}}\right). \tag{3.6.3}$$

The skew-symmetric tensor $\mathbf{W}$ (i.e., $\mathbf{W}^{\mathrm{T}} = -\mathbf{W}$), has only three independent scalar components; they can be used to define the scalar components of a vector $\mathbf{w}$, called the *axial vector of* $\mathbf{W}$, as follows:

$$W_{ij} = -e_{ijk}w_k, \quad [W] = \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix}. \tag{3.6.4}$$

The scalar components of $\mathbf{w}$ can be expressed in terms of the scalar components of $\mathbf{W}$ as

$$w_i = -\tfrac{1}{2}e_{ijk}W_{jk} = \tfrac{1}{2}e_{ijk}\frac{\partial v_k}{\partial x_j} \quad \text{or} \quad \mathbf{w} = \tfrac{1}{2}\boldsymbol{\nabla} \times \mathbf{v}. \tag{3.6.5}$$

Thus, $\mathbf{w}$ is also known as the *vorticity vector*. Note that $\boldsymbol{\nabla} \cdot \mathbf{w} = 0$ by virtue of the vector identity (that is, divergence of the curl of a vector is zero). Thus, the vorticity vector is divergence-free. As discussed in Section 3.5.3, if a velocity vector $\mathbf{v}$ is of the form $\mathbf{v} = \mathbf{W} \cdot \mathbf{x}$ for some skew symmetric tensor $\mathbf{W}$ that is independent of position $\mathbf{x}$, then the motion is a uniform rigid-body rotation about the origin with angular velocity $\mathbf{w}$. Also note that the first and third principal invariants of $\mathbf{W}$ are zero, and the second principal invariant is equal to $w_1^2 + w_2^2 + w_3^2$.

3.6.2 Relationship Between D and $\dot{\mathbf{E}}$

Note that the rate of deformation tensor $\mathbf{D}$ is not the same as the time rate of change of the infinitesimal strain tensor $\boldsymbol{\varepsilon}$ [see Eq. (3.5.1)], that is, the strain rate $\dot{\boldsymbol{\varepsilon}}$, where the superposed dot signifies the material time derivative. However, $\mathbf{D}$ is related to $\dot{\mathbf{E}}$, the time rate of change of Green–Lagrange strain tensor $\mathbf{E}$, as shown in the following paragraphs.

Taking the total time derivative of the expression in Eq. (3.4.10), we obtain

$$\frac{d}{dt}[(ds)^2 - (dS)^2] = \frac{d}{dt}[(ds)^2] = 2\, d\mathbf{X} \cdot \frac{d\mathbf{E}}{dt} \cdot d\mathbf{X} \equiv 2\, d\mathbf{X} \cdot \dot{\mathbf{E}} \cdot d\mathbf{X}, \tag{3.6.6}$$

where we used the fact that $d\mathbf{X}$ and dS are constants. On the other hand, the instantaneous rate of change of the squared length $(ds)^2$ is

$$\frac{d}{dt}[(ds)^2] = \frac{d}{dt}[d\mathbf{x} \cdot d\mathbf{x}] = 2\, d\mathbf{x} \cdot \frac{d}{dt}(d\mathbf{x}) = 2\, d\mathbf{x} \cdot d\mathbf{v}. \tag{3.6.7}$$

Because

$$\mathbf{L} = (\boldsymbol{\nabla}\mathbf{v})^{\mathrm{T}} \;\Rightarrow\; d\mathbf{v} = \mathbf{L} \cdot d\mathbf{x}, \tag{3.6.8}$$

Eq. (3.6.7) takes the form [we make use of Eq. (3.6.1)]

$$\frac{d}{dt}[(ds)^2] = 2\, d\mathbf{x} \cdot \mathbf{L} \cdot d\mathbf{x} = 2\, d\mathbf{x} \cdot (\mathbf{D} + \mathbf{W}) \cdot d\mathbf{x} = 2\, d\mathbf{x} \cdot \mathbf{D} \cdot d\mathbf{x}. \tag{3.6.9}$$

The second term is zero because of the skew symmetry of $\mathbf{W}$. Now equating the right-hand sides of Eqs. (3.6.6) and (3.6.9) and noting $d\mathbf{x} = \mathbf{F} \cdot d\mathbf{X} = d\mathbf{X} \cdot \mathbf{F}^{\mathrm{T}}$

$$d\mathbf{X} \cdot \dot{\mathbf{E}} \cdot d\mathbf{X} = d\mathbf{x} \cdot \mathbf{D} \cdot d\mathbf{x} = d\mathbf{X} \cdot \mathbf{F}^{\mathrm{T}} \cdot \mathbf{D} \cdot \mathbf{F} \cdot d\mathbf{X},$$

we arrive at the result

$$\dot{\mathbf{E}} = \mathbf{F}^{\mathrm{T}} \cdot \mathbf{D} \cdot \mathbf{F}. \tag{3.6.10}$$

Next consider ($d\mathbf{x} = \mathbf{F} \cdot d\mathbf{X}$)

$$d\mathbf{v} = \frac{d}{dt}(d\mathbf{x}) = \left[\frac{d\mathbf{F}}{dt}\right] \cdot d\mathbf{X} + \mathbf{F} \cdot \frac{d(d\mathbf{X})}{dt} = \dot{\mathbf{F}} \cdot d\mathbf{X} + 0, \tag{3.6.11}$$

because $d\dot{\mathbf{X}} = 0$. Then from Eqs. (3.6.8) and (3.6.11) we have

$$\dot{\mathbf{F}} \cdot d\mathbf{X} = \mathbf{L} \cdot d\mathbf{x}. \tag{3.6.12}$$

We also have

$$\dot{\mathbf{F}} = \frac{d}{dt}(\boldsymbol{\nabla}_0 \mathbf{x})^{\mathrm{T}} = (\boldsymbol{\nabla}_0 \mathbf{v})^{\mathrm{T}}. \tag{3.6.13}$$

Note that $\boldsymbol{\nabla}_0 \mathbf{v}$ is the gradient of the velocity vector $\mathbf{v}$ with respect to the material coordinates $\mathbf{X}$, and it is not the same as $\mathbf{L} = (\boldsymbol{\nabla} \mathbf{v})^{\mathrm{T}}$. From Eq. (3.6.12) or from Eq. (3.6.13) and the identity [see Problem **3.36**]

$$(\boldsymbol{\nabla}_0 \mathbf{v})^{\mathrm{T}} = \mathbf{L} \cdot \mathbf{F}, \tag{3.6.14}$$

we obtain

$$\dot{\mathbf{F}} = \mathbf{L} \cdot \mathbf{F} \quad \text{or} \quad \mathbf{L} = \dot{\mathbf{F}} \cdot \mathbf{F}^{-1}. \tag{3.6.15}$$

3.7 Compatibility Equations

3.7.1 Preliminary Comments

The task of computing strains (infinitesimal or finite) from a given displacement field is a straightforward exercise. However, sometimes we face the problem of finding the displacements from a given strain field. This is not as straightforward because there are *six* independent partial differential equations (that is, strain-displacement relations) for only *three* unknown displacements, which would in general over-determine the solution. We will find some conditions, known as *Saint-Venant's compatibility equations*, that will ensure the computation of a unique displacement field from a given strain field. The derivation is presented for infinitesimal strains. For finite strains the same steps may be followed but the process is so difficult that it is never attempted (although some general compatibility conditions may be stated to ensure integrability of the six nonlinear partial differential equations).

To understand the meaning of strain compatibility, imagine that a material body is cut up into pieces before it is strained, and then each piece is given a certain strain. The strained pieces cannot be fitted back into a single continuous body without further deformation. On the other hand, if the strain in each piece is related to or compatible with the strains in the neighboring pieces, then they can be fitted together to form a continuous body. Mathematically, the six strain–displacement relations that connect six strain components to the three displacement components should be consistent.

3.7.2 Infinitesimal Strains

The infinitesimal strain tensor $\mathbf{E} \approx \boldsymbol{\varepsilon}$ is defined in terms of the displacement vector $\mathbf{u}$ as [see Eq. (3.5.3)]

$$\boldsymbol{\varepsilon} = \tfrac{1}{2}\left[\boldsymbol{\nabla}_0\mathbf{u} + (\boldsymbol{\nabla}_0\mathbf{u})^{\mathrm{T}}\right], \quad \varepsilon_{ij} = \tfrac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right). \tag{3.7.1}$$

We begin with infinitesimal strains in two dimensions. We have the following three strain-displacement relations:

$$\begin{aligned} \frac{\partial u_1}{\partial X_1} &= \varepsilon_{11}, \\ \frac{\partial u_2}{\partial X_2} &= \varepsilon_{22}, \\ \frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1} &= 2\varepsilon_{12}. \end{aligned} \tag{3.7.2}$$

If the given data $(\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{12})$ are compatible (or consistent), any two of the three equations should yield the same displacement components. For example, consider the following infinitesimal strain field:

$$\varepsilon_{11} = \varepsilon_{22} = 0, \quad \varepsilon_{12} = X_1 X_2.$$

In terms of the displacement components u_1 and u_2, we have

$$\frac{\partial u_1}{\partial X_1} = 0, \quad \frac{\partial u_2}{\partial X_2} = 0, \quad \frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1} = 2X_1 X_2.$$

Integration of the first two equations gives

$$u_1 = f(X_2), \quad u_2 = g(X_1).$$

On substitution into the shear strain, we obtain

$$\frac{df}{dX_2} + \frac{dg}{dX_1} = 2X_1 X_2,$$

which cannot be satisfied; if ε_{12} is specified as $\varepsilon_{12} = c_1 X_1 + c_2 X_2$, it would be possible to determine f and g, and then u_1 and u_2. Thus, not all arbitrarily specified strain fields are compatible.

The compatibility of a given strain field can be established as follows. Differentiate the first equation with respect to X_2 twice, the second equation with respect to X_1 twice, and the third equation with respect to X_1 and X_2 each, and obtain

$$\begin{aligned} \frac{\partial^3 u_1}{\partial X_1 \partial X_2^2} &= \frac{\partial^2 \varepsilon_{11}}{\partial X_2^2}, \\ \frac{\partial^3 u_2}{\partial X_2 \partial X_1^2} &= \frac{\partial^2 \varepsilon_{22}}{\partial X_1^2}, \\ \frac{\partial^3 u_1}{\partial X_2^2 \partial X_1} + \frac{\partial^3 u_2}{\partial X_1^2 \partial X_2} &= 2\frac{\partial^2 \varepsilon_{12}}{\partial X_1 \partial X_2}. \end{aligned} \tag{3.7.3}$$

Using the first two equations in the third equation, we arrive at the following relation among the three strains:

$$\frac{\partial^2 \varepsilon_{11}}{\partial X_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial X_1^2} = 2\frac{\partial^2 \varepsilon_{12}}{\partial X_1 \partial X_2}\,. \tag{3.7.4}$$

Equation (3.7.4) is called the strain compatibility condition among the three strains ε_{11}, ε_{22}, and ε_{12} that ensures the integrability of Eqs. (3.7.2) to determine the displacement components (u_1, u_2).

A similar procedure can be followed to obtain the strain compatibility equations for the three-dimensional case. In addition to Eq. (3.7.4), five more such conditions can be derived, as given below:

$$\frac{\partial^2 \varepsilon_{11}}{\partial X_3^2} + \frac{\partial^2 \varepsilon_{33}}{\partial X_1^2} = 2\frac{\partial^2 \varepsilon_{13}}{\partial X_1 \partial X_3}\,, \tag{3.7.5}$$

$$\frac{\partial^2 \varepsilon_{22}}{\partial X_3^2} + \frac{\partial^2 \varepsilon_{33}}{\partial X_2^2} = 2\frac{\partial^2 \varepsilon_{23}}{\partial X_2 \partial X_3}\,, \tag{3.7.6}$$

$$\frac{\partial^2 \varepsilon_{11}}{\partial X_2 \partial X_3} + \frac{\partial^2 \varepsilon_{23}}{\partial X_1^2} = \frac{\partial^2 \varepsilon_{13}}{\partial X_1 \partial X_2} + \frac{\partial^2 \varepsilon_{12}}{\partial X_1 \partial X_3}\,, \tag{3.7.7}$$

$$\frac{\partial^2 \varepsilon_{22}}{\partial X_1 \partial X_3} + \frac{\partial^2 \varepsilon_{13}}{\partial X_2^2} = \frac{\partial^2 \varepsilon_{23}}{\partial X_1 \partial X_2} + \frac{\partial^2 \varepsilon_{12}}{\partial X_2 \partial X_3}\,, \tag{3.7.8}$$

$$\frac{\partial^2 \varepsilon_{33}}{\partial X_1 \partial X_2} + \frac{\partial^2 \varepsilon_{12}}{\partial X_3^2} = \frac{\partial^2 \varepsilon_{13}}{\partial X_2 \partial X_3} + \frac{\partial^2 \varepsilon_{23}}{\partial X_1 \partial X_3}\,. \tag{3.7.9}$$

Equations (3.7.4)–(3.7.9) can be written as a single relation using the index notation

$$\frac{\partial^2 \varepsilon_{mn}}{\partial X_i \partial X_j} + \frac{\partial^2 \varepsilon_{ij}}{\partial X_m \partial X_n} = \frac{\partial^2 \varepsilon_{im}}{\partial X_j \partial X_n} + \frac{\partial^2 \varepsilon_{jn}}{\partial X_i \partial X_m}\,, \tag{3.7.10}$$

which yields $(3)^4 = 81$ equations but only 6, shown in Eqs. (3.7.4)–(3.7.9), are distinctly different. These conditions are both necessary and sufficient to determine a single-valued displacement field. Similar compatibility conditions hold for the rate of deformation tensor $\mathbf{D}$. The vector form of Eqs. (3.7.4)–(3.7.9) is given by [see Problem **3.39**]

$$\boldsymbol{\nabla}_0 \times (\boldsymbol{\nabla}_0 \times \boldsymbol{\varepsilon})^{\mathrm{T}} = \mathbf{0} \quad \text{or} \quad e_{ikr}\, e_{jls}\, \varepsilon_{ij,kl} = 0. \tag{3.7.11}$$

Example 3.7.1 illustrates how to check the compatibility of a given strain field.

Example 3.7.1

Given the following two-dimensional, infinitesimal strain field:

$$\varepsilon_{11} = c_1 X_1 \left(X_1^2 + X_2^2\right), \quad \varepsilon_{22} = \tfrac{1}{3} c_2 X_1^3, \quad \varepsilon_{12} = c_3 X_1^2 X_2,$$

where c_1, c_2, and c_3 are constants, determine if the strain field is compatible.

Solution: Using Eq. (3.7.4) we obtain

$$\frac{\partial^2 \varepsilon_{11}}{\partial X_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial X_1^2} - 2\frac{\partial^2 \varepsilon_{12}}{\partial X_1 \partial X_2} = 2c_1 X_1 + 2c_2 X_1 - 4c_3 X_1.$$

Thus the strain field is not compatible, unless $c_1 + c_2 - 2c_3 = 0$.

Example 3.7.2 illustrates how to determine the displacement field from a compatible strain field.

Example 3.7.2

Consider the problem of an isotropic cantilever beam bent by a load P at the free end, as shown in Fig. 3.7.1. To study the beam problem as a two-dimensional elasticity problem, consider the strain field [strains ε_{11} and ε_{12} are known from a book on mechanics of solids; see, e.g., Fenner and Reddy (2012)]:

$$\varepsilon_{11} = -\frac{PX_1X_2}{EI}, \quad \varepsilon_{12} = -\frac{(1+\nu)P}{2EI}(h^2 - X_2^2), \quad \varepsilon_{22} = -\nu\varepsilon_{11} = \frac{\nu PX_1X_2}{EI}, \tag{3.7.12}$$

where I is the moment of inertia about the X_3-axis ($I = 2bh^3/3$), ν is the Poisson ratio, E is Young's modulus, $2h$ is the height of the beam, and b is the width of the beam. Determine if the strain field is compatible and, if it is compatible, find the two-dimensional displacement field (u_1, u_2) that satisfies the kinematic boundary conditions and, therefore, is free of rigid-body translation and rotation.

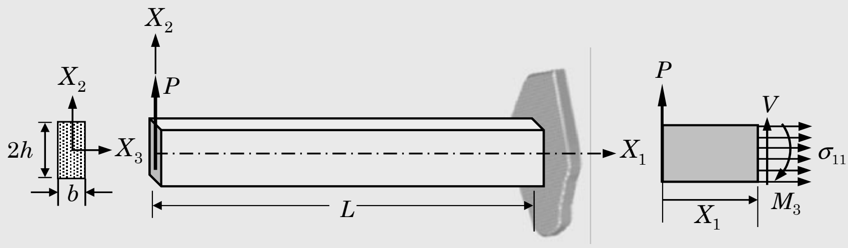


Fig. 3.7.1: Cantilever beam bent by a point load.

Solution: **(a)** Substituting ε_{ij} into the compatibility equation (3.7.4), we obtain $0 + 0 = 0$. Thus the strains in Eq. (3.7.12) satisfy the compatibility conditions for a two-dimensional state of deformation. One can verify that the three-dimensional strains are not compatible; one can show that all of the compatibility equations except Eq. (3.7.9) are satisfied.

(b) Integrating the strain-displacement equations, we obtain

$$\frac{\partial u_1}{\partial X_1} = \varepsilon_{11} = -\frac{PX_1X_2}{EI} \quad \text{or} \quad u_1 = -\frac{PX_1^2X_2}{2EI} + f(X_2), \tag{3.7.13}$$

$$\frac{\partial u_2}{\partial X_2} = \varepsilon_{22} = \frac{\nu PX_1X_2}{EI} \quad \text{or} \quad u_2 = \frac{\nu PX_1X_2^2}{2EI} + g(X_1), \tag{3.7.14}$$

where $f(X_2)$ and $g(X_1)$ are functions of integration. Substituting u_1 and u_2 into the definition of $2\varepsilon_{12}$, we obtain

$$2\varepsilon_{12} = \frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1} = -\frac{PX_1^2}{2EI} + \frac{df}{dX_2} + \frac{\nu PX_2^2}{2EI} + \frac{dg}{dX_1}. \tag{3.7.15}$$

But this must be equal to the shear strain known from Eq. (3.7.12):

$$-\frac{P}{2EI}X_1^2 + \frac{df}{dX_2} + \frac{\nu P}{2EI}X_2^2 + \frac{dg}{dX_1} = -\frac{(1+\nu)}{EI}P(h^2 - X_2^2).$$

Separating the terms that depend only on X_1 and those depend only on X_2 (the constant term can go with either one), we obtain

$$-\frac{dg}{dX_1} + \frac{P}{2EI}X_1^2 - \frac{(1+\nu)Ph^2}{EI} = \frac{df}{dX_2} - \frac{(2+\nu)P}{2EI}X_2^2. \tag{3.7.16}$$

Since the left side depends only on X_1 and the right side depends only on X_2, and yet the equality must hold, it follows that both sides should be equal to a constant, say c_0:

$$\frac{df}{dX_2} - \frac{(2+\nu)P}{2EI}X_2^2 = c_0, \quad -\frac{dg}{dX_1} + \frac{P}{2EI}X_1^2 - \frac{(1+\nu)Ph^2}{EI} = c_0.$$

Integrating the expressions for f and g, we obtain

$$\begin{aligned} f(X_2) &= \frac{(2+\nu)P}{6EI}X_2^3 + c_0 X_2 + c_1, \\ g(X_1) &= \frac{P}{6EI}X_1^3 - \frac{(1+\nu)Ph^2}{EI}X_1 - c_0 X_1 + c_2, \end{aligned} \tag{3.7.17}$$

where c_1 and c_2 are constants of integration. Thus, the most general form of displacement field (u_1, u_2) that corresponds to the strains in Eq. (3.7.12) is given by

$$\begin{aligned} u_1(X_1,X_2) &= -\frac{P}{2EI}X_1^2 X_2 + \frac{(2+\nu)P}{6EI}X_2^3 + c_0 X_2 + c_1, \\ u_2(X_1,X_2) &= -\frac{(1+\nu)Ph^2}{EI}X_1 + \frac{\nu P}{2EI}X_1 X_2^2 + \frac{P}{6EI}X_1^3 - c_0 X_1 + c_2. \end{aligned} \tag{3.7.18}$$

(c) The constants c_0, c_1, and c_2 are determined using suitable boundary conditions. We impose the following boundary conditions that eliminate rigid-body displacements (that is, rigid-body translation and rigid-body rotation):

$$u_1(L,0) = 0, \quad u_2(L,0) = 0, \quad \Omega_{12}\Big|_{X_1=L, X_2=0} = \frac{1}{2}\left(\frac{\partial u_2}{\partial X_1} - \frac{\partial u_1}{\partial X_2}\right)_{X_1=L, X_2=0} = 0. \tag{3.7.19}$$

Imposing the boundary conditions from Eq. (3.7.19) on the displacement field in Eq. (3.7.18), we obtain

$$\begin{aligned} u_1(L,0) = 0 \ &\Rightarrow \ c_1 = 0, \\ u_2(L,0) = 0 \ &\Rightarrow \ c_0 L - c_2 = -\frac{(1+\nu)Ph^2 L}{EI} + \frac{PL^3}{6EI}, \\ \left(\frac{\partial u_2}{\partial X_1} - \frac{\partial u_1}{\partial X_2}\right)_{X_1=L, X_2=0} = 0, \ &\Rightarrow \ c_0 = \frac{PL^2}{2EI} - \frac{(1+\nu)Ph^2}{2EI} \end{aligned} \tag{3.7.20}$$

Thus, we have

$$c_0 = \frac{PL^2}{2EI} - \frac{(1+\nu)Ph^2}{2EI}, \quad c_1 = 0, \quad c_2 = \frac{PL^3}{3EI} + \frac{(1+\nu)Ph^2 L}{2EI}. \tag{3.7.21}$$

Then the final displacement field in Eq. (3.7.18) becomes

$$\begin{aligned} u_1(X_1,X_2) &= \frac{PL^2 X_2}{6EI}\left[3\left(1 - \frac{X_1^2}{L^2}\right) + (2+\nu)\frac{X_2^2}{L^2} - 3(1+\nu)\frac{h^2}{L^2}\right], \\ u_2(X_1,X_2) &= \frac{PL^3}{6EI}\left[2 - 3\frac{X_1}{L}\left(1 - \nu\frac{X_2^2}{L^2}\right) + \frac{X_1^3}{L^3} + 3(1+\nu)\frac{h^2}{L^2}\left(1 - \frac{X_1}{L}\right)\right]. \end{aligned} \tag{3.7.22}$$

In the Euler–Bernoulli beam theory (EBT), where one assumes that $L >> 2h$ and $\nu = 0$, we have $u_1 = 0$, and u_2 is given by

$$u_2^{\text{EBT}}(X_1,X_2) = \frac{PL^3}{6EI}\left(2 - 3\frac{X_1}{L} + \frac{X_1^3}{L^3}\right),$$

while in the Timoshenko beam theory (TBT) we have $u_1 = 0$ [$E = 2(1+\nu)G$, $I = Ah^2/3$, and $A = 2bh$], and u_2 is given by

$$u_2^{\text{TBT}}(X_1,X_2) = \frac{PL^3}{6EI}\left(2 - 3\frac{X_1}{L} + \frac{X_1^3}{L^3}\right) + \frac{PL}{K_s GA}\left(1 - \frac{X_1}{L}\right).$$

Here K_s denotes the shear correction factor. Thus, the Timoshenko beam theory with shear correction factor of $K_s = 4/3$ predicts the same maximum deflection, $u_2(0,0)$, as the two-dimensional elasticity theory [see Reddy (2002) for more details on the Timoshenko beam theory]. Both beam theory solutions, in general, are in error compared to the elasticity solution (primarily because of the Poisson effect).

3.7.3 Finite Strains

In the case of finite strains, the compatibility conditions in terms of the deformation tensor[6] are derived from the mathematical requirement that the curl of a gradient be zero. Since $\mathbf{F}$ is the gradient of $\mathbf{x}$ with respect to $\mathbf{X}$, we require that

$$\boldsymbol{\nabla}_0 \times \mathbf{F}^{\mathrm{T}} = \mathbf{0}, \tag{3.7.23}$$

or, equivalently,

$$F_{iJ,K} = F_{iK,J} \quad \text{or} \quad \frac{\partial^2 x_i}{\partial X_J \partial X_K} = \frac{\partial^2 x_i}{\partial X_K \partial X_J}. \tag{3.7.24}$$

We close this section with a note that the compatibility conditions arise only when the strains (or stresses) are used to formulate the problem and displacements are to be determined. An example of such a situation arises in plane elasticity where stresses are expressed in terms of a single function, called *stress function*. When boundary value problems in mechanics are formulated in terms of the displacements or velocities, the question of strain compatibility never arises.

3.8 Rigid-Body Motions and Material Objectivity

3.8.1 Superposed Rigid-Body Motions

3.8.1.1 Introduction and rigid-body transformation

A rigid-body motion is one that preserves the relative distance between points. Intuitively, rigid-body motions of a whole body should have no effect on the values of computed strains, as they are based on the change of length and orientation of line elements in a small neighborhood of a point. However, mathematically, various measures of strains with superposed rigid-body motions may be expressed in different ways, although the computed values are independent of the rigid-body motion of the body[7].

Consider the motion mapping from Eq. (3.2.1), $\boldsymbol{\chi} : \kappa_0 \rightarrow \kappa$,

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t). \tag{3.8.1}$$

Then a material particle X of a body, occupying position $\mathbf{X}$ in the reference configuration κ_0, now occupies a position $\mathbf{x}$ in κ at time t, as specified by the motion (3.8.1). Under a superposed rigid-body motion, the particle X that is at $\mathbf{x}$ at time t moves to a place $\mathbf{x}^*$ at time $t^* = t + a$, where a is a constant. In the following discussion, we shall use an asterisk (*) on all quantities with the superposed motion. Thus,

$$\mathbf{x}^* = \boldsymbol{\chi}^*(\mathbf{X}, t^*) = \boldsymbol{\chi}^*(\mathbf{X}, t). \tag{3.8.2}$$

[6]The derivation of compatibility conditions in terms of the right Cauchy–Green deformation tensor $\mathbf{C}$ or the Green–Lagrange strain tensor $\mathbf{E}$ is quite involved and not attempted here.

[7]The discussion presented here closely follows that by Naghdi (2001).

Next, consider another material particle Y of the body in the reference configuration that occupies a position $\mathbf{y}$ in κ at time t, as specified by the motion (3.8.1)

$$\mathbf{y} = \boldsymbol{\chi}(\mathbf{Y}, t). \tag{3.8.3}$$

Under the superposed motion the material particle that is at $\mathbf{y}$ at time t moves to a place $\mathbf{y}^*$ at time t^*. Then we have

$$\mathbf{y}^* = \boldsymbol{\chi}^*(\mathbf{Y}, t^*) = \boldsymbol{\chi}^*(\mathbf{Y}, t). \tag{3.8.4}$$

We can use the inverse mapping $\boldsymbol{\chi}^{-1}$ to write $\mathbf{X}$ and $\mathbf{Y}$ in terms of $\mathbf{x}$ and $\mathbf{y}$, respectively. Hence, we have

$$\mathbf{x}^* = \boldsymbol{\chi}^*(\mathbf{X}(\mathbf{x}, t), t) = \overline{\boldsymbol{\chi}}^*(\mathbf{x}, t), \quad \mathbf{y}^* = \boldsymbol{\chi}^*(\mathbf{Y}(\mathbf{y}, t), t) = \overline{\boldsymbol{\chi}}^*(\mathbf{y}, t). \tag{3.8.5}$$

The superposed rigid-body motions of a whole body should preserve the distance between all pairs of material particles of the body for all times $0 \leq t \leq T$, where T is a finite final time; therefore, we have

$$[\overline{\boldsymbol{\chi}}^*(\mathbf{x}, t) - \overline{\boldsymbol{\chi}}^*(\mathbf{y}, t)]^{\mathrm{T}} \cdot [\overline{\boldsymbol{\chi}}^*(\mathbf{x}, t) - \overline{\boldsymbol{\chi}}^*(\mathbf{y}, t)] = (\mathbf{x} - \mathbf{y})^{\mathrm{T}} \cdot (\mathbf{x} - \mathbf{y}), \tag{3.8.6}$$

or

$$(\mathbf{x}^* - \mathbf{y}^*)^{\mathrm{T}} \cdot (\mathbf{x}^* - \mathbf{y}^*) = (\mathbf{x} - \mathbf{y})^{\mathrm{T}} \cdot (\mathbf{x} - \mathbf{y}) \ \text{ for all } \mathbf{x}, \mathbf{y} \text{ in } \kappa \text{ at time } t. \tag{3.8.7}$$

Noting that $\mathbf{x}$ and $\mathbf{y}$ are independent of each other, we can differentiate with respect to $\mathbf{x}$ and $\mathbf{y}$ successively and obtain

$$2\left[\frac{\partial \overline{\boldsymbol{\chi}}^*(\mathbf{x}, t)}{\partial \mathbf{x}}\right]^{\mathrm{T}} \cdot \left[\frac{\partial \overline{\boldsymbol{\chi}}^*(\mathbf{y}, t)}{\partial \mathbf{y}}\right] = 2\mathbf{I}, \tag{3.8.8}$$

or

$$\left[\frac{\partial \overline{\boldsymbol{\chi}}^*(\mathbf{x}, t)}{\partial \mathbf{x}}\right]^{\mathrm{T}} = \left[\frac{\partial \overline{\boldsymbol{\chi}}^*(\mathbf{y}, t)}{\partial \mathbf{y}}\right]^{-1}.$$

Because the left side of the equality depends only on $(\mathbf{x}, t)$ and the right side depends only on $(\mathbf{y}, t)$, both must be equal to a function of time only, say $\mathbf{Q}^{\mathrm{T}}(t)$:

$$\left[\frac{\partial \overline{\boldsymbol{\chi}}^*(\mathbf{x}, t)}{\partial \mathbf{x}}\right]^{\mathrm{T}} = \left[\frac{\partial \overline{\boldsymbol{\chi}}^*(\mathbf{y}, t)}{\partial \mathbf{y}}\right]^{-1} = \mathbf{Q}^{\mathrm{T}}(t), \tag{3.8.9}$$

for all $\mathbf{x}$ and $\mathbf{y}$ in κ at time t. Let us set

$$\frac{\partial \overline{\boldsymbol{\chi}}^*(\mathbf{x}, t)}{\partial \mathbf{x}} = \mathbf{Q}(t) \ \text{ for all } \mathbf{x} \text{ in } \kappa \text{ at time } t. \tag{3.8.10}$$

Then we must also have (because $\mathbf{y}$ is also in κ at time t)

$$\frac{\partial \overline{\boldsymbol{\chi}}^*(\mathbf{y}, t)}{\partial \mathbf{y}} = \mathbf{Q}(t).$$

Therefore, we have

$$\mathbf{Q}^{\mathrm{T}} \cdot \mathbf{Q} = \mathbf{I}. \tag{3.8.11}$$

Thus, $\mathbf{Q}$ is an orthogonal tensor with $|Q| = \pm 1$. Since the motion under consideration must include the special case $\overline{\chi}^* = \mathbf{x}$, for which case $\mathbf{Q} = \mathbf{I}$, we have $|Q| = \pm 1$. Therefore, $\mathbf{Q}$ is a *proper* orthogonal tensor.

Integrating Eq. (3.8.10) with respect to $\mathbf{x}$, we obtain

$$\mathbf{x}^* = \overline{\chi}^*(\mathbf{x}, t) = \mathbf{Q} \cdot \mathbf{x} + \mathbf{c}(t), \tag{3.8.12}$$

where $\mathbf{c}(t)$ is a vector-valued function of time t. Equation (3.8.12) represents a rigid transformation that includes translation $\mathbf{c}$ and rotation $\mathbf{Q}$. Thus, at each instant of time a rigid-body motion is a composition of rigid-body translation $\mathbf{c}$ and a rigid-body rotation $\mathbf{Q}$ about an axis of rotation, as well as a time shift $a = t^* - t$. Figure 3.8.1 shows a sequence of deformation followed by rigid-body transformation. For pure rigid-body rotation, Eq. (3.8.12) reduces to $\mathbf{x}^* = \mathbf{Q} \cdot \mathbf{x}$. The transformation in Eq. (3.8.12) preserves the distance between any two material particles as well as the angle between material lines in the small neighborhood of a material particle, as established next.

Consider the distance between two material particles occupying positions $\mathbf{x}$ and $\mathbf{y}$ in the deformed configuration,

$$\begin{aligned}|\mathbf{x}^* - \mathbf{y}^*|^2 &= (\mathbf{x}^* - \mathbf{y}^*) \cdot (\mathbf{x}^* - \mathbf{y}^*) = \mathbf{Q} \cdot (\mathbf{x} - \mathbf{y}) \cdot \mathbf{Q} \cdot (\mathbf{x} - \mathbf{y}) \\ &= (\mathbf{x} - \mathbf{y}) \cdot (\mathbf{Q}^{\mathrm{T}} \cdot \mathbf{Q}) \cdot (\mathbf{x} - \mathbf{y}) = (\mathbf{x} - \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y}) = |\mathbf{x} - \mathbf{y}|^2.\end{aligned}$$

Thus, the distance between any two points is preserved.

Next, we consider two material line segments in the neighborhood of point $\mathbf{x}$, one connecting $\mathbf{x}$ to $\mathbf{y}$ and the other connecting $\mathbf{x}$ to $\mathbf{z}$. The angle between the two line segments is

$$\cos\theta = \frac{(\mathbf{x} - \mathbf{y}) \cdot (\mathbf{x} - \mathbf{z})}{|\mathbf{x} - \mathbf{y}|\ |\mathbf{x} - \mathbf{z}|}.$$

Now consider the angle between the lines after superposed rigid-body motion:

$$\cos\theta^* = \frac{(\mathbf{x}^* - \mathbf{y}^*)}{|\mathbf{x}^* - \mathbf{y}^*|} \cdot \frac{(\mathbf{x}^* - \mathbf{z}^*)}{|\mathbf{x}^* - \mathbf{z}^*|}.$$

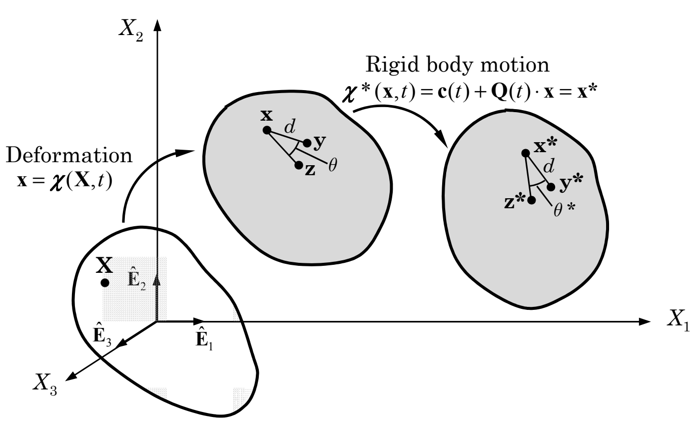


Fig. 3.8.1: Deformation followed by superposed rigid-body motion.

Since the distances are preserved, we have

$$\begin{aligned}\cos\theta^* &= \frac{[\mathbf{Q}\cdot(\mathbf{x}-\mathbf{y})]\cdot[\mathbf{Q}\cdot(\mathbf{x}-\mathbf{z})]}{|\mathbf{x}^*-\mathbf{y}^*|\;|\mathbf{x}^*-\mathbf{z}^*|}\\ &= \frac{(\mathbf{x}-\mathbf{y})\cdot(\mathbf{Q}^{\mathrm{T}}\cdot\mathbf{Q})\cdot(\mathbf{x}-\mathbf{z})}{|\mathbf{x}-\mathbf{y}|\;|\mathbf{x}-\mathbf{z}|}\\ &= \frac{(\mathbf{x}-\mathbf{y})\cdot(\mathbf{x}-\mathbf{z})}{|\mathbf{x}-\mathbf{y}|\;|\mathbf{x}-\mathbf{z}|} = \cos\theta.\end{aligned}$$

The transformation in Eq. (3.8.12) preserves the distance between two material points and the angle between material lines in the neighborhood of every point in the body; hence, the transformation also preserves areas and volumes under the superposed rigid-body motion. Thus, when two frames of references are involved in measuring deformations (and forces) with one frame of reference moving rigidly with respect to the other, the measures will be unaffected.

3.8.1.2 Effect on F

To see the effect of superposed rigid-body motion on the deformation gradient, consider the most general rigid-body mapping in Eq. (3.8.12). Taking the derivative of Eq. (3.8.12) with respect to $\mathbf{X}$, we obtain

$$\frac{\partial \mathbf{x}^*}{\partial \mathbf{X}} = \mathbf{Q}(t)\cdot\frac{\partial \mathbf{x}}{\partial \mathbf{X}},$$

and therefore we have

$$\mathbf{F}^* = \left(\mathbf{Q}(t)\cdot\frac{\partial \mathbf{x}}{\partial \mathbf{X}}\right)^{\mathrm{T}} = \mathbf{F}\cdot\mathbf{Q}^{\mathrm{T}} = \mathbf{Q}\cdot\mathbf{F}. \tag{3.8.13}$$

Thus, the deformation gradients before and after superposed rigid-body motions are related by

$$\mathbf{F}^*(\mathbf{X},t) = \mathbf{Q}(t)\cdot\mathbf{F}(\mathbf{X},t). \tag{3.8.14}$$

Because $\mathbf{F}$ is a two-point tensor from a reference configuration, which is independent of the observer, to the current configuration, it transforms like a vector. The respective Jacobians are given by

$$J = |F|,\quad J^* = |F^*| = |Q|\,|F| = J, \tag{3.8.15}$$

where the fact that $|\mathbf{Q}| = 1$ is used. Thus the volume change is unaffected by superposed rigid-body motion.

3.8.1.3 Effect on C and E

To see how the right Cauchy–Green deformation tensor $\mathbf{C}$ and the Green–Lagrange strain tensor $\mathbf{E}$ change due to superposed rigid-body motion, consider

$$\mathbf{C}^* = (\mathbf{F}^*)^{\mathrm{T}}\cdot\mathbf{F}^* = \left(\mathbf{F}^{\mathrm{T}}\cdot\mathbf{Q}^{\mathrm{T}}\right)\cdot\left(\mathbf{Q}\cdot\mathbf{F}\right) = \mathbf{F}^{\mathrm{T}}\cdot\mathbf{F} = \mathbf{C}, \tag{3.8.16}$$

where Eq. (3.8.14) and the property $\mathbf{Q}^{\mathrm{T}} \cdot \mathbf{Q} = \mathbf{I}$ of an orthogonal matrix $\mathbf{Q}$ is used. Hence, by definition [see Eq. (3.4.11)], the Green–Lagrange strain tensor $\mathbf{E}$ and the right Cauchy–Green deformation tensor $\mathbf{C}$, being defined with respect to the reference configuration, are unaffected by the superposed rigid-body motion:

$$\mathbf{E} = \mathbf{E}^*, \qquad \mathbf{C} = \mathbf{C}^*. \tag{3.8.17}$$

However, the velocities and accelerations of a material point are affected by the superposed rigid-body motion. For example, consider velocity after imposing the rigid-body motion (note that $dt/dt^* = 1$)

$$\mathbf{v}^*(\mathbf{x}^*, t^*) = \frac{d\mathbf{x}^*}{dt^*} = \frac{d}{dt^*}\left(\mathbf{c}(t) + \mathbf{Q}(t)\cdot\mathbf{x}\right) = \dot{\mathbf{c}}(t) + \dot{\mathbf{Q}}(t)\cdot\mathbf{x} + \mathbf{Q}(t)\cdot\mathbf{v}, \tag{3.8.18}$$

which shows that $\mathbf{v}^*$ and $\mathbf{v}$ are not the same, but one can be calculated from the other when $\mathbf{c}$ and $\mathbf{Q}$ are known for the superposed rigid-body motion.

3.8.1.4 Effect on L and D

Here we examine the effect of a superposed rigid-body motion on the velocity gradient tensor $\mathbf{L}$. We begin with Eq. (3.6.15)

$$\begin{aligned}\mathbf{L}^* &= \dot{\mathbf{F}}^* \cdot (\mathbf{F}^*)^{-1} = \left(\dot{\mathbf{Q}}\cdot\mathbf{F} + \mathbf{Q}\cdot\dot{\mathbf{F}}\right)(\mathbf{Q}\cdot\mathbf{F})^{-1}\\ &= \dot{\mathbf{Q}}\cdot\mathbf{Q}^{\mathrm{T}} + \mathbf{Q}\cdot\dot{\mathbf{F}}\cdot\mathbf{F}^{-1}\cdot\mathbf{Q}^{\mathrm{T}} = \dot{\mathbf{Q}}\cdot\mathbf{Q}^{\mathrm{T}} + \mathbf{Q}\cdot\mathbf{L}\cdot\mathbf{Q}^{\mathrm{T}},\end{aligned} \tag{3.8.19}$$

where we have used the following identities:

$$(\mathbf{Q}\cdot\mathbf{F})^{-1} = \mathbf{F}^{-1}\cdot\mathbf{Q}^{-1}, \quad \mathbf{Q}^{-1} = \mathbf{Q}^{\mathrm{T}}, \quad \mathbf{Q}\cdot\mathbf{Q}^{\mathrm{T}} = \mathbf{I}.$$

From Problem **3.48**, it is follows that $\dot{\mathbf{Q}}\cdot\mathbf{Q}^{\mathrm{T}}$ is skew symmetric.

Next consider the symmetric part of $\mathbf{L}$, namely, $\mathbf{D}$. We have

$$\begin{aligned}\mathbf{D}^* &= \tfrac{1}{2}\left[\mathbf{L}^* + (\mathbf{L}^*)^{\mathrm{T}}\right] = \tfrac{1}{2}\left[\dot{\mathbf{Q}}\cdot\mathbf{Q}^{\mathrm{T}} + \mathbf{Q}\cdot\mathbf{L}\cdot\mathbf{Q}^{\mathrm{T}} + \dot{\mathbf{Q}}^{\mathrm{T}}\cdot\mathbf{Q} + \mathbf{Q}\cdot\mathbf{L}^{\mathrm{T}}\cdot\mathbf{Q}^{\mathrm{T}}\right]\\ &= \tfrac{1}{2}\left[\mathbf{Q}\cdot\mathbf{L}\cdot\mathbf{Q}^{\mathrm{T}} + \mathbf{Q}\cdot\mathbf{L}^{\mathrm{T}}\cdot\mathbf{Q}^{\mathrm{T}}\right] = \mathbf{Q}\cdot\mathbf{D}\cdot\mathbf{Q}^{\mathrm{T}}.\end{aligned} \tag{3.8.20}$$

3.8.2 Material Objectivity

3.8.2.1 Observer transformation

In continuum mechanics each frame of reference represents an observer and, therefore, transformations between moving frames are termed *observer transformations.* The concept of frames of reference should not be confused with that of coordinate systems, as they are not the same at all. An observer is free to choose any coordinate system as may be convenient to observe or analyze a system's response. The equations of mechanics are used in different problems and places and, therefore, they must be independent of frames of reference, that is, invariant with respect to an observer transformation. A change of observer may be viewed as certain rigid-body motion superposed on the current configuration, as illustrated in Fig. 3.8.1.

In the analytical description of physical events, the following two requirements must be followed:

(1) Invariance of the equations with respect to stationary coordinate frames of reference

(2) Invariance of the equations with respect to frames of reference that move in arbitrary relative motion

The first requirement is readily met by expressing the equations in vector/tensor form, which is invariant. The assertion that an equation is in "invariant form" refers to the vector form that is independent of the choice of a coordinate system. The coordinate systems used in the present study were assumed to be relatively at rest. The second requirement is that the invariance property holds for reference frames (or observers) moving arbitrarily with respect to each other. This requirement is dictated by the need for forces and deformations to be the same as measured by all observers irrespective of their relative motions. Invariance with respect to changes of observer is termed *material frame indifference* or *material objectivity*.

3.8.2.2 Objectivity of various kinematic measures

Let $\mathcal{F}$ denote a reference frame with origin at O in which the $\mathbf{x}$ is the current position of a particular particle at time t. Let $\mathcal{F}^*$ be another reference frame with origin at O* with time denoted with t^*. Let ϕ be a scalar field when described in the frame $\mathcal{F}$ and ϕ^* is the same scalar field described with respect to the frame $\mathcal{F}^*$, and let $(\mathbf{u}, \mathbf{u}^*)$ and $(\mathbf{S}, \mathbf{S}^*)$ be the vector and tensor fields, respectively, in the two frames. Scalar, vector, and tensor fields are called *frame indifferent* or *objective* if they transform according to the following equations:

1. Events	$\mathbf{x}^* = \mathbf{c}(t) + \mathbf{Q}(t) \cdot \mathbf{x}, \quad t^* = t - a$
2. Scalar field	$\phi^*(\mathbf{x}^*, t^*) = \phi(\mathbf{x}, t)$
3. Displacement vector	$\mathbf{u}^*(\mathbf{x}^*, t^*) = \mathbf{Q}(t) \cdot \mathbf{u}(\mathbf{x}, t)$ (3.8.21)
4. General second-order tensors	$\mathbf{S}^*(\mathbf{x}^*, t^*) = \mathbf{Q}(t) \cdot \mathbf{S}(\mathbf{x}, t) \cdot \mathbf{Q}^{\mathrm{T}}(t)$
5. Two-point second-order tensors	$\mathbf{F}^*(\mathbf{x}^*, t^*) = \mathbf{Q}(t) \cdot \mathbf{F}(\mathbf{x}, t)$

where $\mathbf{Q}(t)$ is a proper orthogonal tensor that rotates frame $\mathcal{F}^*$ into frame $\mathcal{F}$, $\mathbf{c}(t)$ is a vector from O to O* that depends only on time t, and a is a constant. For example, $\mathbf{x}$ and $\mathbf{x}^*$ refer to the same motion, but mathematically $\mathbf{x}^*$ is the motion obtained from $\mathbf{x}$ by superposition of a rigid rotation and translation. The mapping $\mathbf{x}^* = \mathbf{c}(t) + \mathbf{Q}(t) \cdot \mathbf{x}$, derived in Eq. (3.8.12), may be interpreted as one that takes $(\mathbf{x}, t)$ to $(\mathbf{x}^*, t^*)$ as a change of observer from O to O*, so that the event that is observed at place $\mathbf{x}$ at time t by observer O is the *same* event as that observed at place $\mathbf{x}^*$ at time t^* by observer O*, where $t^* = t - a$, and a is a constant. Thus, a change of observer merely changes the *description* of an event. In short, the objectivity ensures that the direction(s) and magnitude are independent of the coordinate frame used to describe them.

We have already established in Section 3.8.1, under the following general rigid-body mapping

$$\mathbf{x}^*(\mathbf{X}, t^*) = \mathbf{c}(t) + \mathbf{Q}(t) \cdot \mathbf{x}, t), \quad t^* = t - a, \tag{3.8.22}$$

between observer O and observer O*, that (if the reference configuration is independent of the observer) the right Cauchy–Green deformation tensor $\mathbf{C}$ and the Green–Lagrange strain tensor $\mathbf{E}$ do not change under the observer transformation, that is, they are objective:

$$\mathbf{E} = \mathbf{E}^*, \qquad \mathbf{C} = \mathbf{C}^*. \tag{3.8.23}$$

In addition, the symmetric part of $\mathbf{L}$, namely, $\mathbf{D}$ is also objective in the sense

$$\mathbf{D}^* = \mathbf{Q} \cdot \mathbf{D} \cdot \mathbf{Q}^{\mathrm{T}}. \tag{3.8.24}$$

We have noted that the two observers' views of the velocity and acceleration of a given motion are different, even though the rate of change at fixed $\mathbf{X}$ is the same in each case. Thus, velocity and acceleration vectors are *not* objective.

3.8.2.3 Time rate of change in a rotating frame of reference

Next, consider two frames of reference with both having the same origin, but one is nonrotating and the other is rotating with respect to the other with an angular velocity $\boldsymbol{\omega}$. Let us use no bars on quantities in the nonrotating system and bars on quantities in the rotating system. Then the time derivatives of a vector-valued function $\mathbf{A}(t)$ in the two coordinate frames are

$$\mathbf{A}(t) = A^i\, \mathbf{e}_i, \quad \frac{D\mathbf{A}}{Dt} = \frac{d\mathbf{A}^i}{dt}\mathbf{e}_i, \quad \text{nonrotating system}, \tag{3.8.25}$$

$$\mathbf{A}(t) = \bar{A}^i \bar{\mathbf{e}}_i, \quad \frac{D\mathbf{A}}{Dt} = \frac{d\bar{A}^i}{dt}\bar{\mathbf{e}}_i + \bar{A}^i \frac{d\bar{\mathbf{e}}_i}{dt}, \quad \text{rotating system}. \tag{3.8.26}$$

The rate of change $d\bar{\mathbf{e}}_i/dt$ is given by

$$\frac{d\bar{\mathbf{e}}_i}{dt} = \boldsymbol{\omega} \times \bar{\mathbf{e}}_i, \tag{3.8.27}$$

because the change is brought about by a rigid-body rotation. To an observer in the rotating frame, however, the basis vectors appear to be constant:

$$\frac{d\bar{A}^i}{dt}\bar{\mathbf{e}}_i \equiv \left(\frac{d\mathbf{A}}{dt}\right)_{\text{rot}}. \tag{3.8.28}$$

The relationship of the time derivatives in the two frames is thus given by

$$\begin{aligned} \left(\frac{d\mathbf{A}}{dt}\right)_{\text{nonrot}} &= \left(\frac{d\mathbf{A}}{dt}\right)_{\text{rot}} + \bar{A}^i(\boldsymbol{\omega} \times \bar{\mathbf{e}}_i) \\ &= \left(\frac{d\mathbf{A}}{dt}\right)_{\text{rot}} + \boldsymbol{\omega} \times \mathbf{A}. \end{aligned} \tag{3.8.29}$$

Thus, in general, the time rates of change of vectors and tensors in the two frames are related by

$$\left(\frac{d}{dt}\right)_{\text{nonrot}} = \left(\frac{d}{dt}\right)_{\text{rot}} + \boldsymbol{\omega} \times \ . \tag{3.8.30}$$

3.9 Polar Decomposition Theorem

3.9.1 Preliminary Comments

Recall that the deformation gradient $\mathbf{F}$ transforms a material vector $d\mathbf{X}$ at $\mathbf{X}$ into the corresponding spatial vector $d\mathbf{x}$, and it characterizes all of the deformation, stretch (elongation) as well as rotation, at $\mathbf{X}$. Therefore, it forms an essential part of the definition of any strain measure. Another role of $\mathbf{F}$ in connection with the strain measures is discussed here with the help of the polar decomposition theorem of Cauchy. The polar decomposition theorem enables one to decompose $\mathbf{F}$ uniquely into the product of a proper orthogonal tensor and a symmetric positive-definite tensor and thereby decompose the general deformation into pure stretch and pure rotation.

3.9.2 Rotation and Stretch Tensors

Suppose that $\mathbf{F}$ is nonsingular so that each line element $d\mathbf{X}$ from the reference configuration is transformed into a unique line element $d\mathbf{x}$ in the current configuration, and conversely. Then the polar decomposition theorem states that $\mathbf{F}$ has a *unique* right and left (multiplicative) decompositions of the form (see Fig. 3.9.1)

$$\mathbf{F} = \mathbf{R} \cdot \mathbf{U} = \mathbf{V} \cdot \mathbf{R} \quad (F_{iI} = R_{iK}U_{KI} = V_{ij}R_{jI}), \tag{3.9.1}$$

so that

$$d\mathbf{x} = \mathbf{F} \cdot d\mathbf{X} = (\mathbf{R} \cdot \mathbf{U}) \cdot d\mathbf{X} = (\mathbf{V} \cdot \mathbf{R}) \cdot d\mathbf{X}, \tag{3.9.2}$$

where $\mathbf{U}$ is the symmetric and positive-definite *right Cauchy stretch tensor* (stretch is the ratio of the final length to the original length), $\mathbf{V}$ is the symmetric and positive-definite *left Cauchy stretch tensor*, and $\mathbf{R}$ is the *orthogonal rotation tensor*,

$$\mathbf{R}^{\mathrm{T}} \cdot \mathbf{R} = \mathbf{I}, \quad \mathbf{U} = \mathbf{U}^{\mathrm{T}}, \quad \mathbf{V} = \mathbf{V}^{\mathrm{T}}. \tag{3.9.3}$$

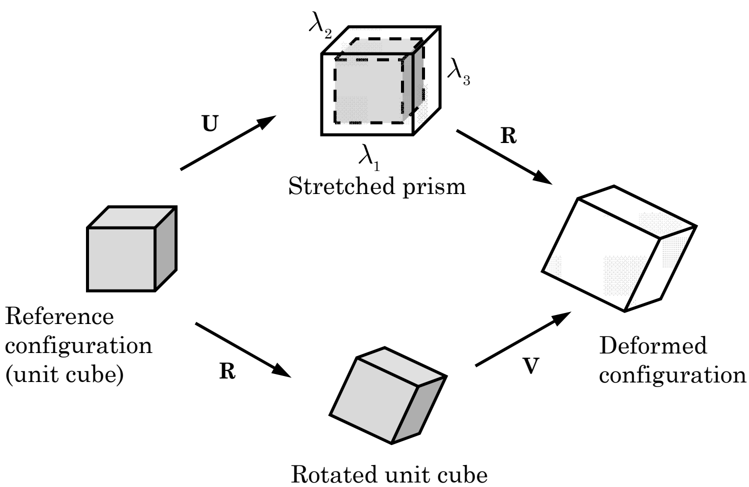


Fig. 3.9.1: The roles of $\mathbf{U}$, $\mathbf{V}$, and $\mathbf{R}$ in stretching and rotating a unit volume of material in the neighborhood of $\mathbf{X}$; λ_I $(I = 1, 2, 3)$ denote the principal stretches.

In Eq. (3.9.2), $\mathbf{U}\cdot d\mathbf{X}$ describes a pure stretch deformation in which there are three mutually perpendicular directions along which the material element $d\mathbf{X}$ stretches (that is, elongates or compresses) but does not rotate. The three directions are provided by the eigenvectors of $\mathbf{U}$. The role of the rotation tensor $\mathbf{R}$ is to rotate the stretched element, $\mathbf{R}\cdot\mathbf{U}\cdot d\mathbf{X}$. These ideas are illustrated in Fig. 3.9.2, which shows[8] the material occupying the spherical volume of radius $|d\mathbf{X}|$ in the undeformed configuration being mapped by the operator $\mathbf{U}$ into an ellipsoid in the deformed configuration at $\mathbf{x}$. Then $\mathbf{R}$ rotates the ellipsoid through a rigid-body rotation.

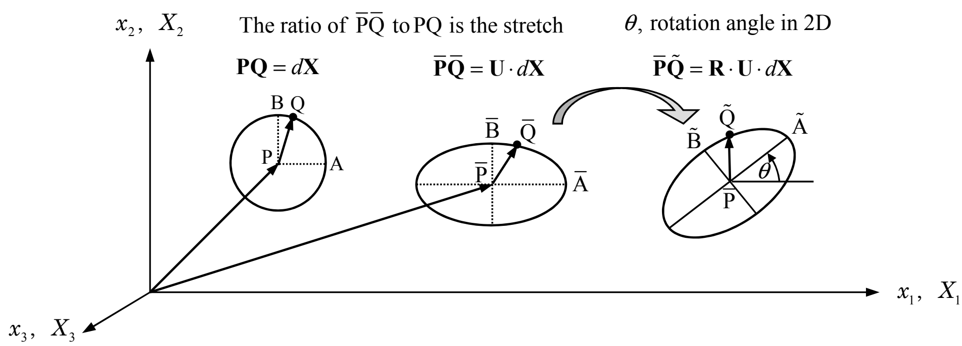


Fig. 3.9.2: The roles of $\mathbf{U}$ and $\mathbf{R}$ in transforming an ellipsoidal volume of material in the neighborhood of $\mathbf{X}$.

From Eqs. (3.9.1) and (3.9.3) it follows that

$$\mathbf{U}=\mathbf{R}^{-1}\cdot\mathbf{F}=\mathbf{R}^{\mathrm{T}}\cdot\mathbf{F},\quad \mathbf{V}=\mathbf{F}\cdot\mathbf{R}^{-1}=\mathbf{F}\cdot\mathbf{R}^{\mathrm{T}}, \tag{3.9.4}$$

$$\begin{aligned}\mathbf{U}^2&=\mathbf{U}\cdot\mathbf{U}=\mathbf{U}^{\mathrm{T}}\cdot\mathbf{U}=\mathbf{F}^{\mathrm{T}}\cdot(\mathbf{R}\cdot\mathbf{R}^{-1})\cdot\mathbf{F}=\mathbf{F}^{\mathrm{T}}\cdot\mathbf{F}=\mathbf{C},\\ \mathbf{V}^2&=\mathbf{V}\cdot\mathbf{V}=\mathbf{V}\cdot\mathbf{V}^{\mathrm{T}}=\mathbf{F}\cdot(\mathbf{R}^{-1}\cdot\mathbf{R})\cdot\mathbf{F}^{\mathrm{T}}=\mathbf{F}\cdot\mathbf{F}^{\mathrm{T}}=\mathbf{B},\end{aligned} \tag{3.9.5}$$

where $\mathbf{C}$ and $\mathbf{B}$ denote the right and left Cauchy–Green deformation tensors, respectively. We also note that

$$\begin{aligned}\mathbf{F}&=\mathbf{R}\cdot\mathbf{U}=(\mathbf{R}\cdot\mathbf{U})\cdot(\mathbf{R}^{\mathrm{T}}\cdot\mathbf{R})=(\mathbf{R}\cdot\mathbf{U}\cdot\mathbf{R}^{\mathrm{T}})\cdot\mathbf{R}=\mathbf{V}\cdot\mathbf{R}\\ &=\mathbf{V}\cdot\mathbf{R}=(\mathbf{R}\cdot\mathbf{R}^{\mathrm{T}})\cdot(\mathbf{V}\cdot\mathbf{R})=\mathbf{R}\cdot(\mathbf{R}^{\mathrm{T}}\cdot\mathbf{V}\cdot\mathbf{R})=\mathbf{R}\cdot\mathbf{U},\end{aligned} \tag{3.9.6}$$

which show that

$$\mathbf{U}=\mathbf{R}^{\mathrm{T}}\cdot\mathbf{V}\cdot\mathbf{R},\quad \mathbf{V}=\mathbf{R}\cdot\mathbf{U}\cdot\mathbf{R}^{\mathrm{T}}. \tag{3.9.7}$$

Since $\mathbf{C}=\mathbf{F}^{\mathrm{T}}\cdot\mathbf{F}$ and $\mathbf{B}=\mathbf{F}\cdot\mathbf{F}^{\mathrm{T}}$, we have $|\mathbf{C}|=|\mathbf{B}|=|\mathbf{F}|^2=J^2$ and therefore $|\mathbf{U}|=|\mathbf{V}|=\sqrt{|\mathbf{C}|}=+J$ (positive because $\mathbf{U}$ and $\mathbf{V}$ are positive-definite matrices). In view of Eq. (3.9.1), it follows that $|\mathbf{R}|=+1$, implying that $\mathbf{R}$ is a *proper* orthogonal tensor. Because $\mathbf{F}^{\mathrm{T}}\cdot\mathbf{F}$ is real and symmetric,

[8]For clarity, the figure shows stretch and rotation only in the X_1-X_2 plane.

there exists an orthogonal matrix $\mathbf{A}$ that transforms $\mathbf{C} = \mathbf{F}^{\mathrm{T}} \cdot \mathbf{F}$ into a diagonal matrix $\bar{\mathbf{C}}$:

$$\bar{\mathbf{C}} = \mathbf{A}^{\mathrm{T}}\mathbf{C}\mathbf{A} = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix} = \sum_{I=1}^{3} \lambda_I^2 \hat{\mathbf{N}}^{(I)}\hat{\mathbf{N}}^{(I)}, \tag{3.9.8}$$

where λ_I^2 are the eigenvalues of $\mathbf{C} = \mathbf{F}^{\mathrm{T}} \cdot \mathbf{F} = \mathbf{U}^2$ and $\mathbf{A}$ is the matrix of normalized eigenvectors $\hat{\mathbf{N}}^{(I)}$ (spectral theorem; see Section 2.5.6.3). The eigenvalues λ_I are called the *principal stretches* and the corresponding mutually orthogonal eigenvectors are called the *principal directions.* The tensors $\mathbf{U}$ and $\mathbf{V}$ have the same eigenvalues [see Eq. (3.9.7) and note $|\mathbf{R}| = 1$], and their eigenvectors differ only by the rotation $\mathbf{R}$; Problem **3.56** for a proof. Thus ($\bar{\mathbf{U}} = \sqrt{\bar{\mathbf{C}}}$)

$$\bar{\mathbf{U}} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}, \quad \mathbf{U} = \mathbf{A}\bar{\mathbf{U}}\mathbf{A}^{\mathrm{T}} = \sum_{I=1}^{3} \lambda_I \hat{\mathbf{N}}^{(I)}\hat{\mathbf{N}}^{(I)}, \tag{3.9.9}$$

where $\hat{\mathbf{N}}^{(I)}$ is the normalized eigenvector associated with eigenvalue λ_I in the reference configuration. Once the stretch tensor $\mathbf{U}$ is known, the rotation tensor $\mathbf{R}$ can be obtained from Eq. (3.9.1) and left stretch tensor $\mathbf{V}$ from Eq. (3.9.4) as

$$\mathbf{R} = \mathbf{F} \cdot \mathbf{U}^{-1}, \quad \mathbf{V} = \mathbf{F} \cdot \mathbf{R}^{\mathrm{T}}. \tag{3.9.10}$$

In view of Eq. (3.9.8), the Lagrangian and Eulerian strain tensors can be expressed in terms of $\mathbf{U}$ and $\mathbf{V}$ as

$$\mathbf{E} = \tfrac{1}{2}\left(\mathbf{U}^2 - \mathbf{I}\right) = \tfrac{1}{2}\sum_{I=1}^{3} \left(\lambda_I^2 - 1\right) \hat{\mathbf{N}}^{(I)}\hat{\mathbf{N}}^{(I)}, \tag{3.9.11}$$

$$\mathbf{e} = \tfrac{1}{2}\left(\mathbf{I} - \mathbf{V}^{-2}\right) = \tfrac{1}{2}\sum_{i=1}^{3} \left(1 - \lambda_i^{-2}\right) \hat{\mathbf{n}}^{(i)}\hat{\mathbf{n}}^{(i)}, \tag{3.9.12}$$

where $\hat{\mathbf{n}}^{(i)}$ is the normalized eigenvector in the current configuration. Next, we consider two examples of the use of the polar decomposition theorem.

Example 3.9.1

Consider the deformation mapping of Example 3.2.1,

$$x_1 = X_1 + At\,X_2, \quad x_2 = X_2 - At\,X_1, \quad x_3 = X_3.$$

It was shown in Example 3.2.1 that this mapping stretches a unit cube in the X_1 and X_2 directions and rotates about the X_3-axis, as shown in Fig. 3.2.5. Use the polar decomposition to determine the components of the symmetric right Cauchy stretch tensor $\mathbf{U}$, the rotation tensor $\mathbf{R}$, and the symmetric left Cauchy stretch tensor $\mathbf{V}$ associated with the deformation for $A = 0.25$ and $t = 2$.

Solution: The matrices associated with the deformation gradient $\mathbf{F}$ and the right Cauchy–Green deformation tensor $\mathbf{C}$ (depend only on A and t, as can be seen from Example 3.2.1) for $A = 0.25$ and $t = 2$ are

$$[F] = \begin{bmatrix} 1.0 & 0.5 & 0.0 \\ -0.5 & 1.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix}, \quad [C] = [F]^{\mathrm{T}}[F] = \begin{bmatrix} 1.25 & 0.00 & 0.00 \\ 0.00 & 1.25 & 0.00 \\ 0.00 & 0.00 & 1.00 \end{bmatrix}.$$

Thus, $[F]$ is independent of the position $\mathbf{X}$ and, therefore, the deformation is homogeneous. The eigenvalues associated with $[C]$ are $\lambda_1^2 = 1.25$, $\lambda_2^2 = 1.25$, and $\lambda_3^2 = 1.0$ for any point in the body. The matrix of normalized eigenvectors associated with these stretches is the identity matrix (the jth column is the eigenvector corresponding to the jth eigenvalue)

$$[A] = \begin{bmatrix} 1.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix}.$$

Then the matrix of the symmetric right stretch tensor $\mathbf{U}$ is determined from ($[C] = [\bar{C}]$ and $[U] = [\bar{U}]$):

$$[U]^2 = [C] = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix} \quad \text{or} \quad [U] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix},$$

where $\lambda_1 = \lambda_2 = 1.1180$ and $\lambda_3 = 1$. The matrix of eigenvectors remains the same, and we have

$$\mathbf{U} = \lambda_1 \hat{\mathbf{e}}_1 \hat{\mathbf{e}}_1 + \lambda_2 \hat{\mathbf{e}}_2 \hat{\mathbf{e}}_2 + \lambda_3 \hat{\mathbf{e}}_3 \hat{\mathbf{e}}_3.$$

We note that $\lambda_1 = 1.1180$ is the stretch of a line parallel to the X_1-axis, $\lambda_2 = 1.1180$ is the stretch of a line parallel to the X_2-axis, and $\lambda_3 = 1$ is the stretch of a line parallel to the X_3-axis (that is, the body did not undergo deformation in the thickness direction) in the undeformed body. The stretches can be verified independently by considering, as an example, the line $X_1 = 0$ (of unit length) in the undeformed body. In the deformed body the line has a length of $l = 1/\cos\alpha = 1.1180$, where $\tan\alpha = At = 0.5$ (or $\alpha = 26.565° = 0.46365$ rad.), as shown in Fig. 3.9.3.

The matrix associated with the rotation tensor $\mathbf{R}$ is determined from Eq. (3.9.10) ($1/\lambda_1 = 0.894427$) as

$$[R] = [F][U]^{-1} = \begin{bmatrix} 1.0 & 0.5 & 0.0 \\ -0.5 & 1.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix} \begin{bmatrix} \frac{1}{\lambda_1} & 0 & 0 \\ 0 & \frac{1}{\lambda_2} & 0 \\ 0 & 0 & \frac{1}{\lambda_3} \end{bmatrix} = \begin{bmatrix} 0.8944 & 0.4472 & 0.0 \\ -0.4472 & 0.8944 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix}.$$

We note that the rotation tensor is of the form ($\theta = -\alpha = -26.565°$)

$$[R] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.8944 & 0.4472 & 0.0 \\ -0.4472 & 0.8944 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix},$$

which agrees with the rotation shown in Fig. 3.9.3.

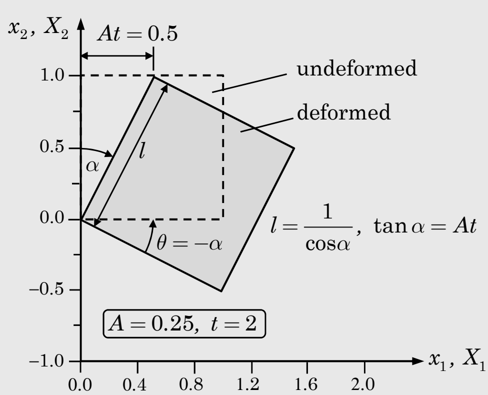


Fig. 3.9.3: Stretch and rotation of a unit square under the mapping, $x_1 = X_1 + At\,X_2$, $x_2 = X_2 - At\,X_1$, $x_3 = X_3$.

The left Cauchy stretch tensor components are determined using Eq. (3.9.10):

$$[V] = [F][R]^{\mathrm{T}} = \begin{bmatrix} 1.0 & 0.5 & 0.0 \\ -0.5 & 1.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix} \begin{bmatrix} 0.8944 & -0.4472 & 0.0 \\ 0.4472 & 0.8944 & 0.0 \\ 0.0000 & 0.0000 & 1.0 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}.$$

Example 3.9.2

Consider the deformation mapping

$$x_1 = \frac{1}{4}\left[4X_1 + (9 - 3X_1 - 5X_2 - X_1X_2)\,t\right], \quad x_2 = \frac{1}{4}\left[4X_2 + (16 + 8X_1)\,t\right], \quad x_3 = X_3.$$

For $(X_1, X_2, X_3) = (0, 0, 0)$ and time $t = 1$,

(a) determine the deformation gradient $\mathbf{F}$ and right Cauchy–Green deformation tensor $\mathbf{C}$,

(b) find the stretches λ_1 and λ_2 and the associated eigenvectors $\hat{\mathbf{N}}^{(1)}$ and $\hat{\mathbf{N}}^{(2)}$,

(c) use the polar decomposition to determine the components of the symmetric right Cauchy stretch tensor $\mathbf{U}$, the rotation tensor $\mathbf{R}$, and the symmetric left Cauchy stretch tensor $\mathbf{V}$, and

(d) use the polar decomposition to determine the components of Green–Lagrange strain tensor $\mathbf{E}$.

Solution: We note that the deformation is nonhomogeneous because of the term X_1X_2 in the mapping. The material point $(X_1, X_2, X_3) = (0, 0, 0)$ in the initial (that is, at $t = 0$) configuration occupies the location $(x_1, x_2, x_3) = (2.25, 4, 0)$ at $t = 1.0$, as shown in Fig. 3.9.4.

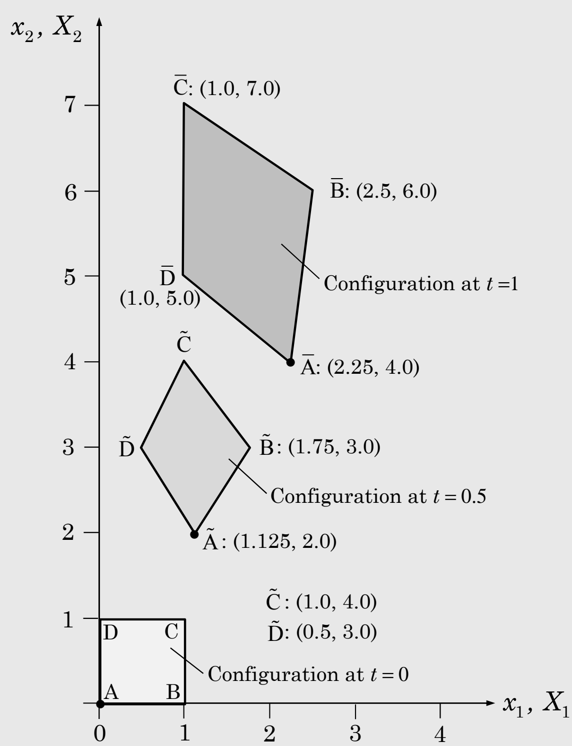


Fig. 3.9.4: Nonhomogeneous deformation of a unit square under the mapping, $x_1 = 0.25\,(9 + X_1 - 5X_2 - X_1X_2)$, $x_2 = 0.25\,(16 + 8X_1 + 4X_2)$, $x_3 = X_3$.

(a) The components of the deformation gradient **F** and right Cauchy–Green strain tensor **C** are

$$[F] = \frac{1}{4}\begin{bmatrix} 1 & -5 & 0 \\ 8 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \quad [C] = [F]^{\mathrm{T}}[F] = \frac{1}{16}\begin{bmatrix} 65 & 27 & 0 \\ 27 & 41 & 0 \\ 0 & 0 & 16 \end{bmatrix}.$$

(b) The eigenvalues λ_1^2, λ_2^2, and λ_3^2 of matrix $[C]$ are determined by setting

$$|[C] - \lambda^2[I]| = 0 \quad \rightarrow \quad \lambda_1^2 = 5.15916, \quad \lambda_2^2 = 1.46584, \quad \lambda_3^2 = 1,$$

and $\lambda_1 = 2.27138$, $\lambda_2 = 1.21072$, and $\lambda_3 = 1$. The eigenvectors are (in component form)

$$\{N^{(1)}\} = \begin{Bmatrix} 0.83849 \\ 0.54491 \\ 0.0 \end{Bmatrix}, \quad \{N^{(2)}\} = \begin{Bmatrix} 0.54491 \\ -0.83849 \\ 0.0 \end{Bmatrix}, \quad \{N^{(3)}\} = \begin{Bmatrix} 0.0 \\ 0.0 \\ 1.0 \end{Bmatrix}.$$

(c) The matrix of the right Cauchy stretch tensor is computed using Eq. (3.9.9):

$$\begin{aligned} [U] &= \begin{bmatrix} 0.83849 & 0.54491 & 0.0 \\ 0.54491 & -0.83849 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} 0.83849 & 0.54491 & 0.0 \\ 0.54491 & -0.83849 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix} \\ &= \begin{bmatrix} 1.95644 & 0.48462 & 0.0 \\ 0.48462 & 1.52566 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix}, \end{aligned}$$

and the principal stretches are $\lambda_1 = 2.27138$, $\lambda_2 = 1.21072$, and $\lambda_3 = 1$.

The tensor form of **U** is

$$\begin{aligned} \mathbf{U} &= \lambda_1 \hat{\mathbf{N}}^{(1)}\hat{\mathbf{N}}^{(1)} + \lambda_2 \hat{\mathbf{N}}^{(2)}\hat{\mathbf{N}}^{(2)} + \lambda_3 \hat{\mathbf{N}}^{(3)}\hat{\mathbf{N}}^{(3)} \\ &= \lambda_1 \left(N_1^{(1)}\hat{\mathbf{e}}_1 + N_2^{(1)}\hat{\mathbf{e}}_2 + N_3^{(1)}\hat{\mathbf{e}}_3\right)\left(N_1^{(1)}\hat{\mathbf{e}}_1 + N_2^{(1)}\hat{\mathbf{e}}_2 + N_3^{(1)}\hat{\mathbf{e}}_3\right) \\ &+ \lambda_2 \left(N_1^{(2)}\hat{\mathbf{e}}_1 + N_2^{(2)}\hat{\mathbf{e}}_2 + N_3^{(2)}\hat{\mathbf{e}}_3\right)\left(N_1^{(2)}\hat{\mathbf{e}}_1 + N_2^{(2)}\hat{\mathbf{e}}_2 + N_3^{(2)}\hat{\mathbf{e}}_3\right) \\ &+ \lambda_3 \left(N_1^{(3)}\hat{\mathbf{e}}_1 + N_2^{(3)}\hat{\mathbf{e}}_2 + N_3^{(3)}\hat{\mathbf{e}}_3\right)\left(N_1^{(3)}\hat{\mathbf{e}}_1 + N_2^{(3)}\hat{\mathbf{e}}_2 + N_3^{(3)}\hat{\mathbf{e}}_3\right), \end{aligned}$$

or

$$\begin{aligned} \mathbf{U} = &\left(\lambda_1[N_1^{(1)}]^2 + \lambda_2[N_1^{(2)}]^2 + \lambda_3[N_1^{(3)}]^2\right)\hat{\mathbf{e}}_1\hat{\mathbf{e}}_1 + \left(\lambda_1[N_2^{(1)}]^2 + \lambda_2[N_2^{(2)}]^2 + \lambda_3[N_2^{(3)}]^2\right)\hat{\mathbf{e}}_2\hat{\mathbf{e}}_2 \\ &+ \left(\lambda_1[N_3^{(1)}]^2 + \lambda_2[N_3^{(2)}]^2 + \lambda_3[N_3^{(3)}]^2\right)\hat{\mathbf{e}}_3\hat{\mathbf{e}}_3 + \left(\lambda_1 N_1^{(1)}N_2^{(1)} + \lambda_2 N_1^{(2)}N_2^{(2)}\right)(\hat{\mathbf{e}}_1\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_2\hat{\mathbf{e}}_1) \\ &+ \left(\lambda_1 N_1^{(1)}N_3^{(1)} + \lambda_3 N_1^{(3)}N_3^{(3)}\right)(\hat{\mathbf{e}}_1\hat{\mathbf{e}}_3 + \hat{\mathbf{e}}_3\hat{\mathbf{e}}_1) + \left(\lambda_2 N_2^{(2)}N_3^{(2)} + \lambda_3 N_2^{(3)}N_3^{(3)}\right)(\hat{\mathbf{e}}_2\hat{\mathbf{e}}_3 + \hat{\mathbf{e}}_3\hat{\mathbf{e}}_2) \\ &= 1.9564\hat{\mathbf{e}}_1\hat{\mathbf{e}}_1 + 0.4846\,(\hat{\mathbf{e}}_1\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_2\hat{\mathbf{e}}_1) + 1.5257\hat{\mathbf{e}}_2\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3\hat{\mathbf{e}}_3. \end{aligned}$$

The matrix of the rotation tensor **R** is determined from Eq. (3.9.10) as

$$[R] = [F][U]^{-1} = \begin{bmatrix} 0.3590 & -0.9333 & 0 \\ 0.9333 & 0.3590 & 0 \\ 0.0 & 0.0 & 1 \end{bmatrix}.$$

It follows that the rotation angle is $\theta = 68.96^\circ$. The left Cauchy stretch tensor components are determined using Eq. (3.9.10):

$$[V] = [F][R]^{\mathrm{T}} = \begin{bmatrix} 0.25 & -1.25 & 0.0 \\ 2.00 & 1.00 & 0.0 \\ 0.00 & 0.00 & 1.0 \end{bmatrix} \begin{bmatrix} 0.3590 & 0.9333 & 0.0 \\ -0.9333 & 0.3590 & 0.0 \\ 0.0000 & 0.0000 & 1.0 \end{bmatrix} = \begin{bmatrix} 1.2564 & -0.2153 & 0.0 \\ -0.2153 & 2.2256 & 0.0 \\ 0.0000 & 0.0000 & 1.0 \end{bmatrix}.$$

(d) The Green–Lagrange strain components at point $(X_1, X_2) = (0, 0)$ are computed using Eq. (3.9.11). We have [which can be verified using $\mathbf{E} = 0.5(\mathbf{C} - \mathbf{I})$]

$$
\begin{aligned}
E_{11} &= \tfrac{1}{2}\left[\left(\lambda_1^2 - 1\right) N_1^{(1)} N_1^{(1)} + \left(\lambda_2^2 - 1\right) N_1^{(2)} N_1^{(2)}\right] \\
&= \tfrac{1}{2}\,(4.15916 \times 0.83849 \times 0.83849 + 0.46584 \times 0.54491 \times 0.54491) = 1.5312, \\
E_{22} &= \tfrac{1}{2}\left[\left(\lambda_1^2 - 1\right) N_2^{(1)} N_2^{(1)} + \left(\lambda_2^2 - 1\right) N_2^{(2)} N_2^{(2)}\right] \\
&= \tfrac{1}{2}\,(4.15916 \times 0.54491 \times 0.54491 + 0.46584 \times 0.83849 \times 0.83849) = 0.7812, \\
E_{12} &= \tfrac{1}{2}\left[\left(\lambda_1^2 - 1\right) N_1^{(1)} N_2^{(1)} + \left(\lambda_2^2 - 1\right) N_1^{(2)} N_2^{(2)}\right] \\
&= \tfrac{1}{2}\,(4.15916 - 0.46584) \times 0.83849 \times 0.54491 = 0.8437.
\end{aligned}
$$

3.9.3 Objectivity of Stretch Tensors

Using the unique polar decomposition in Eq. (3.9.1), we can write

$$\mathbf{F}^* = \mathbf{R}^* \cdot \mathbf{U}^* = \mathbf{R}^* \cdot \mathbf{V}^*, \tag{3.9.13}$$

and objectivity of $\mathbf{F}$ gives

$$\mathbf{F}^* = \mathbf{Q} \cdot \mathbf{F} = \mathbf{Q} \cdot (\mathbf{R} \cdot \mathbf{U}) = \mathbf{Q} \cdot (\mathbf{V} \cdot \mathbf{R}). \tag{3.9.14}$$

Since $\mathbf{Q}$ and $\mathbf{R}$ are (proper) orthogonal tensors, we have

$$(\mathbf{Q} \cdot \mathbf{R}) \cdot (\mathbf{Q} \cdot \mathbf{R})^{\mathrm{T}} = \mathbf{Q} \cdot (\mathbf{R} \cdot \mathbf{R}^{\mathrm{T}}) \cdot \mathbf{Q}^{\mathrm{T}} = \mathbf{I}, \quad |\mathbf{Q} \cdot \mathbf{R}| = |\mathbf{Q}|\,|\mathbf{R}| = 1. \tag{3.9.15}$$

Thus, $\mathbf{Q} \cdot \mathbf{R}$ is also orthogonal. Therefore, in analogy to the two-point tensor $\mathbf{F}$, we can define $\mathbf{R}$ to be objective when [recall from the definition in Eq. (3.9.1) that $\mathbf{R}$ is a two-point tensor]

$$\mathbf{R}^* = \mathbf{Q} \cdot \mathbf{R}. \tag{3.9.16}$$

Then it follows from Eq. (3.9.14) that the right Cauchy stress tensor $\mathbf{U}$, being defined with respect to the reference configuration, remains unaltered by a superposed rigid-body motion and hence objective

$$\mathbf{U}^* = \mathbf{U}. \tag{3.9.17}$$

As far as the left Cauchy stretch tensor is concerned, the transformation law for it to be objective is derived as follows:

$$\mathbf{F}^* = \mathbf{V}^* \cdot \mathbf{R}^* = \mathbf{V}^* \cdot \mathbf{Q} \cdot \mathbf{R} = \mathbf{Q} \cdot \mathbf{F} = \mathbf{Q} \cdot \mathbf{V} \cdot \mathbf{R},$$

from which we arrive at the result

$$\mathbf{V}^* = \mathbf{Q} \cdot \mathbf{V} \cdot \mathbf{Q}^{\mathrm{T}}. \tag{3.9.18}$$

Thus, $\mathbf{V}$ transforms like a second-order tensor defined in the current configuration and is therefore objective.

We shall make use of the ideas presented in this section to develop the constitutive relations among the stress and strain measures (see Chapter 6).

3.10 Summary

In this chapter, deformation mapping χ of a material point occupying position $\mathbf{X}$ in a reference configuration to position $\mathbf{x}$ in the current configuration is introduced; two descriptions of motion, namely, the spatial (Eulerian) and material (Lagrange) descriptions of motion are described; and displacement $\mathbf{u}$ of a material point is defined as $\mathbf{u} = \mathbf{x} - \mathbf{X}$. The deformation gradient $\mathbf{F}$ is introduced as a two-point tensor between the reference configuration and the current configuration, $\mathbf{F} = (\boldsymbol{\nabla}_0 \mathbf{x})^{\mathrm{T}}$. The deformation tensor is nonsingular and hence invertible. Isochoric, homogeneous, and inhomogeneous deformations are discussed in terms of the deformation mapping and the deformation gradient. Changes of volume and surface in going from the reference configuration to the current configuration are derived.

Several strain measures, including the Green–Lagrange strain tensor $\mathbf{E}$, Cauchy strain tensor $\tilde{\mathbf{B}}$, Euler or Almansi strain tensor $\mathbf{e}$, right Cauchy–Green deformation tensor $\mathbf{C}$, the left Cauchy–Green deformation (or Finger) tensor $\mathbf{B}$, and the Cauchy strain tensor $\tilde{\mathbf{B}} = \mathbf{B}^{-1}$ are introduced. A physical interpretation of the normal and shear strain components is also presented. Determinations of the principal strains and principal directions of strain are discussed with the help of the eigenvalue problem of Section 2.5.6. The infinitesimal strain tensor $\boldsymbol{\varepsilon}$ is obtained from $\mathbf{E}$ by retaining terms $|\boldsymbol{\nabla}_0 \mathbf{u}|$ of order $\epsilon = \|\boldsymbol{\nabla}_0 \mathbf{u}\|_\infty$ and omitting terms of order ϵ^2. Thus, in the infinitesimal case, the distinction between the Green–Lagrange strain tensor $\mathbf{E}$ and the Euler strain tensor $\mathbf{e}$ disappears. The displacement gradient tensor $(\boldsymbol{\nabla}\mathbf{u})^{\mathrm{T}}$ is expressed as a sum of the symmetric strain tensor $\tilde{\boldsymbol{\varepsilon}}$ and the skew symmetric rotation tensor $\boldsymbol{\Omega}$. Similarly, the rate of deformation tensor $\mathbf{L} = (\boldsymbol{\nabla}\mathbf{v})^{\mathrm{T}}$, where $\mathbf{v}$ is the velocity vector, is expressed as the sum of the symmetric part, namely, the rate of deformation tensor $\mathbf{D}$, and the skew symmetric part, the vorticity tensor $\mathbf{W}$.

The effect of superposed rigid-body motion and the concept of frame indifference that ensures nondependency on the frame of reference in measuring displacements, velocities, accelerations, and various strain measures is briefly discussed. It is shown that the measures of displacements and various measures of strains obey the frame indifference principle (i.e., they are independent of the coordinate frame of reference), while the velocities and accelerations are dependent on the coordinate of frame of reference. It is also found that the time rate of change of the displacement as well as strain measures are not objective, unless the rigid-body rotation $\mathbf{Q}$ is independent of time.

Finally, the polar decomposition theorem is presented that allows the unique (multiplicative) left and right decompositions of the deformation gradient $\mathbf{F}$ into pure stretch and pure rotation, $\mathbf{F} = \mathbf{R}\cdot\mathbf{U}$ and $\mathbf{F} = \mathbf{V}\cdot\mathbf{R}$, where $\mathbf{U}$ and $\mathbf{V}$ are the symmetric, positive-definite right and left Cauchy stretch tensors, respectively, and $\mathbf{R}$ is the proper orthogonal rotation tensor. Compatibility conditions on infinitesimal strain tensor $\boldsymbol{\varepsilon}$ and deformation tensor $\mathbf{F}$ that ensure a unique determination of displacements from a given strain field are also presented.

Numerous examples are presented throughout the chapter to illustrate the concepts introduced.

Problems

Descriptions of Motion

3.1 Given the motion

$$\boldsymbol{\chi}(\mathbf{x}, t) = \mathbf{x} = (1+t)X_1\,\hat{\mathbf{e}}_1 + (1+t)X_2\,\hat{\mathbf{e}}_2 + X_3\,\hat{\mathbf{e}}_3, \quad 0 \le t < \infty,$$

(a) determine the velocity and acceleration fields of the motion, and

(b) sketch deformations of the line $X_2 = 2X_1$, for fixed $X_3 = 1$ at $t = 1, 2$, and 3.

3.2 Determine the deformation mapping that maps a unit square into the quadrilateral shape shown in Fig. P3.2. Assume that the mapping is a complete polynomial in X_1 and X_2 up to the term X_1X_2 (note that the constant term is zero for this case).

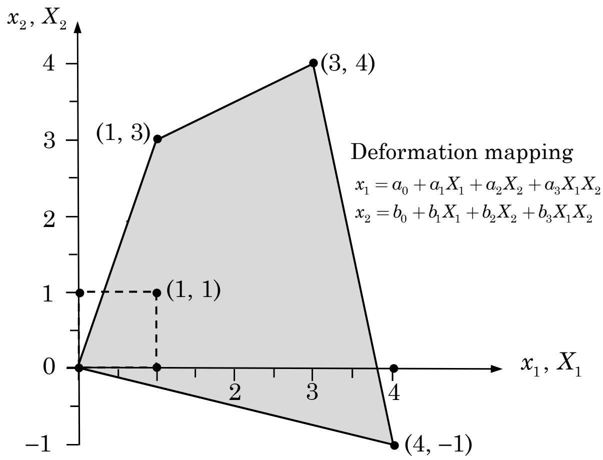


Fig. P3.2

3.3 Show that in the spatial description the acceleration components in the cylindrical coordinates are

$$a_r = \frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z\frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r},$$
$$a_\theta = \frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z\frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r},$$
$$a_z = \frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}.$$

3.4 Show that in the spatial description the acceleration components in the spherical coordinates are

$$a_R = \frac{\partial v_R}{\partial t} + v_R\frac{\partial v_R}{\partial R} + \frac{v_\phi}{R}\left(\frac{\partial v_R}{\partial \phi} - v_\phi\right) + \frac{v_\theta}{R\sin\phi}\left(\frac{\partial v_R}{\partial \theta} - v_\theta\sin\phi\right),$$
$$a_\phi = \frac{\partial v_\phi}{\partial t} + v_R\frac{\partial v_\phi}{\partial R} + \frac{v_\phi}{R}\left(\frac{\partial v_\phi}{\partial \phi} + v_R\right) + \frac{v_\theta}{R\sin\phi}\left(\frac{\partial v_\phi}{\partial \theta} - v_\theta\cos\phi\right),$$
$$a_\theta = \frac{\partial v_\theta}{\partial t} + v_R\frac{\partial v_\theta}{\partial R} + \frac{v_\phi}{R}\frac{\partial v_\theta}{\partial \phi} + \frac{v_\theta}{R\sin\phi}\left(\frac{\partial v_\theta}{\partial \theta} + v_R\sin\phi + v_\phi\cos\phi\right).$$

Analysis of Deformation and Strain Measures

3.5 The motion of a continuous medium is given by

$$x_1 = \left(1 + e^{at}\right)X_1, \quad x_2 = \left(1 + e^{-2at}\right)X_2, \quad x_3 = X_3, \quad 0 \le t < \infty,$$

where a is a positive constant. Determine

(a) the components of the deformation gradient **F** and the inverse mapping,

(b) the velocity components in the spatial description,

(c) the velocity components in the material description, and

(d) the acceleration components in the spatial description.

(e) Then verify the results of (d) by calculating first the acceleration components in the material coordinates and then using the inverse transformation in (a) to obtain the components in the spatial description.

3.6 For the deformation shown in Problem **3.2** (see Fig. P3.2), determine

(a) the components of the deformation gradient **F** and its inverse, and

(b) the components of the displacement vector.

3.7 The motion of a body is described by the mapping

$$\boldsymbol{\chi}(\mathbf{X}) = (X_1 + t^2 X_2)\,\hat{\mathbf{e}}_1 + (X_2 + t^2 X_1)\,\hat{\mathbf{e}}_2 + X_3\,\hat{\mathbf{e}}_3, \quad 0 \leq t < \infty.$$

Determine

(a) the components of the deformation gradient **F** and its inverse,

(b) the components of the displacement, velocity, and acceleration vectors,

(c) the position (X_1, X_2, X_3) of the particle in undeformed configuration that occupies the position $(x_1, x_2, x_3) = (9, 6, 1)$ at time $t = 2$ in the deformed configuration, and

(d) the location at time $t = 2$ of the particle that later will be located at $\mathbf{x} = (2, 3, 1)$ at time $t = 3$.

(e) Then plot the deformed shape of a body at times $t = 0, 1, 2,$ and 3, assuming that it is initially a unit cube.

3.8 *Homogeneous stretch.* Consider a body with deformation mapping of the form

$$\boldsymbol{\chi}(\mathbf{X}) = k_1 X_1\,\hat{\mathbf{e}}_1 + k_2 X_2\,\hat{\mathbf{e}}_2 + k_3 X_3\,\hat{\mathbf{e}}_3,$$

where $k_i \neq 0$ are constants. Determine the components of

(a) the deformation gradient **F**, and

(b) the right and left Cauchy–Green deformation tensors **C** and **B**.

3.9 *Homogeneous stretch followed by simple shear.* Consider a body with deformation mapping of the form

$$\boldsymbol{\chi}(\mathbf{X}) = (k_1 X_1 + e_0 k_2 X_2)\,\hat{\mathbf{e}}_1 + k_2 X_2\,\hat{\mathbf{e}}_2 + k_3 X_3\,\hat{\mathbf{e}}_3,$$

where $k_i \neq 0$ and e_0 are constants. Determine the components of

(a) the deformation gradient **F**, and

(b) the right and left Cauchy–Green deformation tensors **C** and **B**.

(c) Then plot representative shapes of a deformed unit square (let $k_1 = k_3 = 1$) that are achievable with this mapping; the suggested cases are (i) $k_2/k_1 = 1.5$, $e_0 = 0.1$; (ii) $k_2/k_1 = 1.5$, $e_0 = 0.25$; (iii) $k_2/k_1 = 1.25$, $e_0 = 0.5$; and (iv) $k_2/k_1 = 1.25$, $e_0 = 1.0$.

3.10 Suppose that the motion of a continuous medium is given by

$$\begin{aligned} x_1 &= X_1 \cos At + X_2 \sin At, \\ x_2 &= -X_1 \sin At + X_2 \cos At, \\ x_3 &= (1 + Bt) X_3, \quad 0 \leq t < \infty, \end{aligned}$$

where A and B are constants. Determine the components of

(a) the displacement vector in the material description,

(b) the displacement vector in the spatial description,

(c) displacement vector components in the spatial description with respect to a cylindrical basis, and

(d) the Green–Lagrange and Eulerian strain tensors in the Cartesian coordinate system.

3.11 If the deformation mapping of a body is given by

$$\boldsymbol{\chi}(\mathbf{X}) = (X_1 + AX_2)\,\hat{\mathbf{e}}_1 + (X_2 + BX_1)\,\hat{\mathbf{e}}_2 + X_3\,\hat{\mathbf{e}}_3,$$

where A and B are constants, determine

(a) the displacement components in the material description,

(b) the displacement components in the spatial description, and

(c) the components of the Green–Lagrange and Eulerian strain tensors.

3.12 For the deformation mapping in Problem **3.2**, determine the components of the Green–Lagrange strain tensor.

3.13 For the deformation field given in Problem **3.7**, determine the Green–Lagrange strain tensor components.

3.14 For the deformation mapping given in Problem **3.9**, determine the current positions (x_1, x_2) of material particles that were on the circle $X_1^2 + X_2^2 = R^2$ with radius R in the undeformed body.

3.15 The motion of a continuous medium is given by

$$\begin{aligned} x_1 &= \tfrac{1}{2}(X_1 + X_2)e^t + \tfrac{1}{2}(X_1 - X_2)e^{-t}, \\ x_2 &= \tfrac{1}{2}(X_1 + X_2)e^t - \tfrac{1}{2}(X_1 - X_2)e^{-t}, \\ x_3 &= X_3, \end{aligned}$$

for $0 \leq t < \infty$. Determine

(a) the velocity components in the material description,

(b) the velocity components in the spatial description, and

(c) the components of the rate of deformation and vorticity tensors.

3.16 *Nanson's formula* Let the differential area in the reference configuration be dA. Then

$$\hat{\mathbf{N}}\,dA = d\mathbf{X}^{(1)} \times d\mathbf{X}^{(2)} \quad \text{or} \quad N_I\,dA = e_{IJK}\,dX_J^{(1)} dX_K^{(2)},$$

where $d\mathbf{X}^{(1)}$ and $d\mathbf{X}^{(2)}$ are two nonparallel differential vectors in the reference configuration. The mapping from the undeformed configuration to the deformed configuration maps $d\mathbf{X}^{(1)}$ and $d\mathbf{X}^{(2)}$ into $d\mathbf{x}^{(1)}$ and $d\mathbf{x}^{(2)}$, respectively. Then $\hat{\mathbf{n}}\,da = d\mathbf{x}^{(1)} \times d\mathbf{x}^{(2)}$. Show that

$$\hat{\mathbf{n}}\,da = J\mathbf{F}^{-\mathrm{T}} \cdot \hat{\mathbf{N}}\,dA.$$

3.17 Consider a rectangular block of material of thickness h and sides $3b$ and $4b$, and having a triangular hole as shown in Fig. P3.17. If the block is subjected to the deformation mapping given in Eq. (3.3.14),

$$\boldsymbol{\chi}(\mathbf{X}) = (X_1 + \gamma X_2)\hat{\mathbf{e}}_1 + X_2\,\hat{\mathbf{e}}_2 + X_3\,\hat{\mathbf{e}}_3,$$

determine

(a) the equation of the line BC in the undeformed and deformed configurations,

(b) the angle ABC in the undeformed and deformed configurations, and

(c) the area of the triangle ABC in the undeformed and deformed configurations.

$$\chi(\mathbf{X}) = (X_1 + \gamma X_2)\hat{\mathbf{e}}_1 + X_2\,\hat{\mathbf{e}}_2 + X_3\,\hat{\mathbf{e}}_3$$

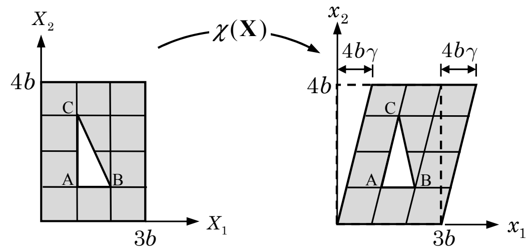


Fig. P3.17

NOTE: In Problems 3.18–3.22, undeformed and deformed configurations of bodies in equilibrium are given. In all cases, the deformation mapping can be determined uniquely with the suggested form of the mapping and the boundary data. Therefore, details of material constitution, material homogeneity, and loads causing deformation are not required to determine the kinematics of deformation.

3.18 Consider a unit square block of material of thickness h (into the plane of the paper), as shown in Fig. P3.18. If the block is subjected to a loading that deforms the square block into the shape shown (with no change in the thickness), (a) determine the deformation mapping, assuming that it is a complete polynomial in X_1 and X_2 up to the term X_1X_2, (b) compute the components of the right Cauchy–Green deformation tensor $\mathbf{C}$ and Green–Lagrange strain tensor $\mathbf{E}$ at the point $\mathbf{X} = (1, 1, 0)$, and (c) compute the principal strains and directions at $\mathbf{X} = (1, 1, 0)$ for $\gamma = 1$.

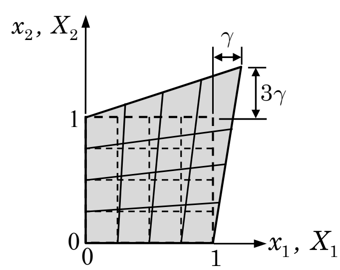
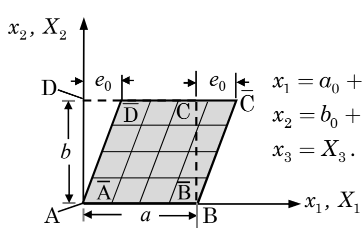


Fig. P3.18

3.19 Determine the displacements and Green–Lagrange strain tensor components for the deformed configuration shown in Fig. P3.19. The undeformed configuration is shown in dashed lines. Assume that the deformation mapping is a linear polynomial of X_1 and X_2 (note that for this case the constant terms are zero).



$$x_1 = a_0 + a_1X_1 + a_2X_2 + a_3X_1X_2,$$
$$x_2 = b_0 + b_1X_1 + b_2X_2 + b_3X_1X_2,$$
$$x_3 = X_3\,.$$

Fig. P3.19

3.20 Determine the displacements and Green–Lagrange strain components for the deformed configuration shown in Fig. P3.20. The undeformed configuration is shown in dashed lines. Use the suggested form of the deformation mapping, as implied by the deformed configuration.

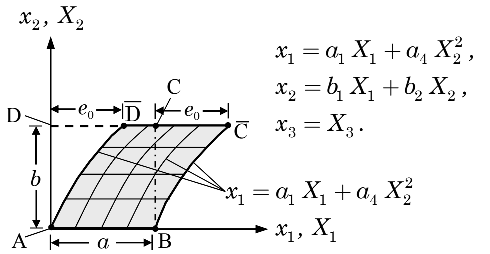


Fig. P3.20

3.21 Determine the displacements and Green–Lagrange strains in the (x_1, x_2, x_3) system for the deformed configuration shown in Fig. P3.21. The undeformed configuration is shown in dashed lines. Use the suggested form of the deformation mapping (for this case the constant terms are zero).

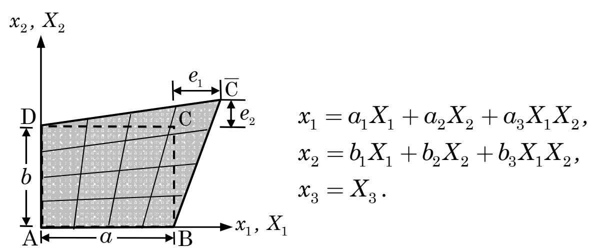
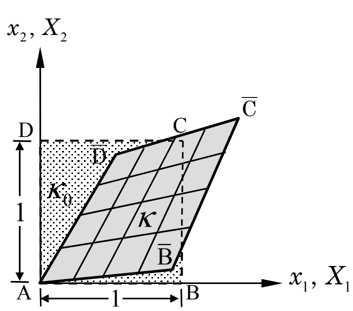


Fig. P3.21

3.22 Determine the displacements and Green–Lagrange strains for the deformed configuration shown in Fig. P3.22. The undeformed configuration is shown in dashed lines. Use the suggested form of the deformation mapping (note that constant terms are zero for this case).



κ_0	κ
A: $(X_1, X_2) = (0,0)$	$\bar{A}$: $(x_1, x_2) = (0,0)$
B: $(X_1, X_2) = (1,0)$	$\bar{B}$: $(x_1, x_2) = (0.8, 0.2)$
C: $(X_1, X_2) = (1,1)$	$\bar{C}$: $(x_1, x_2) = (1.3, 1.2)$
D: $(X_1, X_2) = (0,1)$	$\bar{D}$: $(x_1, x_2) = (0.5, 0.9)$

$$x_1 = a_0 + a_1X_1 + a_2X_2 + a_3X_1X_2;\ x_2 = b_0 + b_1X_1 + b_2X_2 + b_4X_1X_2;\ x_3 = X_3$$

Fig. P3.22

3.23 Given the following displacement vector in a material description using a cylindrical coordinate system

$$\mathbf{u} = Ar\hat{\mathbf{e}}_r + Brz\hat{\mathbf{e}}_\theta + C\sin\theta\hat{\mathbf{e}}_z,$$

where A, B, and C are constants, determine the infinitesimal strains. Here (r, θ, z) denote the material coordinates.

3.24 Show that the components of the Green–Lagrange strain tensor in cylindrical coordinate system are given by

$$E_{rr} = \frac{\partial u_r}{\partial r} + \tfrac{1}{2}\left[\left(\frac{\partial u_r}{\partial r}\right)^2 + \left(\frac{\partial u_\theta}{\partial r}\right)^2 + \left(\frac{\partial u_z}{\partial r}\right)^2\right],$$

$$E_{r\theta} = \tfrac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial r}\frac{\partial u_r}{\partial \theta} + \frac{1}{r}\frac{\partial u_\theta}{\partial r}\frac{\partial u_\theta}{\partial \theta}\right.$$
$$\left. + \frac{1}{r}\frac{\partial u_z}{\partial r}\frac{\partial u_z}{\partial \theta} + \frac{u_r}{r}\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\frac{\partial u_r}{\partial r}\right),$$

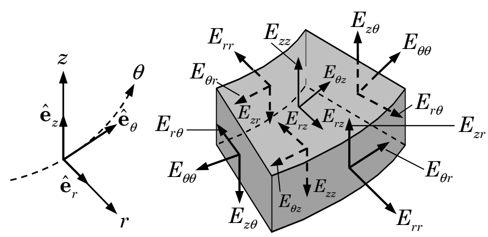
$$E_{rz} = \tfrac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial r}\frac{\partial u_r}{\partial z} + \frac{\partial u_\theta}{\partial r}\frac{\partial u_\theta}{\partial z} + \frac{\partial u_z}{\partial r}\frac{\partial u_z}{\partial z}\right),$$

$$E_{\theta\theta} = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{1}{2}\left[\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta}\right)^2 + \left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}\right)^2 + \left(\frac{1}{r}\frac{\partial u_z}{\partial \theta}\right)^2\right.$$
$$\left. - \frac{2}{r^2}u_\theta\frac{\partial u_r}{\partial \theta} + \frac{2}{r^2}u_r\frac{\partial u_\theta}{\partial \theta} + \left(\frac{u_\theta}{r}\right)^2 + \left(\frac{u_r}{r}\right)^2\right],$$

$$E_{\theta z} = \tfrac{1}{2}\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\frac{\partial u_r}{\partial z} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}\frac{\partial u_\theta}{\partial z}\right.$$
$$\left. + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\frac{\partial u_z}{\partial z} - \frac{u_\theta}{r}\frac{\partial u_r}{\partial z} + \frac{u_r}{r}\frac{\partial u_\theta}{\partial z}\right),$$

$$E_{zz} = \frac{\partial u_z}{\partial z} + \tfrac{1}{2}\left[\left(\frac{\partial u_r}{\partial z}\right)^2 + \left(\frac{\partial u_\theta}{\partial z}\right)^2 + \left(\frac{\partial u_z}{\partial z}\right)^2\right].$$

Here (r, θ, z) denote the material coordinates (see Fig. P3.24).



$E_{\xi\eta}$ is the component of strain in the ξ-direction on the η-plane

Fig. P3.24

3.25 The two-dimensional displacement field in a body is given by

$$u_1(\mathbf{X}) = X_1\left[X_1^2 X_2 + c_1\left(2c_2^3 + 3c_2^2 X_2 - X_2^3\right)\right],$$
$$u_2(\mathbf{X}) = -X_2\left(2c_2^3 + \frac{3}{2}c_2^2 X_2 - \frac{1}{4}X_2^3 + \frac{3}{2}c_1 X_1^2 X_2\right),$$

where c_1 and c_2 are constants. Find the linear and nonlinear Green–Lagrange strains.

3.26 Find the axial strain in the diagonal element, $\bar{\mathrm{A}}\bar{\mathrm{C}}$, of Problem **3.19**, using

(a) the basic definition of normal strain, and

(b) the strain transformation equations.

3.27 The biaxial state of strain at a point is given by $\varepsilon_{11} = 800\times10^{-6}$ in./in., $\varepsilon_{22} = 200\times10^{-6}$ in./in., $\varepsilon_{12} = 400 \times 10^{-6}$ in./in. Find the principal strains and their directions.

3.28 Show that the invariants J_1, J_2, and J_3 of the Green–Lagrange strain tensor $\mathbf{E}$ can be expressed in terms of the principal values λ_i of $\mathbf{E}$ as

$$J_1 = \lambda_1 + \lambda_2 + \lambda_3, \quad J_2 = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1, \quad J_3 = \lambda_1\lambda_2\lambda_3.$$

Of course, the above result holds for any second-order tensor.

3.29 Given the displacement field in the cylindrical coordinate system

$$u_r = U(r), \quad u_\theta = 0, \quad u_z = 0,$$

where $U(r)$ is a function of only r, determine the Green–Lagrange strain components.

3.30 Given the displacement field in the spherical coordinate system

$$u_R = U(R), \quad u_\phi = 0, \quad u_\theta = 0,$$

where $U(r)$ is a function of only r, determine the Green–Lagrange strain components.

Velocity Gradient, Rate of Deformation, and Vorticity Tensors

3.31 Show that the components of the spin tensor $\mathbf{W}$ in the cylindrical coordinate system are

$$W_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r} - \frac{\partial v_\theta}{\partial r}\right) = -W_{\theta r},$$
$$W_{rz} = \frac{1}{2}\left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right) = -W_{zr},$$
$$W_{z\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z}\right) = -W_{\theta z}.$$

3.32 If $\mathbf{D} = \mathbf{0}$, show that

$$\mathbf{v} = \mathbf{w} \times \mathbf{x} + \mathbf{c} \quad (v_i = e_{ijk}\, w_j\, x_k + c_i),$$

where both $\mathbf{w}$ (vorticity vector) and $\mathbf{c}$ are constant vectors.

3.33 Show that

$$\dot{\mathbf{E}} = \frac{1}{2}\dot{\mathbf{C}} = \frac{1}{2}\left(\dot{\mathbf{F}}^{\mathrm{T}} \cdot \mathbf{F} + \mathbf{F}^{\mathrm{T}} \cdot \dot{\mathbf{F}}\right),$$

and

$$\mathbf{W} = \frac{1}{2}\left(\dot{\mathbf{F}} \cdot \mathbf{F}^{-1} - \mathbf{F}^{-\mathrm{T}} \cdot \dot{\mathbf{F}}^{\mathrm{T}}\right).$$

3.34 Verify that

$$\dot{\mathbf{v}} = \frac{\partial \mathbf{v}}{\partial t} + \frac{1}{2}\boldsymbol{\nabla}(\mathbf{v}\cdot\mathbf{v}) + 2\mathbf{W}\cdot\mathbf{v}$$
$$= \frac{\partial \mathbf{v}}{\partial t} + \frac{1}{2}\boldsymbol{\nabla}(\mathbf{v}\cdot\mathbf{v}) + 2\mathbf{w}\times\mathbf{v},$$

where $\mathbf{W}$ is the spin tensor and $\mathbf{w}$ is the vorticity vector [see Eq. (3.6.5)].

3.35 Show that

$$\frac{DJ}{Dt} = (\boldsymbol{\nabla}\cdot\mathbf{v})\, J.$$

Hints: $\frac{Dx_i}{Dt} = v_i$ and $\frac{\partial v_i}{\partial X_j} = \frac{\partial v_i}{\partial x_k}\frac{\partial x_k}{\partial X_j}$. See also the list of properties of determinants highlighted in Section 2.3.6.

3.36 Establish the identities

$$d\mathbf{v} = \mathbf{L}\cdot d\mathbf{x}, \quad \text{and} \quad (\boldsymbol{\nabla}_0\mathbf{v})^{\mathrm{T}} = \mathbf{L}\cdot\mathbf{F}.$$

3.37 Show that $\dot{\mathbf{C}} = 2\mathbf{F}^{\mathrm{T}}\cdot\mathbf{D}\cdot\mathbf{F}$, where $\mathbf{C}$, $\mathbf{D}$, and $\mathbf{F}$ are the right Cauchy–Green deformation tensor, rate of deformation tensor, and deformation gradient, respectively.

3.38 Show that the Eulerian strain rate is given by

$$\dot{\mathbf{e}} = \mathbf{D} - \left(\mathbf{e}\cdot\mathbf{L} + \mathbf{L}^{\mathrm{T}}\cdot\mathbf{e}\right),$$

and [see Eq. (3.5.10) for the definition of $\tilde{\boldsymbol{\varepsilon}}$]

$$\dot{\tilde{\boldsymbol{\varepsilon}}} = \mathbf{D}.$$

Compatibility Conditions

3.39 Use the index notation to establish the compatibility conditions in Eq. (3.7.11)

$$\boldsymbol{\nabla}_0 \times (\boldsymbol{\nabla}_0 \times \boldsymbol{\varepsilon})^{\mathrm{T}} = \mathbf{0}$$

for the infinitesimal strains. *Hint:* Begin with $\boldsymbol{\nabla}_0 \times \boldsymbol{\varepsilon}$ and use Eq. (3.5.15).

3.40 Show that the following second-order tensor is symmetric:

$$\mathbf{S} = \boldsymbol{\nabla}_0 \times (\boldsymbol{\nabla}_0 \times \boldsymbol{\varepsilon})^{\mathrm{T}}.$$

3.41 Let [see the compatibility conditions in Eqs. (3.7.4)–(3.7.9)]

$$-S_{33} = R_3 = \frac{\partial^2 \varepsilon_{11}}{\partial X_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial X_1^2} - 2\frac{\partial^2 \varepsilon_{12}}{\partial X_1 \partial X_2}, \tag{1}$$

$$-S_{22} = R_2 = \frac{\partial^2 \varepsilon_{11}}{\partial X_3^2} + \frac{\partial^2 \varepsilon_{33}}{\partial X_1^2} - 2\frac{\partial^2 \varepsilon_{13}}{\partial X_1 \partial X_3}, \tag{2}$$

$$-S_{11} = R_1 = \frac{\partial^2 \varepsilon_{22}}{\partial X_3^2} + \frac{\partial^2 \varepsilon_{33}}{\partial X_2^2} - 2\frac{\partial^2 \varepsilon_{23}}{\partial X_2 \partial X_3}, \tag{3}$$

$$-S_{23} = U_1 = -\frac{\partial^2 \varepsilon_{11}}{\partial X_2 \partial X_3} + \frac{\partial}{\partial X_1}\left(-\frac{\partial \varepsilon_{23}}{\partial X_1} + \frac{\partial \varepsilon_{13}}{\partial X_2} + \frac{\partial \varepsilon_{12}}{\partial X_3}\right), \tag{4}$$

$$-S_{31} = U_2 = -\frac{\partial^2 \varepsilon_{22}}{\partial X_1 \partial X_3} + \frac{\partial}{\partial X_2}\left(\frac{\partial \varepsilon_{23}}{\partial X_1} - \frac{\partial \varepsilon_{13}}{\partial X_2} + \frac{\partial \varepsilon_{12}}{\partial X_3}\right), \tag{5}$$

$$-S_{12} = U_3 = -\frac{\partial^2 \varepsilon_{33}}{\partial X_1 \partial X_2} + \frac{\partial}{\partial X_3}\left(\frac{\partial \varepsilon_{23}}{\partial X_1} + \frac{\partial \varepsilon_{13}}{\partial X_2} - \frac{\partial \varepsilon_{12}}{\partial X_3}\right). \tag{6}$$

Show that

$$\begin{aligned}
\frac{\partial R_1}{\partial X_1} + \frac{\partial U_3}{\partial X_2} + \frac{\partial U_2}{\partial X_3} &= 0,\\
\frac{\partial U_3}{\partial X_1} + \frac{\partial R_2}{\partial X_2} + \frac{\partial U_1}{\partial X_3} &= 0,\\
\frac{\partial U_2}{\partial X_1} + \frac{\partial U_1}{\partial X_2} + \frac{\partial R_3}{\partial X_3} &= 0.
\end{aligned} \tag{7}$$

These relations are known as the *Bianchi formulas.*

3.42 Consider the following infinitesimal strain field:

$$\varepsilon_{11} = c_1 X_2^2, \quad \varepsilon_{22} = c_1 X_1^2, \quad 2\varepsilon_{12} = c_2 X_1 X_2,$$
$$\varepsilon_{31} = \varepsilon_{32} = \varepsilon_{33} = 0,$$

where c_1 and c_2 are constants. Determine

(a) c_1 and c_2 such that there exists a continuous, single-valued displacement field that corresponds to this strain field,

(b) the most general form of the corresponding displacement field using c_1 and c_2 obtained in (a), and

(c) the constants of integration introduced in (b) for the boundary conditions $\mathbf{u} = \mathbf{0}$ and $\boldsymbol{\Omega} = \mathbf{0}$ at $\mathbf{X} = \mathbf{0}$ (i.e., $u_1 = u_2 = 0$ and $\frac{\partial u_1}{\partial X_2} - \frac{\partial u_2}{\partial X_1} = 0$ at $X_1 = X_2 = 0$).

3.43 Determine whether the following strain fields, under the assumption of infinitesimal strains, are possible in a continuous body:

$$\text{(a) } [\varepsilon] = \begin{bmatrix} (X_1^2 + X_2^2) & X_1X_2 \\ X_1X_2 & X_2^2 \end{bmatrix}. \quad \text{(b) } [\varepsilon] = \begin{bmatrix} X_3(X_1^2 + X_2^2) & 2X_1X_2X_3 & X_3 \\ 2X_1X_2X_3 & X_2^2 & X_1 \\ X_3 & X_1 & X_3^2 \end{bmatrix}.$$

3.44 Evaluate the compatibility conditions $\boldsymbol{\nabla}_0 \times (\boldsymbol{\nabla}_0 \times \mathbf{E})^{\mathrm{T}} = \mathbf{0}$ in cylindrical coordinates.

3.45 Given the infinitesimal strain components

$$\varepsilon_{11} = f(X_2, X_3), \quad \varepsilon_{22} = \varepsilon_{33} = -\nu f(X_2, X_3), \quad \varepsilon_{12} = \varepsilon_{13} = \varepsilon_{23} = 0,$$

determine the form of $f(X_2, X_3)$ in order that the strain field is compatible.

3.46 Given the strain tensor $\mathbf{E} = E_{rr}\hat{\mathbf{e}}_r\hat{\mathbf{e}}_r + E_{\theta\theta}\hat{\mathbf{e}}_\theta\hat{\mathbf{e}}_\theta$ in an axisymmetric body (i.e., E_{rr} and $E_{\theta\theta}$ are functions of r and z only), determine the compatibility conditions on E_{rr} and $E_{\theta\theta}$. *Hint:* See Example 2.5.1.

Rigid-Body Motion and Objectivity

3.47 Determine the effect of the superposed rigid-body motion on the left Cauchy–Green deformation tensor $\mathbf{B} = \mathbf{F} \cdot \mathbf{F}^{\mathrm{T}}$.

3.48 If $\mathbf{Q}(t)$ is an orthogonal tensor-valued function of a scalar t [i.e., $\mathbf{Q}^{-1} = \mathbf{Q}^{\mathrm{T}}$], show that $\dot{\mathbf{Q}} \cdot \mathbf{Q}^{\mathrm{T}} = -(\dot{\mathbf{Q}} \cdot \mathbf{Q}^{\mathrm{T}})^{\mathrm{T}}$. That is, show that $\dot{\mathbf{Q}} \cdot \mathbf{Q}^{\mathrm{T}}$ is skew symmetric.

3.49 Show that the spin tensor $\mathbf{W}$ under superposed rigid-body motion becomes

$$\mathbf{W}^* = \mathbf{Q} \cdot \mathbf{W} \cdot \mathbf{Q}^{\mathrm{T}} + \boldsymbol{\Omega},$$

where $\boldsymbol{\Omega}$ is the skew symmetric rotation tensor, $\boldsymbol{\Omega} = \dot{\mathbf{Q}} \cdot \mathbf{Q}^{\mathrm{T}}$ [see also Eq. (3.8.19)].

3.50 Suppose that the second-order tensor $\mathbf{T}$ is objective in the sense that it satisfies the condition $\mathbf{T}^* = \mathbf{Q} \cdot \mathbf{T} \cdot \mathbf{Q}$, where quantities with and without an asterisk belong to two different frames of reference. Then show that the following second-order tensor $\mathbf{S}$ is objective (i.e., show that $\mathbf{S}^* = \mathbf{S}$):

$$\mathbf{S} = \mathbf{F}^{-1} \cdot \mathbf{T} \cdot \mathbf{F}^{-\mathrm{T}},$$

where $\mathbf{Q}$ is a proper orthogonal tensor.

3.51 Prove or disprove if the following second tensor satisfies objectivity:

$$\mathbf{T} = \mathbf{S} \cdot \mathbf{U} + \mathbf{U} \cdot \mathbf{S},$$

where $\mathbf{S}^* = \mathbf{R} \cdot \mathbf{S} \cdot \mathbf{R}^{\mathrm{T}}$, $\mathbf{U} = \mathbf{R}^{\mathrm{T}} \cdot \mathbf{F}$ is the right Cauchy stretch tensor, and $\mathbf{F}$ is the deformation gradient.

3.52 Using the transformation rule $\mathbf{F}^* = \mathbf{Q} \cdot \mathbf{F}$, show that the Euler stain tensor $\mathbf{e}$ transforms according to the rule under superposed rigid-body motion

$$\mathbf{e}^* = \mathbf{Q} \cdot \mathbf{e} \cdot \mathbf{Q}^{\mathrm{T}}.$$

3.53 Show that the spatial gradient of a vector $\mathbf{u}(\mathbf{x}, t)$ is objective, that is, prove

$$\boldsymbol{\nabla}^*\mathbf{u}^*(\mathbf{x}^*, t^*) = \mathbf{Q}(t) \cdot \boldsymbol{\nabla}\mathbf{u}(\mathbf{x}, t) \cdot \mathbf{Q}^{\mathrm{T}}(t).$$

3.54 Show that the material time derivatives of objective vector and tensor fields, $\mathbf{u}$ and $\mathbf{S}$, are not objective.

Polar Decomposition

3.55 Establish the uniqueness of the decomposition $\mathbf{F} = \mathbf{R} \cdot \mathbf{U} = \mathbf{V} \cdot \mathbf{R}$. For example, if $\mathbf{F} = \mathbf{R}_1 \cdot \mathbf{U}_1 = \mathbf{R}_2 \cdot \mathbf{U}_2$, then show that $\mathbf{R}_1 = \mathbf{R}_2$ and $\mathbf{U}_1 = \mathbf{U}_2$.

3.56 Show that the eigenvalues of the left and right Cauchy stretch tensors $\mathbf{U}$ and $\mathbf{V}$ are the same and that the eigenvector of $\mathbf{V}$ is given by $\mathbf{R} \cdot \mathbf{n}$, where $\mathbf{n}$ is the eigenvector of $\mathbf{U}$.

3.57 (a) If λ is the eigenvalue and $\mathbf{n}$ is the eigenvector of $\mathbf{U}$, show that the eigenvalue of $\mathbf{C}$ is λ^2 and the eigenvector is the same as that of $\mathbf{U}$. (b) Show that a line element in the principal direction $\mathbf{n}$ of $\mathbf{C}$ becomes an element in the direction of $\mathbf{R} \cdot \mathbf{n}$ in the deformed configuration.

3.58 Show that the spin tensor $\mathbf{W}$ can be written as

$$2\mathbf{W} = 2\dot{\mathbf{R}} \cdot \mathbf{R}^{\mathrm{T}} + \mathbf{R} \cdot \left(\dot{\mathbf{U}} \cdot \mathbf{U}^{-1} - \mathbf{U}^{-1} \cdot \dot{\mathbf{U}}\right) \cdot \mathbf{R}^{\mathrm{T}},$$

where $\mathbf{R}$ is the (proper) orthogonal rotation tensor $\mathbf{R}^{-1} = \mathbf{R}^{\mathrm{T}}$ and $\mathbf{U}$ is the symmetric positive-definite right Cauchy stretch tensor. Also show that for rigid-body motion, one has $\mathbf{W} = \dot{\mathbf{R}} \cdot \mathbf{R}^{\mathrm{T}}$.

3.59 Prove the symmetry and positive-definiteness of the right Cauchy–Green deformation tensor $\mathbf{C} = \mathbf{F}^{\mathrm{T}} \cdot \mathbf{F}$.

3.60 Calculate $\sqrt{\mathbf{C}}$ when

$$[C] = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 9 \end{bmatrix}.$$

3.61 Given that

$$[F] = \frac{1}{5}\begin{bmatrix} 2 & -5 \\ 11 & 2 \end{bmatrix},$$

determine the right and left stretch tensors.

3.62 Given that

$$[F] = \begin{bmatrix} \sqrt{3} & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

determine the right and left stretch tensors.

3.63 Calculate the left and right Cauchy stretch tensors $\mathbf{U}$ and $\mathbf{V}$ associated with $\mathbf{F}$ of Problem **3.11** for the choice of $A = 2$ and $B = 0$.