4
STRESS MEASURES

Most of the fundamental ideas of science are essentially simple, and may, as a rule, be expressed in a language comprehensible to everyone. — Albert Einstein (1879–1955)

4.1 Introduction

In the beginning of Chapter 3, we briefly discussed the need for studying deformations and stresses in material systems that we may design for engineering applications. All materials have certain thresholds to withstand forces, beyond which they "fail" to perform their intended function. The force per unit area, called *stress*, is a measure of the capacity of the material to carry loads, and all designs are based on the criterion that the materials used have the capacity to carry the working loads of the system. Thus, it is necessary to determine the state of stress in a material.

In this chapter we study the concept of stress and its various measures. For instance, stress can be measured per unit deformed area or undeformed area. As we shall see shortly, stress at a point in a three-dimensional continuum can be measured in terms of nine quantities, three per plane, on three mutually perpendicular planes at the point. These nine quantities may be viewed as the components of a second-order tensor, called a *stress tensor*. Coordinate transformations and principal values associated with the stress tensor and stress equilibrium equations are also discussed.

4.2 Cauchy Stress Tensor and Cauchy's Formula

4.2.1 Stress Vector

First we introduce the true stress, that is, the stress in the deformed configuration κ that is measured per unit area of the deformed configuration κ. The surface force acting on a small element of (surface) area in a continuous medium depends not only on the magnitude of the area but also on the orientation of the area. It is customary to denote the direction of a plane area by means of a unit vector drawn normal to that plane. The direction of the normal is taken by convention as that in which a right-handed screw advances as it is rotated according to the direction of travel along the boundary curve or contour. Let the unit normal vector be denoted by $\hat{\mathbf{n}}$. Then the area is expressed as $\mathbf{A} = A\hat{\mathbf{n}}$.

If we denote by $d\mathbf{f}(\hat{\mathbf{n}})$ the force on a small area $\hat{\mathbf{n}}\,da$ located at position $\mathbf{x}$, the *stress vector* can be defined, shown graphically in Fig. 4.2.1, as

$$\mathbf{t}(\hat{\mathbf{n}}) = \lim_{\Delta a \to 0} \frac{\Delta \mathbf{f}(\hat{\mathbf{n}})}{\Delta a}. \tag{4.2.1}$$

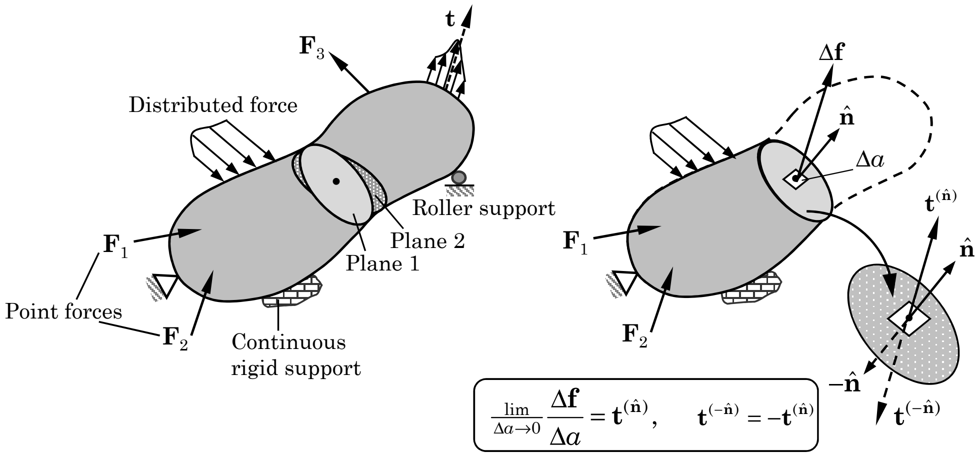


Fig. 4.2.1: Cuts through a point of a material body by planes of different orientations. The figure also shows a stress vector on a plane whose normal is $\hat{\mathbf{n}}$.

We see that the stress vector $\mathbf{t}$ is a point function of the unit normal $\hat{\mathbf{n}}$, which denotes the orientation of the plane on which $\mathbf{t}$ acts. Because of Newton's third law for action and reaction, we see that $\mathbf{t}(-\hat{\mathbf{n}}) = -\mathbf{t}(\hat{\mathbf{n}})$. It is fruitful to establish a relationship between $\mathbf{t}$ and $\hat{\mathbf{n}}$.

4.2.2 Cauchy's Formula

To establish the relationship between $\mathbf{t}$ and $\hat{\mathbf{n}}$ for the infinitesimal deformation[1], we set up an infinitesimal tetrahedron in Cartesian coordinates. The tetrahedron can come either from an interior point or from a boundary point, as indicated in Fig. 4.2.2(a). If $-\mathbf{t}_1, -\mathbf{t}_2, -\mathbf{t}_3$, and $\mathbf{t}$ denotes the stress vectors in the outward directions on the faces of the infinitesimal tetrahedron whose areas are Δa_1, Δa_2, Δa_3, and Δa, respectively, as shown in Fig. 4.2.2(b) (i.e., $-\mathbf{t}_j$ acts on the plane perpendicular to the negative x_j-axis), we have by Newton's second law for the mass inside the tetrahedron,

$$\mathbf{t}\,\Delta a - \mathbf{t}_1\,\Delta a_1 - \mathbf{t}_2\,\Delta a_2 - \mathbf{t}_3\,\Delta a_3 + \rho\,\Delta v\,\mathbf{f} = \rho\,\Delta v\,\mathbf{a}, \tag{4.2.2}$$

where Δv is the volume of the tetrahedron, ρ is the density, $\mathbf{f}$ is the body force per unit mass, and $\mathbf{a}$ is the acceleration. Because the total vector area of a closed surface is zero (by the gradient theorem), we have

$$\Delta a\,\hat{\mathbf{n}} - \Delta a_1\,\hat{\mathbf{e}}_1 - \Delta a_2\,\hat{\mathbf{e}}_2 - \Delta a_3\,\hat{\mathbf{e}}_3 = \mathbf{0}. \tag{4.2.3}$$

It follows that

$$\Delta a_1 = (\hat{\mathbf{e}}_1 \cdot \hat{\mathbf{n}})\Delta a, \quad \Delta a_2 = (\hat{\mathbf{e}}_2 \cdot \hat{\mathbf{n}})\Delta a, \quad \Delta a_3 = (\hat{\mathbf{e}}_3 \cdot \hat{\mathbf{n}})\Delta a. \tag{4.2.4}$$

[1]The Cauchy formula can be established for the finite deformation case by considering a tetrahedron with curved faces.

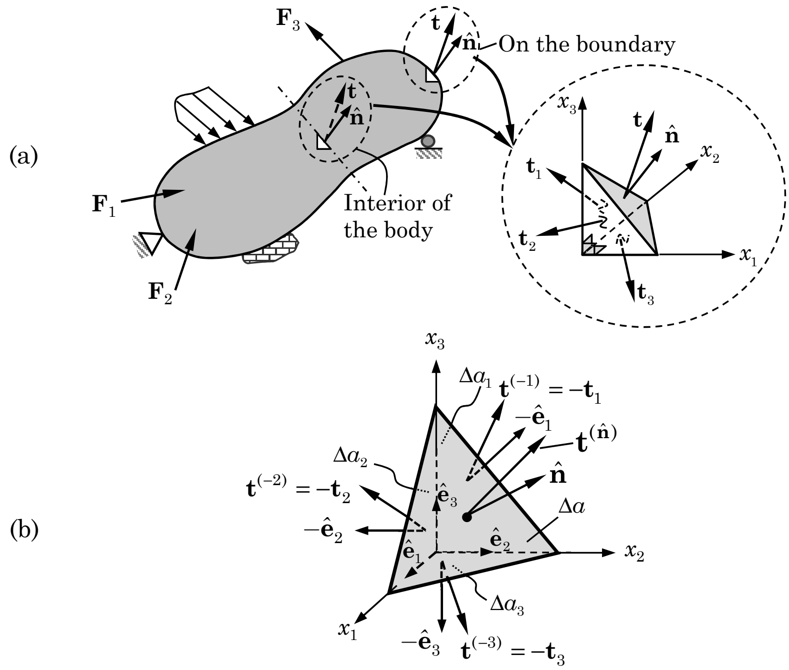


Fig. 4.2.2: A tetrahedral element with stress vectors on all its faces.

The volume of the element Δv can be expressed as

$$\Delta v = \frac{\Delta h}{3}\Delta a, \tag{4.2.5}$$

where Δh is the perpendicular distance from the origin to the slant face.

Substitution of Eqs. (4.2.4) and (4.2.5) into Eq. (4.2.2) and dividing throughout by Δa yields

$$\mathbf{t} = \mathbf{t}_1(\hat{\mathbf{e}}_1 \cdot \hat{\mathbf{n}}) + \mathbf{t}_2(\hat{\mathbf{e}}_2 \cdot \hat{\mathbf{n}}) + \mathbf{t}_3(\hat{\mathbf{e}}_3 \cdot \hat{\mathbf{n}}) + \rho\tfrac{\Delta h}{3}(\mathbf{a} - \mathbf{f}). \tag{4.2.6}$$

In the limit as the tetrahedron is shrunk to a point, $\Delta h \to 0$, we obtain

$$\mathbf{t} = \mathbf{t}_1(\hat{\mathbf{e}}_1 \cdot \hat{\mathbf{n}}) + \mathbf{t}_2(\hat{\mathbf{e}}_2 \cdot \hat{\mathbf{n}}) + \mathbf{t}_3(\hat{\mathbf{e}}_3 \cdot \hat{\mathbf{n}}) = \mathbf{t}_i(\hat{\mathbf{e}}_i \cdot \hat{\mathbf{n}}). \tag{4.2.7}$$

4.2.3 Cauchy Stress Tensor

It is convenient to display Eq. (4.2.7) as

$$\mathbf{t} = (\mathbf{t}_1\,\hat{\mathbf{e}}_1 + \mathbf{t}_2\,\hat{\mathbf{e}}_2 + \mathbf{t}_3\,\hat{\mathbf{e}}_3) \cdot \hat{\mathbf{n}}. \tag{4.2.8}$$

The terms in the parentheses should be treated as a dyadic, called *stress dyadic* or *stress tensor* (because its components transform like a second-order tensor), denoted $\boldsymbol{\sigma}$:

$$\boldsymbol{\sigma} \equiv \mathbf{t}_1\,\hat{\mathbf{e}}_1 + \mathbf{t}_2\,\hat{\mathbf{e}}_2 + \mathbf{t}_3\,\hat{\mathbf{e}}_3 = \mathbf{t}_j\,\hat{\mathbf{e}}_j. \tag{4.2.9}$$

The stress tensor is a property of the medium that is independent of the unit outward normal vector $\hat{\mathbf{n}}$. Thus, from Eqs. (4.2.8) and (4.2.9), we have

$$\mathbf{t}(\hat{\mathbf{n}}) = \boldsymbol{\sigma} \cdot \hat{\mathbf{n}} = \hat{\mathbf{n}} \cdot \boldsymbol{\sigma}^{\mathrm{T}} \qquad (t_i = \sigma_{ij}\, n_j), \tag{4.2.10}$$

and the dependence of $\mathbf{t}$ on $\hat{\mathbf{n}}$ has been explicitly displayed. Equation (4.2.10) is known as the *Cauchy stress formula*, and $\boldsymbol{\sigma}$ is termed the *Cauchy stress tensor*. Thus, the Cauchy stress tensor $\boldsymbol{\sigma}$ is defined to be the *current force* per unit *deformed area*, $d\mathbf{f} = \mathbf{t}\, da = \boldsymbol{\sigma} \cdot d\mathbf{a}$, where Cauchy's formula, $\mathbf{t} = \boldsymbol{\sigma} \cdot \hat{\mathbf{n}}$, and $d\mathbf{a} = \hat{\mathbf{n}}\, da$ are used.

In Cartesian component form, the Cauchy formula in Eq. (4.2.10) can be written as $t_i = \sigma_{ij} n_j$, and it can be expressed in matrix form as

$$\begin{Bmatrix} t_1 \\ t_2 \\ t_3 \end{Bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix}. \tag{4.2.11}$$

It is useful to resolve the stress vectors $\mathbf{t}_1, \mathbf{t}_2$, and $\mathbf{t}_3$ into their orthogonal components in a rectangular Cartesian system:

$$\mathbf{t}_j = \hat{\mathbf{e}}_1\, \sigma_{1j} + \hat{\mathbf{e}}_2\, \sigma_{2j} + \hat{\mathbf{e}}_3\, \sigma_{3j} = \hat{\mathbf{e}}_i\, \sigma_{ij}, \quad j = 1, 2, 3. \tag{4.2.12}$$

Hence, the stress tensor can be expressed in the Cartesian basis as

$$\boldsymbol{\sigma} = \mathbf{t}_j\, \hat{\mathbf{e}}_j = \sigma_{ij} \hat{\mathbf{e}}_i \hat{\mathbf{e}}_j. \tag{4.2.13}$$

The component σ_{ij} represents the stress *in the x_i-coordinate direction and on a plane perpendicular to the x_j coordinate,* as shown in Fig. 4.2.3 on the faces of a point cube (i.e., the faces of the cube can be imagined as the planes passing through a point). We note that the symmetry of $\boldsymbol{\sigma}$ is *not* assumed.

The stress tensor can be expressed in any coordinate system. For example, in the cylindrical coordinate system, the nonion form of $\boldsymbol{\sigma}$ is (see Fig. 4.2.4)

$$\begin{aligned} \boldsymbol{\sigma} = {} & \sigma_{rr} \hat{\mathbf{e}}_r \hat{\mathbf{e}}_r + \sigma_{r\theta} \hat{\mathbf{e}}_r \hat{\mathbf{e}}_\theta + \sigma_{\theta r} \hat{\mathbf{e}}_\theta \hat{\mathbf{e}}_r + \sigma_{rz} \hat{\mathbf{e}}_r \hat{\mathbf{e}}_z + \sigma_{zr} \hat{\mathbf{e}}_z \hat{\mathbf{e}}_r \\ & + \sigma_{\theta\theta} \hat{\mathbf{e}}_\theta \hat{\mathbf{e}}_\theta + \sigma_{\theta z} \hat{\mathbf{e}}_\theta \hat{\mathbf{e}}_z + \sigma_{z\theta} \hat{\mathbf{e}}_z \hat{\mathbf{e}}_\theta + \sigma_{zz} \hat{\mathbf{e}}_z \hat{\mathbf{e}}_z. \end{aligned} \tag{4.2.14}$$

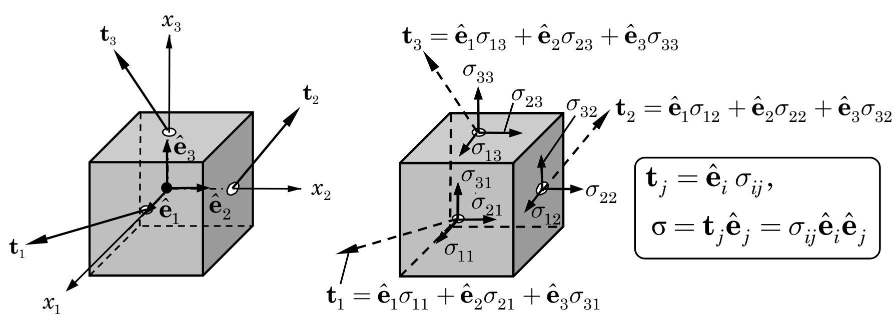
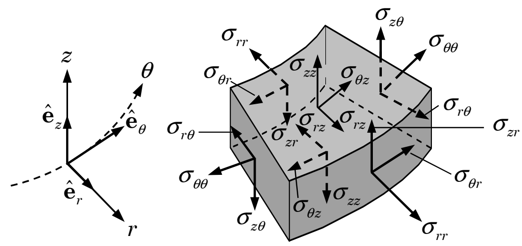


Fig. 4.2.3: Display of stress components in Cartesian rectangular coordinates.



$\sigma_{\xi\eta}$ denotes the component of stress in the ξ-direction on the η-plane

Fig. 4.2.4: Display of stress components in cylindrical coordinates.

Note that $\mathbf{t}(\hat{\mathbf{n}})$, in general, is not in the direction of $\hat{\mathbf{n}}$. The component of $\mathbf{t}$ that is in the direction of $\hat{\mathbf{n}}$ is called the *normal stress*. The component of $\mathbf{t}$ that is normal to $\hat{\mathbf{n}}$ (i.e., the component lies in the surface) is termed the (projected) *shear* stress. According to the vector identity in Eq. (2.2.26), the stress vector $\mathbf{t}$ can be represented as the sum of vectors along and perpendicular to the unit normal vector $\hat{\mathbf{n}}$, as shown in Fig. 4.2.5:

$$\mathbf{t} = (\mathbf{t}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}} + \hat{\mathbf{n}}\times(\mathbf{t}\times\hat{\mathbf{n}}) \equiv \mathbf{t}_{nn} + \mathbf{t}_{ns}. \tag{4.2.16}$$

Stress vectors on a plane are called *traction vectors*. The traction vector $\mathbf{t}_{nn}$ normal to the plane and its magnitude t_{nn} are

$$\mathbf{t}_{nn} = (\mathbf{t}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}; \quad t_{nn} = \mathbf{t}\cdot\hat{\mathbf{n}} = t_i n_i = n_j\,\sigma_{ji}\,n_i = \sigma_{ij}\,n_i\,n_j, \tag{4.2.17}$$

and the shear traction vector $\mathbf{t}_{ns}$ (i.e., projection of $\mathbf{t}$ along the plane) and its magnitude t_{ns} are

$$\mathbf{t}_{ns} = \mathbf{t} - \mathbf{t}_{nn}; \quad |\mathbf{t}_{ns}| = t_{ns} = \sqrt{|\mathbf{t}|^2 - t_{nn}^2}\,. \tag{4.2.18}$$

The tangential component lies in the $\hat{\mathbf{n}} - \mathbf{t}$ plane but perpendicular to $\hat{\mathbf{n}}$, as shown in Fig. 4.2.5. Example 4.2.1 illustrates the ideas presented here.

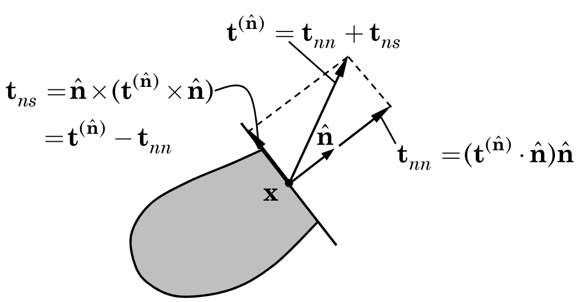


Fig. 4.2.5: The normal and shear stress vectors at a point $\mathbf{x}$ on a plane.

Example 4.2.1

With reference to a rectangular Cartesian system (x_1, x_2, x_3), the components of the stress dyadic at a certain point of a continuous medium $\mathcal{B}$ are given by (see Fig. 4.2.6)

$$[\sigma] = \begin{bmatrix} 200 & 400 & 300 \\ 400 & 0 & 0 \\ 300 & 0 & -100 \end{bmatrix} \text{ psi.}$$

Determine the stress vector $\mathbf{t}$ and its normal and tangential components at the point on the plane, $\phi(x_1, x_2, x_3) \equiv x_1 + 2x_2 + 2x_3 = \text{constant}$, which is passing through the point.

Solution: First, we should find the unit normal to the plane on which we are required to find the stress vector. The unit normal to the plane defined by $\phi(x_1, x_2, x_3) = \text{constant}$ is

$$\hat{\mathbf{n}} = \frac{\nabla\phi}{|\nabla\phi|} = \tfrac{1}{3}(\hat{\mathbf{e}}_1 + 2\hat{\mathbf{e}}_2 + 2\hat{\mathbf{e}}_3).$$

The components of the stress vector are

$$\begin{Bmatrix} t_1 \\ t_2 \\ t_3 \end{Bmatrix} = \begin{bmatrix} 200 & 400 & 300 \\ 400 & 0 & 0 \\ 300 & 0 & -100 \end{bmatrix} \frac{1}{3} \begin{Bmatrix} 1 \\ 2 \\ 2 \end{Bmatrix} = \frac{1}{3} \begin{Bmatrix} 1600 \\ 400 \\ 100 \end{Bmatrix} \text{ psi,}$$

or

$$\mathbf{t}(\hat{\mathbf{n}}) = \tfrac{100}{3}(16\,\hat{\mathbf{e}}_1 + 4\,\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3) \text{ psi.}$$

The traction vector normal to the plane is given by

$$\mathbf{t}_{nn} = (\mathbf{t}(\hat{\mathbf{n}}) \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} = \tfrac{2600}{9}\hat{\mathbf{n}} = \tfrac{2600}{27}(\hat{\mathbf{e}}_1 + 2\hat{\mathbf{e}}_2 + 2\hat{\mathbf{e}}_3) \text{ psi,}$$

and the traction vector projected onto the plane (i.e., shear traction) is given by

$$\mathbf{t}_{ns} = \mathbf{t}(\hat{\mathbf{n}}) - \mathbf{t}_{nn} = \tfrac{100}{27}(118\hat{\mathbf{e}}_1 - 16\hat{\mathbf{e}}_2 - 43\hat{\mathbf{e}}_3) \text{ psi.}$$

The magnitudes are

$$|\mathbf{t}_{nn}| = t_{nn} = \tfrac{2600}{9} = 288.89 \text{ psi}, \quad |\mathbf{t}_{ns}| = t_{ns} = 468.91 \text{ psi.}$$

One can also determine t_{ns} from

$$t_{ns} = \sqrt{|\mathbf{t}|^2 - t_{nn}^2} = \tfrac{100}{9}\sqrt{(256 + 16 + 1)9 - 26 \times 26} \text{ psi} = 468.91 \text{ psi.}$$

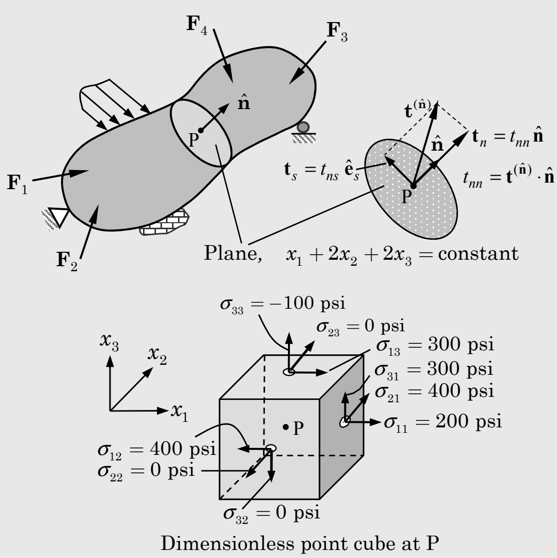


Fig. 4.2.6: Stress vector and its normal and shear components.

4.3 Transformations of Stress Components and Principal Stresses

4.3.1 Transformation of Stress Components

Since the Cauchy stress tensor $\boldsymbol{\sigma}$ is a second-order tensor, all of the properties of a second-order tensor that were discussed in Chapter 2 apply. In particular, we can define the principal invariants I_1, I_2, and I_3; transformation laws for the components of $\boldsymbol{\sigma}$; and eigenvalues (principal values) and eigenvectors (principal planes) of the Cauchy stress tensor.

4.3.1.1 Invariants

The invariants of stress tensor $\boldsymbol{\sigma}$ are defined by [see Eqs. (2.5.16) and (2.5.17)]

$$I_1 = \mathrm{tr}\,[\boldsymbol{\sigma}], \quad I_2 = \tfrac{1}{2}\left[(\mathrm{tr}\,[\boldsymbol{\sigma}])^2 - \mathrm{tr}\,([\boldsymbol{\sigma}]^2)\right], \quad I_3 = |\sigma|, \tag{4.3.1}$$

and in terms of the rectangular Cartesian components

$$I_1 = \sigma_{ii}, \quad I_2 = \tfrac{1}{2}\left(\sigma_{ii}\,\sigma_{jj} - \sigma_{ij}\,\sigma_{ji}\right), \quad I_3 = |\sigma|. \tag{4.3.2}$$

4.3.1.2 Transformation equations

The components of the Cauchy stress tensor $\boldsymbol{\sigma}$ in one rectangular Cartesian coordinate system $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ are related to the components in another rectangular Cartesian system (x_1, x_2, x_3) according to the transformation law in Eq. (2.5.21):

$$\bar{\sigma}_{ij} = \ell_{ik}\,\ell_{j\ell}\,\sigma_{k\ell} \quad \text{or} \quad [\bar{\sigma}] = [L][\sigma][L]^{\mathrm{T}}, \tag{4.3.3}$$

where ℓ_{ij} are the direction cosines

$$\ell_{ij} = \hat{\bar{\mathbf{e}}}_i \cdot \hat{\mathbf{e}}_j. \tag{4.3.4}$$

In Examples 4.3.1 and 4.3.2, symmetry of the stress tensor, which will be established in Chapter 5, is used in deriving the stress transformation equations (4.3.7) and (4.3.8) for a special coordinate transformation, whereas Eq. (4.3.3) is valid for the rectangular components of any second-order tensor and for a general coordinate transformation.

Example 4.3.1

Consider a rectangular, unidirectional fiber-reinforced composite layer shown in Fig. 4.3.1, where the fibers are symbolically shown as black lines. The rectangular coordinates (x, y, z) are taken such that the z-coordinate is normal to the plane of the layer, and the x and y coordinates are in the plane of the layer but parallel to the edges of the layer. Now suppose we define a new rectangular coordinate system (x_1, x_2, x_3) such that the x_3-coordinate coincides with the z-coordinate and the x_1-axis is taken along the fiber direction; that is, the x_1x_2-plane is obtained by rotating the xy-plane about the z-axis in a counterclockwise direction by an angle θ. Determine the relations between the stress components referred to the (x, y, z) system and those referred to the coordinates system (x_1, x_2, x_3).

Solution: The coordinates of a material point in the two coordinate systems are related as follows $(z = x_3)$:

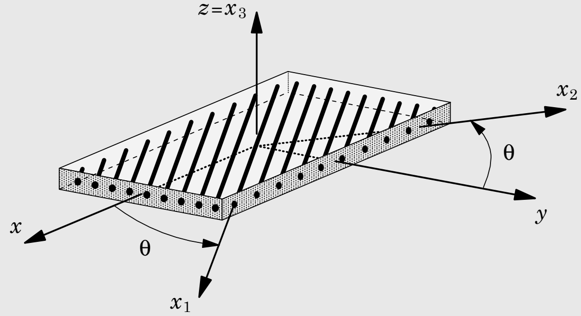


Fig. 4.3.1: Stress components in a fiber-reinforced layer referred to different rectangular Cartesian coordinate systems: (x, y, z) are parallel to the sides of the rectangular lamina, while (x_1, x_2, x_3) are taken along and transverse to the fiber direction.

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = [L] \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}. \tag{4.3.5}$$

Next, we establish the relationship between the components of stress in the (x, y, z) and (x_1, x_2, x_3) coordinate systems. Let σ_{ij} be the components of the stress tensor $\boldsymbol{\sigma}$ in the (x_1, x_2, x_3) coordinate system, and $\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$, etc. be the stress components in the (x, y, z) coordinate system. If we view (x_1, x_2, x_3) as the barred coordinate system, then ℓ_{ij} are the direction cosines defined by

$$\ell_{ij} = \hat{\bar{\mathbf{e}}}_i \cdot \hat{\mathbf{e}}_j,$$

where $\hat{\bar{\mathbf{e}}}_\mathbf{i}$ and $\hat{\mathbf{e}}_i$ are the orthonormal basis vectors in coordinate systems (x_1, x_2, x_3) and (x, y, z), respectively. Then using Eq. (4.3.3), we can write

$$[\bar{\sigma}] = [L][\sigma][L]^{\mathrm{T}}, \quad [\sigma] = [L]^{\mathrm{T}}[\bar{\sigma}][L], \tag{4.3.6}$$

where $[L]$ is the 3×3 matrix of direction cosines defined in Eq. (4.3.5). Carrying out the indicated matrix multiplications in Eq. (4.3.6) and rearranging the equations in terms of the column vectors of stress components ($\sigma_{xy} = \sigma_{yx}$, $\sigma_{xz} = \sigma_{zx}$, $\sigma_{yz} = \sigma_{zy}$, $\sigma_{12} = \sigma_{21}$, $\sigma_{13} = \sigma_{31}$, and $\sigma_{23} = \sigma_{32}$), we obtain

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 0 & 0 & 0 & \sin 2\theta \\ \sin^2\theta & \cos^2\theta & 0 & 0 & 0 & -\sin 2\theta \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 & \sin\theta & \cos\theta & 0 \\ -\frac{1}{2}\sin 2\theta & \frac{1}{2}\sin 2\theta & 0 & 0 & 0 & \cos 2\theta \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix}. \tag{4.3.7}$$

and

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 0 & 0 & 0 & -\sin 2\theta \\ \sin^2\theta & \cos^2\theta & 0 & 0 & 0 & \sin 2\theta \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta & \sin\theta & 0 \\ 0 & 0 & 0 & -\sin\theta & \cos\theta & 0 \\ \frac{1}{2}\sin 2\theta & -\frac{1}{2}\sin 2\theta & 0 & 0 & 0 & \cos 2\theta \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix}, \tag{4.3.8}$$

The result in Eq. (4.3.8) can also be obtained from Eq. (4.3.7) by replacing θ with $-\theta$.

Example 4.3.2

Consider a thin, closed, filament-wound circular cylindrical pressure vessel shown in Fig. 4.3.2. The vessel has an internal diameter $D_i = 63.5$ cm (25 in.) and thickness $h = 2$ cm, and is pressurized to $p = 1.379$ MPa (200 psi). If the filament winding angle is $\theta = 53.13°$ from the longitudinal axis of the pressure vessel, determine the shear and normal forces per unit length of the filament winding. Assume that the material used is graphite–epoxy with the following material properties [material properties are not needed to solve the problem; see Reddy (2004)]:

$$E_1 = 140 \text{ MPa } (20.3 \times 10^6 \text{ psi}), \quad E_2 = 10 \text{ MPa } (1.45 \times 10^6 \text{ psi}), \\ G_{12} = 7 \text{ MPa } (1.02 \times 10^6 \text{ psi}), \quad \nu_{12} = 0.3, \tag{4.3.9}$$

where MPa denotes mega (10^6) Pascal (Pa) and Pa = N/m^2 (1 psi = 6,894.76 Pa).

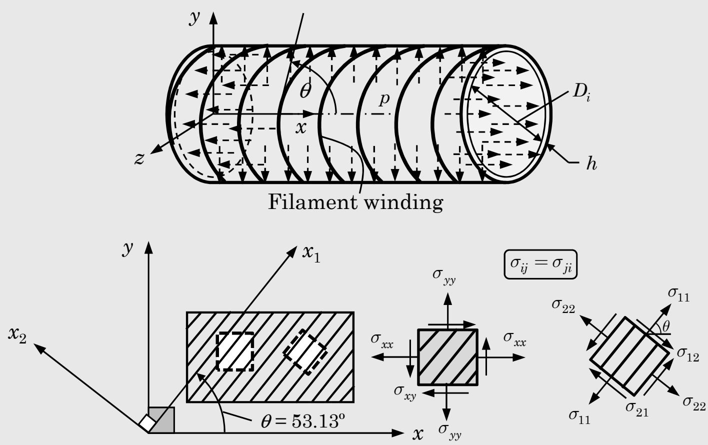


Fig. 4.3.2: A filament-wound cylindrical pressure vessel.

Solution: First, we compute the stresses in the pressure vessel using the formulas from a book on mechanics of materials [see, e.g., Fenner and Reddy (2012)]. The longitudinal (σ_{xx}) and circumferential (σ_{yy}) stresses are given by (the shear stress σ_{xy} is zero)

$$\sigma_{xx} = \frac{pD_i}{4h}, \quad \sigma_{yy} = \frac{pD_i}{2h}, \tag{4.3.10}$$

where p is internal pressure, D_i is the internal diameter, and h is the thickness of the pressure vessel. Note that the stresses are independent of material properties and depend only on the geometry and loads. We calculate the longitudinal and circumferential stresses to be

$$\sigma_{xx} = \frac{1.379 \times 0.635}{4 \times 0.02} = 10.946 \text{ MPa}, \quad \sigma_{yy} = \frac{1.379 \times 0.635}{2 \times 0.02} = 21.892 \text{ MPa}.$$

Next, we determine the shear stress σ_{12} along the fiber–matrix interface and the normal stress σ_{11} in the fiber direction using the transformation equations in Eq. (4.3.7). Noting that $\sin\theta = 0.8$, $\cos\theta = 0.6$, and $\sin 2\theta = 0.96$ for $\theta = 53.13°$, we obtain

$$\begin{aligned} \sigma_{11} &= 10.946 \times (0.6)^2 + 21.892 \times (0.8)^2 = 17.951 \text{ MPa}, \\ \sigma_{22} &= 10.946 \times (0.8)^2 + 21.892 \times (0.6)^2 = 14.886 \text{ MPa}, \\ \sigma_{12} &= \tfrac{1}{2}\,(21.892 - 10.946) \times 0.96 = 5.254 \text{ MPa}. \end{aligned}$$

Thus, the normal and shear forces per unit length along the fiber–matrix interface are $F_{22} = 14.886\,h$ MN and $F_{12} = 5.254\,h$ MN, whereas the force per unit length in the fiber direction is $F_{11} = 17.951\,h$ MN (MN $= 10^6$N).

4.3.2 Principal Stresses and Principal Planes

For a given state of stress, the determination of maximum normal stresses and shear stresses at a point is of considerable interest in the design of structures because failures occur when the magnitudes of stresses exceed the allowable (normal or shear) stress values, called strengths, of the material. In this regard it is of interest to determine the values and the planes on which the stresses are the maximum. Thus, we must determine the eigenvalues and eigenvectors associated with the stress tensor (see Section 2.5.6 for details).

It is clear from Fig. 4.2.5 that the normal component of a stress vector is largest when $\mathbf{t}$ is parallel to the unit outward normal $\hat{\mathbf{n}}$; that is, $\mathbf{t}_n = \mathbf{t} = |\mathbf{t}|\hat{\mathbf{n}}$. This amounts to finding the plane (i.e., $\hat{\mathbf{n}}$) on which $\mathbf{t}_n$ is largest. It turns out there are three such planes on which the normal stress is the largest (and the projected shear stress is zero). If we denote this value of the normal stress by λ, then we can write $\mathbf{t} = \lambda\hat{\mathbf{n}}$, and by Cauchy's formula, $\mathbf{t} = \boldsymbol{\sigma}\cdot\hat{\mathbf{n}}$. Thus, we have

$$\mathbf{t} = \boldsymbol{\sigma}\cdot\hat{\mathbf{n}} = \lambda\hat{\mathbf{n}} \quad \text{or} \quad (\boldsymbol{\sigma} - \lambda\mathbf{I})\cdot\hat{\mathbf{n}} = \mathbf{0}. \tag{4.3.11}$$

This is a homogeneous set of equations for the components of vector $\hat{\mathbf{n}}$; hence, a nontrivial solution will not exist unless the determinant of the matrix $[\sigma] - \lambda[I]$ vanishes. The vanishing of this determinant yields a cubic equation for λ, called the *characteristic equation* [see Eq. (2.5.42)]:

$$-\lambda^3 + I_1\lambda^2 - I_2\lambda + I_3 = 0. \tag{4.3.12}$$

The solution of this cubic equation yields three values of λ, which are called the *principal stresses*, and the associated eigenvectors are called the *principal planes*. That is, for a given state of stress at a given point in the body $\mathcal{B}$, there exists a set of planes $\hat{\mathbf{n}}$ on which the stress vector is normal to the planes (i.e., there is no shear component on the planes).

The computation of the eigenvalues of the stress tensor is made easy by seeking the eigenvalues of the deviatoric stress tensor [see Eq. (2.5.53)]:

$$\boldsymbol{\sigma}' = \boldsymbol{\sigma} - \tfrac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})\,\mathbf{I} \quad \left(\sigma'_{ij} \equiv \sigma_{ij} - \tfrac{1}{3}\sigma_{kk}\,\delta_{ij}\right). \tag{4.3.13}$$

Let σ_m denote the mean normal stress

$$\sigma_m = \tfrac{1}{3}\mathrm{tr}\,[\boldsymbol{\sigma}] = \tfrac{1}{3}I_1 \quad (\sigma_m = \tfrac{1}{3}\sigma_{kk}). \tag{4.3.14}$$

Then the stress tensor can be expressed as the sum of the *spherical* or the *hydrostatic* part and the *deviatoric* part of the stress tensor:

$$\boldsymbol{\sigma} = \sigma_m\mathbf{I} + \boldsymbol{\sigma}'. \tag{4.3.15}$$

Thus, the deviatoric stress tensor is defined by

$$\boldsymbol{\sigma}' = \boldsymbol{\sigma} - \tfrac{1}{3}I_1\mathbf{I} \quad (\sigma'_{ij} = \sigma_{ij} - \tfrac{1}{3}\delta_{ij}\sigma_{kk}). \tag{4.3.16}$$

The invariants I'_1, I'_2, and I'_3 of the deviatoric stress tensor are

$$I'_1 = 0, \quad I'_2 = \tfrac{1}{2}\sigma'_{ij}\sigma'_{ij}, \quad I'_3 = \tfrac{1}{3}\sigma'_{ij}\,\sigma'_{jk}\,\sigma'_{ki}. \tag{4.3.17}$$

The deviatoric stress invariants are particularly important in the determination of the principal stresses, as discussed in Section 2.5.6. Example 4.3.3 illustrates the computation of principal stresses and principal planes.

Example 4.3.3

The components of a stress dyadic at a point, referred to the (x_1, x_2, x_3) system, are:

$$[\sigma] = \begin{bmatrix} 12 & 9 & 0 \\ 9 & -12 & 0 \\ 0 & 0 & 6 \end{bmatrix} \text{MPa.}$$

Find the principal stresses and the principal plane associated with the maximum stress.

Solution: Clearly, $\lambda = 6$ is an eigenvalue. Expanding the determinant $|\sigma - \lambda I|$ with the last row or column, we obtain

$$(6-\lambda)[(12-\lambda)(-12-\lambda) - 81] = 0 \quad \Rightarrow \quad (\lambda^2 - 225)(6-\lambda) = 0.$$

The remaining two eigenvalues are obtained from $\lambda^2 - 225 = 0 \;\to\; \lambda = \pm 15$; thus, the three principal stresses are

$$\sigma_1 = \lambda_1 = 15\,\text{MPa}, \quad \sigma_2 = \lambda_2 = 6\,\text{MPa}, \quad \sigma_3 = \lambda_3 = -15\,\text{MPa}.$$

The plane associated with the maximum principal stress $\lambda_1 = 15$ MPa can be calculated from

$$\begin{bmatrix} 12-15 & 9 & 0 \\ 9 & -12-15 & 0 \\ 0 & 0 & 6-15 \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix},$$

which gives

$$-3n_1 + 9n_2 = 0, \quad 9n_1 - 27n_2 = 0, \quad -9n_3 = 0 \quad \to \quad n_3 = 0, \quad n_1 = 3n_2$$

$$\mathbf{n}^{(1)} = 3\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 \quad \text{or} \quad \hat{\mathbf{n}}^{(1)} = \tfrac{1}{\sqrt{10}}\big(3\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2\big).$$

The eigenvector associated with $\lambda_2 = 6$ MPa is $\mathbf{n}^{(2)} = \hat{\mathbf{e}}_3$. Finally, the eigenvector associated with $\lambda_3 = -15$ MPa is

$$\mathbf{n}^{(3)} = \pm\big(\hat{\mathbf{e}}_1 - 3\hat{\mathbf{e}}_2\big) \quad \text{or} \quad \hat{\mathbf{n}}^{(3)} = \pm\tfrac{1}{\sqrt{10}}\big(\hat{\mathbf{e}}_1 - 3\hat{\mathbf{e}}_2\big).$$

The principal plane 1 is depicted in Fig. 4.3.3.

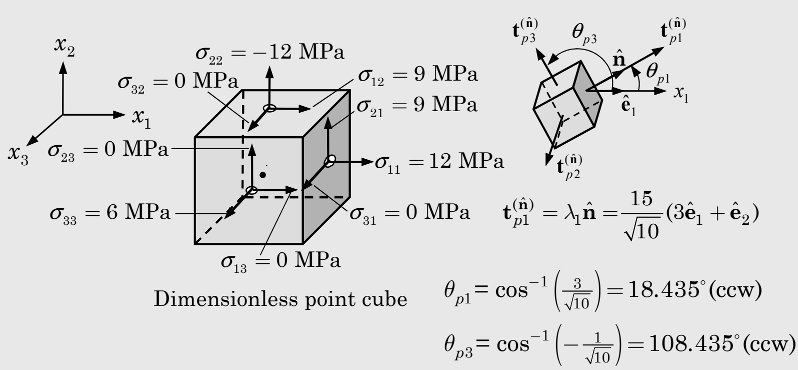


Fig. 4.3.3: Stresses on a point cube at the point of interest and orientation of the first principal plane.

4.3.3 Maximum Shear Stress

In the previous section, we studied the procedure to determine the maximum normal stresses at a point. The eigenvalues of the stress tensor at the point are the maximum normal stresses on three perpendicular planes (whose normals are the eigenvectors), and the largest of these three stresses is the true maximum normal stress. Recall that the shear stresses are zero on the principal planes. In this section, we wish to determine the maximum shear stresses and their planes.

Let λ_1, λ_2, and λ_3 denote the principal (normal) stresses and $\hat{\mathbf{n}}$ be an arbitrary unit normal vector. Then the stress vector is $\mathbf{t} = \lambda_1 n_1 \hat{\mathbf{e}}_1 + \lambda_2 n_2 \hat{\mathbf{e}}_2 + \lambda_3 n_3 \hat{\mathbf{e}}_3$ and $t_{nn} = t_i n_i = \lambda_1 n_1^2 + \lambda_2 n_2^2 + \lambda_3 n_3^2$. The square of the magnitude of the shear stress on the plane with unit normal $\hat{\mathbf{n}}$ is given by

$$t_{ns}^2(\hat{\mathbf{n}}) = |\mathbf{t}|^2 - t_{nn}^2 = \lambda_1^2 n_1^2 + \lambda_2^2 n_2^2 + \lambda_3^2 n_3^2 - \left(\lambda_1 n_1^2 + \lambda_2 n_2^2 + \lambda_3 n_3^2\right)^2. \quad (4.3.18)$$

We wish to determine the plane $\hat{\mathbf{n}}$ on which t_{ns} is the maximum. Thus, we seek the maximum of the function $F(n_1, n_2, n_3) = t_{ns}^2(n_1, n_2, n_3)$ subject to the constraint

$$n_1^2 + n_2^2 + n_3^2 - 1 = 0. \quad (4.3.19)$$

One way to determine the extremum of a function subjected to a constraint is to use the Lagrange multiplier method, in which we seek the stationary value of the modified function

$$F_L(n_1, n_2, n_3) = t_{ns}^2(n_1, n_2, n_3) + \lambda_L\left(n_1^2 + n_2^2 + n_3^2 - 1\right), \quad (4.3.20)$$

where λ_L is the Lagrange multiplier, which is to be determined along with n_1, n_2, and n_3. The necessary condition for the stationarity of F_L is

$$0 = dF_L = \frac{\partial F_L}{\partial n_1} dn_1 + \frac{\partial F_L}{\partial n_2} dn_2 + \frac{\partial F_L}{\partial n_3} dn_3 + \frac{\partial F_L}{\partial \lambda_L} d\lambda_L,$$

or, because the increments dn_1, dn_2, dn_3, and $d\lambda_L$ are linearly independent of each other, we have

$$\frac{\partial F_L}{\partial n_1} = 0, \quad \frac{\partial F_L}{\partial n_2} = 0, \quad \frac{\partial F_L}{\partial n_3} = 0, \quad \frac{\partial F_L}{\partial \lambda_L} = 0. \quad (4.3.21)$$

The last of the four relations in Eq. (4.3.21) is the same as that in Eq. (4.3.19). The remaining three equations in Eq. (4.3.21) yield the following two sets of solutions (not derived here):

$$(n_1, n_2, n_3) = (1, 0, 0), \quad (0, 1, 0), \quad (0, 0, 1), \quad (4.3.22)$$

$$(n_1, n_2, n_3) = \left(\tfrac{1}{\sqrt{2}}, \pm\tfrac{1}{\sqrt{2}}, 0\right), \quad \left(\tfrac{1}{\sqrt{2}}, 0, \pm\tfrac{1}{\sqrt{2}}\right), \quad \left(0, \tfrac{1}{\sqrt{2}}, \pm\tfrac{1}{\sqrt{2}}\right). \quad (4.3.23)$$

The first set of solutions corresponds to the principal planes, on which the shear stresses are the minimum, namely zero. The second set of solutions corresponds to the maximum shear stress planes. The maximum shear stresses on the planes are given by

$$
\begin{aligned}
t_{ns}^2 &= \tfrac{1}{4}\big(\lambda_1 - \lambda_2\big)^2 \quad \text{for} \quad \hat{\mathbf{n}} = \tfrac{1}{\sqrt{2}}\big(\hat{\mathbf{e}}_1 \pm \hat{\mathbf{e}}_2\big),\\
t_{ns}^2 &= \tfrac{1}{4}\big(\lambda_1 - \lambda_3\big)^2 \quad \text{for} \quad \hat{\mathbf{n}} = \tfrac{1}{\sqrt{2}}\big(\hat{\mathbf{e}}_1 \pm \hat{\mathbf{e}}_3\big),\\
t_{ns}^2 &= \tfrac{1}{4}\big(\lambda_2 - \lambda_3\big)^2 \quad \text{for} \quad \hat{\mathbf{n}} = \tfrac{1}{\sqrt{2}}\big(\hat{\mathbf{e}}_2 \pm \hat{\mathbf{e}}_3\big).
\end{aligned}
\tag{4.3.24}
$$

The largest shear stress is given by the largest of the three values given above. Thus, we have

$$
(t_{ns})_{\max} = \tfrac{1}{2}\big(\lambda_{\max} - \lambda_{\min}\big), \tag{4.3.25}
$$

where $\lambda_{\max}$ and $\lambda_{\min}$ are the maximum and minimum principal values of stress, respectively. The plane of the maximum shear stress lies between the planes of the maximum and minimum principal stresses (i.e., oriented at $\pm 45°$ to both planes).

Example 4.3.4

For the state of stress given in Example 4.3.3, determine the maximum shear stress.

Solution: From Example 4.3.3, the principal stresses are (ordered from the minimum to the maximum)

$$
\lambda_1 = -15\,\text{MPa}, \quad \lambda_2 = 6\,\text{MPa}, \quad \lambda_3 = 15\,\text{MPa}.
$$

Hence, the maximum shear stress is given by

$$
(t_{ns})_{\max} = \tfrac{1}{2}\big(\lambda_3 - \lambda_1\big) = \tfrac{1}{2}\big[15 - (-15)\big] = 15\,\text{MPa}.
$$

The planes of the maximum principal stress ($\lambda_1 = 15$ MPa) and the minimum principal stress ($\lambda_3 = -15$ MPa) are given by their normal vectors (not unit vectors):

$$
\mathbf{n}^{(1)} = 3\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2, \quad \mathbf{n}^{(3)} = \hat{\mathbf{e}}_1 - 3\hat{\mathbf{e}}_2.
$$

Then the plane of the maximum shear stress is given by the vector

$$
\mathbf{n}_s = \big(\mathbf{n}^{(1)} - \mathbf{n}^{(3)}\big) = 2\hat{\mathbf{e}}_1 + 4\hat{\mathbf{e}}_2 \quad \text{or} \quad \hat{\mathbf{n}}_s = \tfrac{1}{\sqrt{5}}\big(\hat{\mathbf{e}}_1 + 2\hat{\mathbf{e}}_2\big).
$$

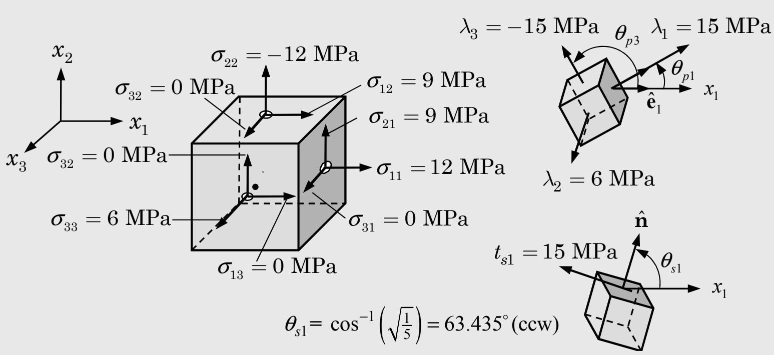


Fig. 4.3.4: Stresses on a point cube at the point of interest and orientation of the maximum shear stress plane.

4.4 Other Stress Measures

4.4.1 Preliminary Comments

The Cauchy stress tensor is the most natural and physical measure of the state of stress at a point in the deformed configuration and, measured per unit area of the deformed configuration. It is the quantity most commonly used in spatial descriptions of problems in fluid mechanics. In order to use the Lagrangian description, which is common in solid mechanics, the equations of motion or equilibrium of a material body that are derived in the deformed configuration must be expressed in terms of the known reference configuration. In doing so we introduce various other measures of stress. These measures emerge in a natural way as we transform volumes and areas from the deformed configuration to the reference configuration. These measures are purely mathematical in nature but facilitate the analysis.

4.4.2 First Piola–Kirchhoff Stress Tensor

Consider a continuum $\mathcal{B}$ subjected to a deformation mapping $\boldsymbol{\chi}$ that results in the deformed configuration κ, as shown in Fig. 4.4.1. Let the force vector on an elemental area da with normal $\hat{\mathbf{n}}$ in the deformed configuration be $d\mathbf{f}$. The force $d\mathbf{f}$ can be expressed in terms of a stress vector $\mathbf{t}$ times the deformed area da as

$$d\mathbf{f} = \mathbf{t}^{(\mathbf{n})}\, da = \boldsymbol{\sigma} \cdot \hat{\mathbf{n}}\, da = \boldsymbol{\sigma} \cdot d\mathbf{a}, \tag{4.4.1}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress tensor, and Cauchy's formula (4.2.10) is invoked in arriving at the last result. Now suppose that the area element in the undeformed configuration that corresponds to da is dA. We define a stress vector $\mathbf{T}^{(\mathbf{N})}$ over the area element dA with normal $\mathbf{N}$ in the undeformed configuration such that it results in the *same* total force

$$d\mathbf{f} = \mathbf{t}^{(\mathbf{n})}\, da = \mathbf{T}^{(\mathbf{N})} dA. \tag{4.4.2}$$

Clearly, both stress vectors have the same direction but different magnitudes owing to the different areas. The stress vector $\mathbf{T}^{(\mathbf{N})}$ is measured per unit undeformed area, while the stress vector $\mathbf{t}^{(\mathbf{n})}$ is measured per unit deformed area.

Analogous to Cauchy's formula relating the Cauchy stress tensor $\boldsymbol{\sigma}$ to the stress vector $\mathbf{t}^{(\mathbf{n})}$, we can introduce a stress tensor $\mathbf{P}$, called the *first Piola–Kirchhoff stress tensor*, to the stress vector $\mathbf{T}^{(\mathbf{N})}$; that is

$$\mathbf{t}^{(\mathbf{n})} = \boldsymbol{\sigma} \cdot \hat{\mathbf{n}}; \qquad \mathbf{T}^{(\mathbf{N})} = \mathbf{P} \cdot \hat{\mathbf{N}}. \tag{4.4.3}$$

Then using Eq. (4.4.2) and Cauchy's formulas for $\mathbf{t}^{(\mathbf{n})}$ and $\mathbf{T}^{(\mathbf{N})}$, we can write

$$d\mathbf{f} = \boldsymbol{\sigma} \cdot d\mathbf{a} = \mathbf{P} \cdot d\mathbf{A}; \quad d\mathbf{a} = da\, \hat{\mathbf{n}}, \quad d\mathbf{A} = dA\, \hat{\mathbf{N}}. \tag{4.4.4}$$

The first Piola–Kirchhoff stress tensor, also referred to as the *nominal stress tensor* or *Lagrangian stress tensor*, gives the *current force* per unit *undeformed area*.

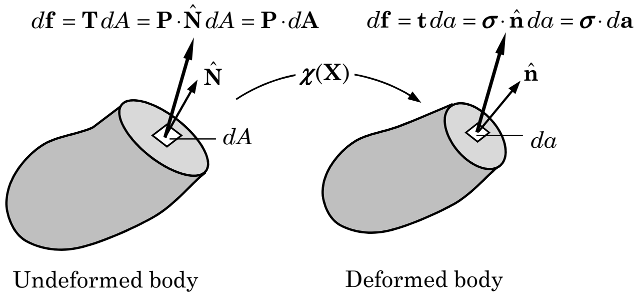


Fig. 4.4.1: Definition of the first Piola–Kirchhoff stress tensor.

The stress vector $\mathbf{T}^{(\mathbf{N})}$ is known as the *pseudo stress vector* associated with the first Piola–Kirchhoff stress tensor. The Cartesian component representation of tensor $\mathbf{P}$ is given by

$$\mathbf{P} = P_{iI}\,\hat{\mathbf{e}}_i\,\hat{\mathbf{E}}_I. \tag{4.4.5}$$

Clearly, the first Piola–Kirchhoff stress tensor is a *two-point tensor* (like $\mathbf{F}$) in the sense that it connects a point in the undeformed body to the corresponding point in the deformed body.

To express the first Piola–Kirchhoff stress tensor $\mathbf{P}$ in terms of the Cauchy stress tensor $\boldsymbol{\sigma}$, we must write $d\mathbf{a}$ in terms of $d\mathbf{A}$. From Nanson's formula in Eq. (3.3.25), we recall such a relation between $d\mathbf{a}$ in terms of $d\mathbf{A}$:

$$d\mathbf{a} = J\,\mathbf{F}^{-\mathrm{T}} \cdot d\mathbf{A} = J\,d\mathbf{A} \cdot \mathbf{F}^{-1}, \tag{4.4.6}$$

where J is the Jacobian, $J = |\mathbf{F}|$. Substituting the relation into Eq. (4.4.4), we obtain

$$\mathbf{P} \cdot d\mathbf{A} = \boldsymbol{\sigma} \cdot d\mathbf{a} = J\,\boldsymbol{\sigma} \cdot \mathbf{F}^{-\mathrm{T}} \cdot d\mathbf{A}. \tag{4.4.7}$$

Thus, we arrive at the relation

$$\mathbf{P} = J\boldsymbol{\sigma} \cdot \mathbf{F}^{-\mathrm{T}} \quad \text{or} \quad \boldsymbol{\sigma} = \frac{1}{J}\,\mathbf{P} \cdot \mathbf{F}^{\mathrm{T}}. \tag{4.4.8}$$

In general, the first Piola–Kirchhoff stress tensor $\mathbf{P}$ is unsymmetric even when the Cauchy stress tensor $\boldsymbol{\sigma}$ is symmetric (which is not yet established).

4.4.3 Second Piola–Kirchhoff Stress Tensor

Similar to the relationship between $d\mathbf{x}$ and $d\mathbf{X}$, $d\mathbf{X} = \mathbf{F}^{-1} \cdot d\mathbf{x}$, the force $d\mathbf{f}$ on the deformed elemental area $d\mathbf{a}$ can be related, by analogy to the relation between $d\mathbf{x}$ and $d\mathbf{X}$, to a force $d\mathcal{F}$ on the undeformed elemental area $d\mathbf{A}$ by

$$d\mathcal{F} = \mathbf{F}^{-1} \cdot d\mathbf{f}. \tag{4.4.9}$$

Then we can think of the existence of a stress tensor $\mathbf{S}$, in the same way as in Eq. (4.4.1), such that

$$d\mathcal{F} = \mathbf{S} \cdot d\mathbf{A}, \tag{4.4.10}$$

where $\mathbf{S}$ is called the *second Piola–Kirchhoff stress tensor* $\mathbf{S}$. Thus, the second Piola–Kirchhoff stress tensor $\mathbf{S}$ gives the *transformed current force* per unit *undeformed area.*

The second Piola–Kirchhoff stress tensor $\mathbf{S}$ can be related to the first Piola-Kirchhoff stress tensor $\mathbf{P}$ with the help of Eqs. (4.4.4), (4.4.9), and (4.4.10) as

$$\mathbf{S} = \mathbf{F}^{-1} \cdot \mathbf{P}. \tag{4.4.11}$$

The relation between $\mathbf{S}$ and $\boldsymbol{\sigma}$ can also be established using Eqs. (4.4.4), (4.4.6), (4.4.9), and (4.4.10) as

$$\mathbf{S} \cdot d\mathbf{A} = \mathbf{F}^{-1} \cdot \boldsymbol{\sigma} \cdot d\mathbf{a} = J\,\mathbf{F}^{-1} \cdot \boldsymbol{\sigma} \cdot \mathbf{F}^{-\mathrm{T}} \cdot d\mathbf{A},$$

or

$$\mathbf{S} = J\,\mathbf{F}^{-1} \cdot \boldsymbol{\sigma} \cdot \mathbf{F}^{-\mathrm{T}} \quad \text{or} \quad \boldsymbol{\sigma} = \frac{1}{J}\mathbf{F} \cdot \mathbf{S} \cdot \mathbf{F}^{\mathrm{T}}. \tag{4.4.12}$$

Clearly, $\mathbf{S}$ is symmetric (i.e., $\mathbf{S} = \mathbf{S}^{\mathrm{T}}$) whenever $\boldsymbol{\sigma}$ is symmetric. Cartesian component representation of the tensor $\mathbf{S}$ is

$$\mathbf{S} = S_{IJ}\,\hat{\mathbf{E}}_I\,\hat{\mathbf{E}}_J. \tag{4.4.13}$$

All of the discussion in Section 4.3 concerning the transformation of components and determination of eigenvalues and eigenvectors is valid for $\mathbf{S}$.

We can introduce the pseudo stress vector $\tilde{\mathbf{T}}$ associated with the second Piola–Kirchhoff stress tensor by

$$d\mathcal{F} = \tilde{\mathbf{T}}\,dA = \mathbf{S} \cdot \hat{\mathbf{N}}\,dA = \mathbf{S} \cdot d\mathbf{A} \quad \text{or} \quad \tilde{\mathbf{T}} = \mathbf{S} \cdot \hat{\mathbf{N}}. \tag{4.4.14}$$

An interpretation of the second Piola–Kirchhoff stress tensor is possible in the case of rigid-body motion, for which the polar decomposition theorem gives $\mathbf{F} = \mathbf{R}$ and $J = 1$. Hence, we have

$$\mathbf{S} = J\,\mathbf{F}^{-1} \cdot \boldsymbol{\sigma} \cdot \mathbf{F}^{-\mathrm{T}} = \mathbf{R}^{\mathrm{T}} \cdot \boldsymbol{\sigma} \cdot \mathbf{R}, \tag{4.4.15}$$

which resembles the stress transformation equation (4.3.3). That is, the second Piola–Kirchhoff stress components are the same as the components of the Cauchy stress tensor expressed in the local set of orthogonal axes that are obtained from rotating the global Cartesian coordinates by the rotation matrix $[L] = [R]^{\mathrm{T}}$.

We close this section with an example that illustrates the meaning of the first and second Piola–Kirchhoff stress tensors and the computation of the first and second Piola–Kirchhoff stress tensor components from the Cauchy stress tensor components [see Hjelmstad (1997)].

Example 4.4.1

Consider a bar of cross-sectional area $A = bH$ and length L. The initial configuration of the bar is such that its longitudinal axis is along the X_1 axis, as shown in Fig. 4.4.2(a). Suppose that the bar is subjected to uniaxial tensile stress that produces a pure stretch λ along the length and a pure stretch μ along the height of the bar and then rotates it, without bending, by an angle θ, as shown in Fig. 4.4.2(a). Assume that the width b of the bar does not change during the deformation. Therefore, μ denotes the ratio of deformed to undeformed height (or cross-sectional area) of the bar. Determine (a) the deformation mapping and the components of the deformation gradient and (b) the components of the Cauchy stress tensor as well as the first and second Piola–Kirchhoff stress tensors.

Solution: (a) We use the deformed geometry to determine the deformation mapping. We have from Fig. 4.4.2(b),

$$\boldsymbol{\chi}(\mathbf{X}) = \big(\lambda X_1 \cos\theta - \mu X_2 \sin\theta\big)\hat{\mathbf{e}}_1 + \big(\lambda X_1 \sin\theta + \mu X_2 \cos\theta\big)\hat{\mathbf{e}}_2 + X_3\, \hat{\mathbf{e}}_3.$$

The ratio of volume in the deformed to undeformed configuration is $(\lambda L\, \mu H\, b)/(LHb) = \mu\,\lambda$.

The components of the deformation gradient and its inverse are

$$[F] = \begin{bmatrix} \lambda\cos\theta & -\mu\sin\theta & 0 \\ \lambda\sin\theta & \mu\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad [F]^{-1} = \frac{1}{J}\begin{bmatrix} \mu\cos\theta & \mu\sin\theta & 0 \\ -\lambda\sin\theta & \lambda\cos\theta & 0 \\ 0 & 0 & \lambda\mu \end{bmatrix},$$

and the Jacobian is equal to $J = \mu\,\lambda$, which is the ratio of volumes in the deformed and undeformed configurations (i.e., $v = J\,V$).

(b) The unit vector normal to the undeformed cross-sectional area is $\hat{\mathbf{N}} = \hat{\mathbf{E}}_1$, and the unit vector normal to the cross-sectional area of the deformed configuration is

$$\hat{\mathbf{n}} = \cos\theta\, \hat{\mathbf{e}}_1 + \sin\theta\, \hat{\mathbf{e}}_2.$$

The Cauchy stress tensor is $\sigma = \sigma_0\, \hat{\mathbf{n}}\hat{\mathbf{n}}$ and associated stress vector is $\mathbf{t} = \sigma_0\, \hat{\mathbf{n}}$, as shown in Fig. 4.4.3(a). The components of the Cauchy stress tensor are

$$[\sigma] = \begin{Bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{Bmatrix} \sigma_0\, \big\{ \cos\theta\ \sin\theta\ 0 \big\} = \sigma_0 \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta & 0 \\ \cos\theta\sin\theta & \sin^2\theta & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

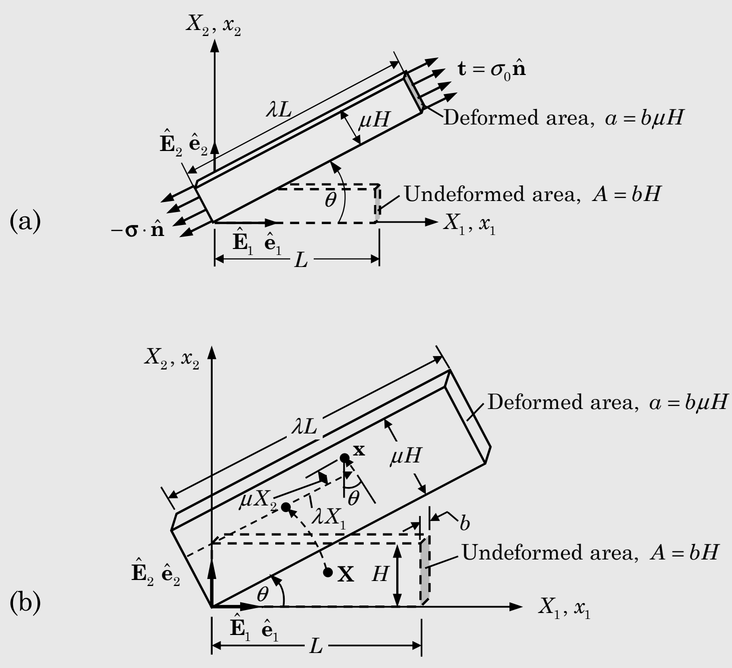


Fig. 4.4.2: (a) Undeformed and (b) deformed configurations of the bar of Example 4.4.1.

The components of the first Piola–Kirchhoff stress tensor are computed using Eq. (4.4.8)

$$[P] = J[\sigma][F]^{-\mathrm{T}} = \sigma_0 \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta & 0 \\ \cos\theta\sin\theta & \sin^2\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mu\cos\theta & -\lambda\sin\theta & 0 \\ \mu\sin\theta & \lambda\cos\theta & 0 \\ 0 & 0 & \lambda\mu \end{bmatrix}$$

$$= \mu\sigma_0 \begin{bmatrix} \cos\theta & 0 & 0 \\ \sin\theta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Clearly, the matrix representing $\mathbf{P}$ is not symmetric. The first Piola–Kirchhoff stress tensor is

$$\mathbf{P} = \mu\sigma_0 \begin{Bmatrix} \hat{\mathbf{e}}_1 \\ \hat{\mathbf{e}}_2 \\ \hat{\mathbf{e}}_3 \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} \cos\theta & 0 & 0 \\ \sin\theta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{E}}_1 \\ \hat{\mathbf{E}}_2 \\ \hat{\mathbf{E}}_3 \end{Bmatrix}$$

$$= \mu\sigma_0 \,(\cos\theta\,\hat{\mathbf{e}}_1 + \sin\theta\,\hat{\mathbf{e}}_2)\hat{\mathbf{E}}_1.$$

The associated stress vector is ($\hat{\mathbf{N}} = \hat{\mathbf{E}}_1$)

$$\mathbf{T} = \mathbf{P}\cdot\hat{\mathbf{N}} = \mu\sigma_0(\cos\theta\,\hat{\mathbf{e}}_1 + \sin\theta\,\hat{\mathbf{e}}_2) = \mu\sigma_0\,\hat{\mathbf{n}},$$

as shown in Fig. 4.4.3(b).

The second Piola–Kirchhoff stress tensor components can be computed either using Eq. (4.4.12) or (4.4.13). Using Eq. (4.4.12), we obtain

$$[S] = [F]^{-1}[P] = \frac{\mu\sigma_0}{J} \begin{bmatrix} \mu\cos\theta & \mu\sin\theta & 0 \\ -\lambda\sin\theta & \lambda\cos\theta & 0 \\ 0 & 0 & \lambda\mu \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & 0 \\ \sin\theta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \frac{\mu\sigma_0}{\lambda} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The second Piola–Kirchhoff stress tensor and the associated pseudo stress vector are [see Fig. 4.4.3(c)]

$$\mathbf{S} = \frac{\mu\sigma_0}{\lambda}\,\hat{\mathbf{E}}_1\hat{\mathbf{E}}_1, \quad \tilde{\mathbf{T}} = \mathbf{S}\cdot\hat{\mathbf{E}} = \frac{\mu\sigma_0}{\lambda}\,\hat{\mathbf{E}}_1.$$

In closing this example we note that the forces (occurring in the deformed body) that produce the Cauchy stress tensor and the second Piola–Kirchhoff stress tensor (recall that the Cauchy stress tensor is measured as the *current force per unit deformed area* while the second Piola–Kirchhoff stress tensor is measured as the *transformed current force per unit undeformed area*) are in equilibrium [see Figs. 4.4.3(a) and 4.4.3(c)], as expected. On the other hand, there is no reason to expect pseudo forces due to the first Piola–Kirchhoff stress tensor, which is measured as the *current force per unit undeformed area*, to satisfy the equilibrium conditions in the undeformed body [see Fig. 4.4.3(b)].

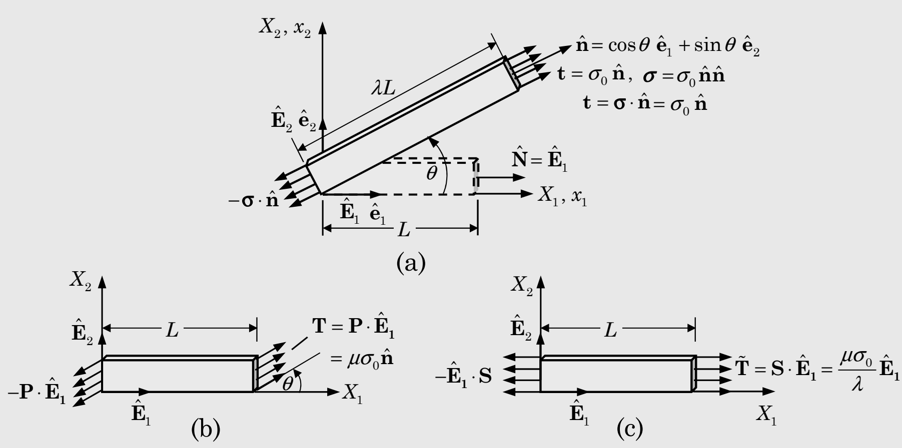


Fig. 4.4.3: Various stresses in the bar of Example 4.4.1.

4.5 Equilibrium Equations for Small Deformations

The principle of conservation of linear momentum, which is commonly known as Newton's second law of motion, is discussed along with other principles of mechanics in Chapter 5. To make the present chapter on stresses complete, we derive the equations of equilibrium of a continuous medium undergoing *small deformations* (that is, strains are infinitesimal $\mathbf{E} \approx \boldsymbol{\varepsilon}$, and the difference between $\boldsymbol{\sigma}$ and $\mathbf{S}$ and between $\mathbf{X}$ and $\mathbf{x}$ vanishes) using Newton's second law of motion.

We isolate from the continuum an infinitesimal parallelepiped element with dimensions dx_1, dx_2, and dx_3 along coordinate x_1, x_2, and x_3, respectively, centered at point $\mathbf{x}$. The stresses acting on various faces of the parallelepiped element are shown in Fig. 4.5.1. The element is also subjected to body force $\rho_0 \mathbf{f}$ (measured per unit mass), where ρ_0 denotes the mass density. The body force components are $\rho_0 f_1$, $\rho_0 f_2$, and $\rho_0 f_3$ along the x_1-, x_2-, and x_3-coordinates, respectively. Setting the sum of all forces in the x_1-direction to zero, we obtain

$$\begin{aligned} 0 &= \Big(\sigma_{11} + \frac{\partial \sigma_{11}}{\partial x_1}\, dx_1\Big) dx_2\, dx_3 - \sigma_{11}\, dx_2\, dx_3 + \Big(\sigma_{12} + \frac{\partial \sigma_{12}}{\partial x_2}\, dx_2\Big) dx_1\, dx_3 \\ &\quad - \sigma_{12}\, dx_1\, dx_3 + \Big(\sigma_{13} + \frac{\partial \sigma_{13}}{\partial x_3}\, dx_3\Big) dx_1\, dx_2 - \sigma_{13}\, dx_1\, dx_2 + \rho_0 f_1\, dx_1\, dx_2\, dx_3 \\ &= \Big(\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} + \rho_0 f_1\Big) dx_1\, dx_2\, dx_3. \end{aligned} \tag{4.5.1}$$

On dividing throughout by $dx_1\, dx_2\, dx_3$, we obtain

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} + \rho_0 f_1 = 0 \quad \text{or} \quad \frac{\partial \sigma_{1j}}{\partial x_j} + \rho_0 f_1 = 0, \tag{4.5.2}$$

for $j = 1, 2$, and 3. Similarly, by setting the sum of forces in the x_2- and x_3-directions to zero separately, we obtain

$$\frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} + \rho_0 f_2 = 0 \quad \text{or} \quad \frac{\partial \sigma_{2j}}{\partial x_j} + \rho_0 f_2 = 0, \tag{4.5.3}$$

$$\frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + \rho_0 f_3 = 0 \quad \text{or} \quad \frac{\partial \sigma_{3j}}{\partial x_j} + \rho_0 f_3 = 0. \tag{4.5.4}$$

Equations (4.5.2)–(4.5.4) can be expressed in a single equation as

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \rho_0 f_i = 0, \quad i, j = 1, 2, 3. \tag{4.5.5}$$

Noting that $\frac{\partial \sigma_{ij}}{\partial x_j} = (\boldsymbol{\sigma} \cdot \overleftarrow{\nabla})_i = (\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}^{\mathrm{T}})_i$, we can express Eq. (4.5.5) in vector form (another example of the use of the backward gradient operator):

$$\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}^{\mathrm{T}} + \rho_0 \mathbf{f} = \mathbf{0}. \tag{4.5.6}$$

The principle of conservation of angular momentum (i.e., Newton's second law for moments) can be used to establish the symmetry of the stress tensor when no body couples exist in the continuum. Consider the moment of all forces acting on the parallelepiped about the x_3-axis (see Fig. 4.5.1). Using the right-handed screw rule for positive moment, we obtain

$$\left[\left(\sigma_{21}+\frac{\partial\sigma_{21}}{\partial x_1}\,dx_1\right)dx_2\,dx_3\right]\frac{dx_1}{2}+(\sigma_{21}dx_2\,dx_3)\,\frac{dx_1}{2}$$
$$-\left[\left(\sigma_{12}+\frac{\partial\sigma_{12}}{\partial x_2}\,dx_2\right)dx_1\,dx_3\right]\frac{dx_2}{2}-(\sigma_{12}\,dx_1\,dx_3)\,\frac{dx_2}{2}=0.$$

Note that the body force components do not have a moment because they pass through the origin of the coordinate system. Dividing throughout by $\frac{1}{2}dx_1\,dx_2\,dx_3$ and taking the limit $dx_1\to 0$ and $dx_2\to 0$, we obtain

$$\sigma_{21}-\sigma_{12}=0. \tag{4.5.7}$$

Similar considerations of moments about the x_1-axis and x_2-axis give, respectively, the relations

$$\sigma_{32}-\sigma_{23}=0,\quad \sigma_{31}-\sigma_{13}=0. \tag{4.5.8}$$

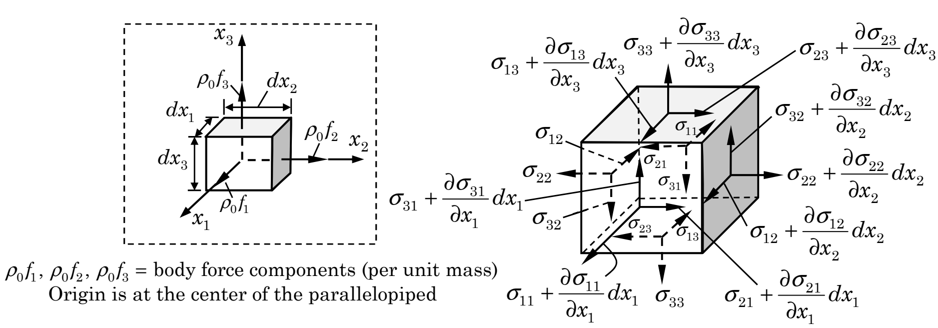


Fig. 4.5.1: Stress components on the faces of a parallelepiped element of dimensions dx_1, dx_2, and dx_3.

Thus, the stress tensor is symmetric ($\sigma_{ij}=\sigma_{ji}$). Equations (4.5.7) and (4.5.8) can be expressed in a single equation using the index notation as

$$\sigma_{ji}\,e_{ijk}=0\ \Rightarrow\ \sigma_{ij}=\sigma_{ji}\ \text{ or }\ \boldsymbol{\sigma}^{\mathrm{T}}=\boldsymbol{\sigma}, \tag{4.5.9}$$

where e_{ijk} are the components of the third-order permutation tensor defined in Eqs. (2.2.49)–(2.2.51). The symmetry of stress tensor with real-valued components has real principal values, and the principal directions associated with distinct principal stresses are orthogonal (see Section 2.5.6). Next, we consider two examples of application of the stress equilibrium equations.

Example 4.5.1

Given the following state of stress ($\sigma_{ij}=\sigma_{ji}$) in a kinematically infinitesimal deformation,

$$\sigma_{11}=-2x_1^2,\quad \sigma_{12}=-7+4x_1x_2+x_3,\quad \sigma_{13}=1+x_1-3x_2,$$
$$\sigma_{22}=3x_1^2-2x_2^2+5x_3,\quad \sigma_{23}=0,\quad \sigma_{33}=-5+x_1+3x_2+3x_3,$$

determine the body force components for which the stress field describes a state of equilibrium.

Solution: The body force components are

$$\rho_0 f_1 = -\left(\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3}\right) = -[(-4x_1) + (4x_1) + 0] = 0,$$
$$\rho_0 f_2 = -\left(\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3}\right) = -[(4x_2) + (-4x_2) + 0] = 0,$$
$$\rho_0 f_3 = -\left(\frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3}\right) = -[1 + 0 + 3] = -4.$$

Thus, the body is in equilibrium for the body force components $\rho_0 f_1 = 0$, $\rho_0 f_2 = 0$, and $\rho_0 f_3 = -4$.

Example 4.5.2

Determine if the following stress field ($\sigma_{ij} = \sigma_{ji}$) in a kinematically infinitesimal deformation satisfies the equations of equilibrium:

$$\sigma_{11} = x_2^2 + k(x_1^2 - x_2^2), \quad \sigma_{12} = -2kx_1x_2, \quad \sigma_{13} = 0,$$
$$\sigma_{22} = x_1^2 + k(x_2^2 - x_1^2), \quad \sigma_{23} = 0, \quad \sigma_{33} = k(x_1^2 + x_2^2).$$

Solution: We have

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} + \rho_0 f_1 = (2kx_1) + (-2kx_1) + 0 + \rho_0 f_1 = 0,$$
$$\frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} + \rho_0 f_2 = (-2kx_2) + (2kx_2) + 0 + \rho_0 f_2 = 0,$$
$$\frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + \rho_0 f_3 = 0 + 0 + 0 + \rho_0 f_3 = 0.$$

Thus the given stress field is in equilibrium in the absence of any body forces; that is, $\rho_0 \mathbf{f} = \mathbf{0}$.

4.6 Objectivity of Stress Tensors

4.6.1 Cauchy Stress Tensor

The Cauchy stress tensor is objective if we can show that $\boldsymbol{\sigma}^* = \mathbf{Q} \cdot \boldsymbol{\sigma} \cdot \mathbf{Q}^{\mathrm{T}}$ [see Eq. (3.8.21) for the definition of the objectivity of various order tensors]. We begin with the relations

$$\mathbf{t} = \boldsymbol{\sigma} \cdot \mathbf{n}, \quad \mathbf{t}^* = \boldsymbol{\sigma}^* \cdot \mathbf{n}^*; \qquad \mathbf{t}^* = \mathbf{Q} \cdot \mathbf{t}, \quad \mathbf{n}^* = \mathbf{Q} \cdot \mathbf{n}. \tag{4.6.1}$$

Then

$$\mathbf{t}^* = \boldsymbol{\sigma}^* \cdot \mathbf{n}^* = \boldsymbol{\sigma}^* \cdot (\mathbf{Q} \cdot \mathbf{n}),$$
$$\mathbf{t}^* = \mathbf{Q} \cdot \mathbf{t} = \mathbf{Q} \cdot \boldsymbol{\sigma} \cdot \mathbf{n}.$$

Then, we have

$$\boldsymbol{\sigma}^* \cdot \mathbf{Q} = \mathbf{Q} \cdot \boldsymbol{\sigma},$$

from which it follows that

$$\boldsymbol{\sigma}^* = \mathbf{Q} \cdot \boldsymbol{\sigma} \cdot \mathbf{Q}^{\mathrm{T}}. \tag{4.6.2}$$

Thus, the Cauchy stress tensor is objective.

4.6.2 First Piola–Kirchhoff Stress Tensor

The first Piola–Kirchhoff stress tensor $\mathbf{P}$ is a two-point tensor, and it transforms like the other two-point tensor $\mathbf{F}$. To establish this, we begin with the relation between $\mathbf{P}$ and $\boldsymbol{\sigma}$ after superposed rigid-body motion and make use of the relations $\mathbf{F}^* = \mathbf{Q}\cdot\mathbf{F}$ and $J^* = J$,

$$\begin{aligned}\mathbf{P}^* &= J^*\,\boldsymbol{\sigma}^*\cdot(\mathbf{F}^*)^{-\mathrm{T}} = J(\mathbf{Q}\cdot\boldsymbol{\sigma}\cdot\mathbf{Q}^{\mathrm{T}})\cdot(\mathbf{Q}\cdot\mathbf{F})^{-\mathrm{T}}\\ &= J\,\mathbf{Q}\cdot\boldsymbol{\sigma}\cdot(\mathbf{Q}^{\mathrm{T}}\cdot\mathbf{Q}^{-\mathrm{T}})\cdot\mathbf{F}^{-\mathrm{T}} = J\,\mathbf{Q}\cdot\boldsymbol{\sigma}\cdot\mathbf{F}^{-\mathrm{T}} = \mathbf{Q}\cdot\mathbf{P}.\end{aligned}\qquad(4.6.3)$$

Thus $\mathbf{P}$, being a two-point tensor, transforms like a vector under superposed rigid-body motion, and hence is objective.

4.6.3 Second Piola–Kirchhoff Stress Tensor

The second Piola–Kirchhoff stress tensor $\mathbf{S}$ is the stress tensor of choice in the study of solid mechanics. Because it is defined with respect to the reference configuration, rigid-body motion should not alter it. Using the relations $\mathbf{F}^* = \mathbf{Q}\cdot\mathbf{F}$ and $\boldsymbol{\sigma}^* = \mathbf{Q}\cdot\boldsymbol{\sigma}\cdot\mathbf{Q}^{\mathrm{T}}$, we obtain

$$\begin{aligned}\mathbf{S}^* &= J^*\,(\mathbf{F}^*)^{-1}\cdot\boldsymbol{\sigma}^*\cdot(\mathbf{F}^*)^{-\mathrm{T}} = J\,(\mathbf{F}^{-1}\cdot\mathbf{Q}^{-1})\cdot(\mathbf{Q}\cdot\boldsymbol{\sigma}\cdot\mathbf{Q}^{\mathrm{T}})\cdot(\mathbf{Q}^{-\mathrm{T}}\cdot\mathbf{F}^{-\mathrm{T}})\\ &= J\,\mathbf{F}^{-1}\cdot\boldsymbol{\sigma}\cdot\mathbf{F}^{-\mathrm{T}} = \mathbf{S}.\end{aligned}\qquad(4.6.4)$$

Thus, $\mathbf{S}$ is not affected by the superposed rigid-body motion and, therefore, it is objective.

4.7 Summary

In this chapter, the concept of stress in a continuum is introduced and stress vector at a point is defined. It is shown that the stress vector $\mathbf{t}$ at a point depends on the orientation of the plane ($\hat{\mathbf{n}}$) on which it acts. Then a relation between the stress vector $\mathbf{t}$ acting on a plane with unit normal $\hat{\mathbf{n}}$ and stress vectors ($\mathbf{t}_1$, $\mathbf{t}_2$, $\mathbf{t}_3$) acting on three mutually perpendicular planes whose normals are $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$, $\hat{\mathbf{e}}_3$ is established. It is here the Cauchy stress tensor $\boldsymbol{\sigma}$ is introduced as a dyadic with respect to the Cartesian basis $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$:

$$\boldsymbol{\sigma} \equiv \mathbf{t}_1\,\hat{\mathbf{e}}_1 + \mathbf{t}_2\,\hat{\mathbf{e}}_2 + \mathbf{t}_3\,\hat{\mathbf{e}}_3 = \mathbf{t}_j\,\hat{\mathbf{e}}_j,\quad \mathbf{t}_j = \sigma_{ij}\,\hat{\mathbf{e}}_i \quad\to\quad \boldsymbol{\sigma} = \sigma_{ij}\,\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j.$$

The stress tensor $\boldsymbol{\sigma}$ at a point $\mathbf{x}$ is shown to be related to the stress vector $\mathbf{t}$ on a plane $\hat{\mathbf{n}}$ by $\mathbf{t} = \boldsymbol{\sigma}\cdot\hat{\mathbf{n}}$, which is known as Cauchy's formula.

We encounter stress vectors $\mathbf{t}$ in two instances: (1) stress vector at a point $\mathbf{x}$ in the interior of the body on a plane whose outward normal is $\hat{\mathbf{n}}$; and (2) stress vector at a point on the surface of the body, which is either specified or to be determined. Cauchy's formula is useful not only in relating the surface traction to the state of stress inside the body at a boundary point, $\bar{\mathbf{t}} = \boldsymbol{\sigma}\cdot\hat{\mathbf{n}}$, but it also implies that when a material volume Ω_0 is removed from a body, then it is possible to maintain Ω_0 in its equilibrium state by merely applying a suitable distribution of $\mathbf{t}$ on the boundary Γ_0 of the volume Ω_0, as shown in Fig. 4.7.1.

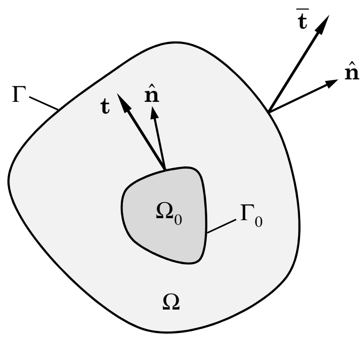


Fig. 4.7.1: Surface traction vector $\bar{\mathbf{t}}$ and internal stress vector $\mathbf{t}(\hat{\mathbf{n}})$.

Transformation relations for the components of the stress tensor in one coordinate system to its components in another coordinate system are established, and the determination of the principal values and principal planes of a stress tensor is detailed. Then two other measures of stress, namely, the first and second Piola–Kirchhoff stress tensors, $\mathbf{P}$ and $\mathbf{S}$, are introduced. Whereas the Cauchy stress tensor $\boldsymbol{\sigma}$ is measured as the current force per unit deformed area, the first Piola–Kirchhoff stress tensor $\mathbf{P}$ is measured as the current force per unit undeformed area and the second Piola–Kirchhoff stress tensor $\mathbf{S}$ is measured as the transformed current force per unit undeformed area. The first and second Piola–Kichhoff stress tensors are related to the Cauchy stress tensor by

$$\mathbf{P} = J\boldsymbol{\sigma}\cdot\mathbf{F}^{-\mathrm{T}} = \mathbf{F}\cdot\mathbf{S}, \quad \mathbf{S} = \mathbf{F}^{-1}\cdot\mathbf{P} = J\,\mathbf{F}^{-1}\cdot\boldsymbol{\sigma}\cdot\mathbf{F}^{-\mathrm{T}}. \tag{4.7.1}$$

The stress equilibrium equations $\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}^{\mathrm{T}} + \rho_0\mathbf{f} = \mathbf{0}$ in the case of infinitesimal deformations are derived, and symmetry of the stress tensor, $\boldsymbol{\sigma}^{\mathrm{T}} = \boldsymbol{\sigma}$, in the absence of body couples, is established. Several examples are presented to illustrate the concepts and ideas introduced.

It is also shown that under superposed rigid-body transformation $\mathbf{x}^* = \mathbf{c}(t) + \mathbf{Q}(t)\cdot\mathbf{x}$, where $\mathbf{c}(t)$ is a constant vector characterizing the rigid-body translation and $\mathbf{Q}(t)$ is a proper orthogonal tensor characterizing the rigid-body rotation, the three stress tensors introduced in this chapter transform according to the following relations and are objective:

$$\boldsymbol{\sigma}^* = \mathbf{Q}\cdot\boldsymbol{\sigma}\cdot\mathbf{Q}^{\mathrm{T}}, \quad \mathbf{P}^* = \mathbf{Q}\cdot\mathbf{P}, \quad \mathbf{S} = \mathbf{S}^*. \tag{4.7.2}$$

Problems

Cauchy Stress Tensor and Cauchy's Formula

4.1 Suppose that $\mathbf{t}^{\hat{\mathbf{n}}_1}$ and $\mathbf{t}^{\hat{\mathbf{n}}_2}$ are stress vectors acting on planes with unit normals $\hat{\mathbf{n}}_1$ and $\hat{\mathbf{n}}_2$, respectively, and passing through a point with the stress state $\boldsymbol{\sigma}$. Show that the component of $\mathbf{t}^{\hat{\mathbf{n}}_1}$ along $\hat{\mathbf{n}}_2$ is equal to the component of $\mathbf{t}^{\hat{\mathbf{n}}_2}$ along the normal $\hat{\mathbf{n}}_1$ *if and only if* $\boldsymbol{\sigma}$ is symmetric.

4.2 Write the stress vectors on each boundary surface in terms of the given values and base vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ for the system shown in Fig. P4.2.

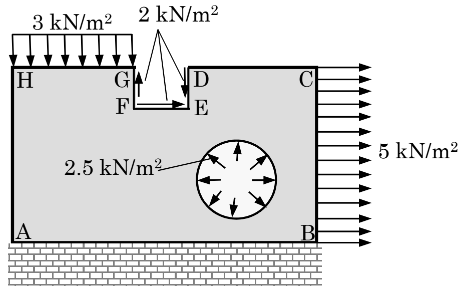


Fig. P4.2

4.3 The components of a stress tensor at a point, with respect to the (x_1, x_2, x_3) system, are (in MPa):

$$\text{(i)} \begin{bmatrix} 12 & 9 & 0 \\ 9 & -12 & 0 \\ 0 & 0 & 6 \end{bmatrix}, \quad \text{(ii)} \begin{bmatrix} 9 & 0 & 12 \\ 0 & -25 & 0 \\ 12 & 0 & 16 \end{bmatrix}, \quad \text{(iii)} \begin{bmatrix} 1 & -3 & \sqrt{2} \\ -3 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 4 \end{bmatrix}.$$

Find the following:

(a) The stress vector acting on a plane perpendicular to the vector $2\hat{\mathbf{e}}_1 - 2\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3$

(b) The magnitude of the stress vector and the angle between the stress vector and the normal to the plane

(c) The magnitudes of the normal and tangential components of the stress vector

4.4 Consider a (kinematically infinitesimal) stress field whose matrix of scalar components in the vector basis $\{\hat{\mathbf{e}}_i\}$ is

$$\begin{bmatrix} 1 & 0 & 2x_2 \\ 0 & 1 & 4x_1 \\ 2x_2 & 4x_1 & 1 \end{bmatrix} \text{ (MPa)},$$

where the Cartesian coordinate variables X_i are in meters (m) and the units of stress are MPa (10^6 Pa $= 10^6$ N/m^2).

(a) Determine the traction vector acting at point $\mathbf{X} = \hat{\mathbf{e}}_1 + 2\hat{\mathbf{e}}_2 + 3\hat{\mathbf{e}}_3$ on the plane $x_1 + x_2 + x_3 = 6$.

(b) Determine the normal and projected shear tractions acting at this point on this plane.

4.5 The three-dimensional state of stress at a point $(1, 1, -2)$ within a body relative to the coordinate system (x_1, x_2, x_3) is

$$\begin{bmatrix} 2.0 & 3.5 & 2.5 \\ 3.5 & 0.0 & -1.5 \\ 2.5 & -1.5 & 1.0 \end{bmatrix} \text{ MPa}.$$

Determine the normal and shear stresses at the point and on the surface of an internal sphere whose equation is $x_1^2 + (x_2 - 2)^2 + x_3^2 = 6$.

4.6 The components of a stress tensor at a point, with respect to the (x_1, x_2, x_3) system, are

$$\begin{bmatrix} 25 & 0 & 0 \\ 0 & -30 & -60 \\ 0 & -60 & 5 \end{bmatrix} \text{ MPa}.$$

Determine

(a) the stress vector acting on a plane perpendicular to the vector $2\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 + 2\hat{\mathbf{e}}_3$, and

(b) the magnitude of the normal and tangential components of the stress vector.

4.7 For the state of stress given in Problem **4.5**, determine the normal and shear stresses on a plane intersecting the point where the plane is defined by the points $(0,0,0)$, $(2,-1,3)$, and $(-2,0,1)$.

4.8 The Cauchy stress tensor components at a point P in the deformed body with respect to the coordinate system (x_1, x_2, x_3) are given by

$$[\sigma] = \begin{bmatrix} 1 & 4 & -2 \\ 4 & 0 & 0 \\ -2 & 0 & 3 \end{bmatrix} \text{ MPa.}$$

(a) Determine the Cauchy stress vector $\mathbf{t}^{\hat{\mathbf{n}}}$ at the point P on a plane passing through the point and parallel to the plane $2x_1 + 3x_2 + x_3 = 4$.

(b) Find the length of $\mathbf{t}^{\hat{\mathbf{n}}}$ and the angle between $\mathbf{t}^{\hat{\mathbf{n}}}$ and the vector normal to the plane.

(c) Determine the components of the Cauchy stress tensor in a rectangular coordinate system $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ whose orthonormal base vectors $\hat{\bar{\mathbf{e}}}_{\mathbf{i}}$ are given in terms of the base vectors $\hat{\mathbf{e}}_i$ of the coordinate system (x_1, x_2, x_3)

$$\hat{\bar{\mathbf{e}}}_{\mathbf{2}} = \tfrac{1}{\sqrt{2}}\left(\hat{\mathbf{e}}_{\mathbf{1}} - \hat{\mathbf{e}}_{\mathbf{3}}\right), \quad \hat{\bar{\mathbf{e}}}_{\mathbf{3}} = \tfrac{1}{3}\left(\mathbf{2}\hat{\mathbf{e}}_{\mathbf{1}} - \hat{\mathbf{e}}_{\mathbf{2}} + \mathbf{2}\hat{\mathbf{e}}_{\mathbf{3}}\right).$$

4.9 The Cauchy stress tensor components at a point P in the deformed body with respect to the coordinate system (x_1, x_2, x_3) are given by

$$[\sigma] = \begin{bmatrix} 2 & 5 & 3 \\ 5 & 1 & 4 \\ 3 & 4 & 3 \end{bmatrix} \text{ MPa.}$$

(a) Determine the Cauchy stress vector $\mathbf{t}^{(\hat{\mathbf{n}})}$ at the point P on a plane passing through the point whose normal is $\mathbf{n} = 3\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 - 2\hat{\mathbf{e}}_3$.

(b) Find the length of $\mathbf{t}^{(\hat{\mathbf{n}})}$ and the angle between $\mathbf{t}^{(\hat{\mathbf{n}})}$ and the vector normal to the plane.

(c) Find the normal and shear components of $\mathbf{t}^{\hat{\mathbf{n}}}$ on the plane.

4.10 Suppose that at a point on the surface of a body the unit outward normal is $\hat{\mathbf{n}} = (\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 - \hat{\mathbf{e}}_3)/\sqrt{3}$ and the traction vector is $P(\hat{\mathbf{e}}_1 + 2\hat{\mathbf{e}}_2)$, where P is a constant. Determine

(a) the normal traction vector $\mathbf{t}_n$ and the shear traction vector $\mathbf{t}_{ns}$ at this point on the surface of the body, and

(b) the conditions between the stress tensor components and the traction vector components.

4.11 Determine the traction free planes (defined by their unit normal vectors) passing through a point in the body where the stress state with respect to the rectangular Cartesian basis is

$$[\sigma] = \begin{bmatrix} 1 & 2 & 1 \\ 2 & \sigma_0 & 0 \\ 1 & 0 & -3 \end{bmatrix} \text{ MPa.}$$

What is the value of σ_0?

Transformation Equations

4.12 Use equilibrium of forces to derive the relations between the normal and shear stresses σ_n and σ_s on a plane whose normal is $\hat{\mathbf{n}} = \cos\theta\hat{\mathbf{e}}_1 + \sin\theta\hat{\mathbf{e}}_2$ to the stress components σ_{11}, σ_{22}, and $\sigma_{12} = \sigma_{21}$ on the $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ planes, as shown in Fig. P4.12:

$$\begin{aligned} \sigma_n &= \sigma_{11}\cos^2\theta + \sigma_{22}\sin^2\theta + \sigma_{12}\sin 2\theta, \\ \sigma_s &= -\tfrac{1}{2}(\sigma_{11} - \sigma_{22})\sin 2\theta + \sigma_{12}\cos 2\theta. \end{aligned} \tag{1}$$

Note that θ is the angle measured from the positive x_1-axis to the normal to the inclined plane (the same as that shown in Fig. 4.3.2). Then show that (a) the principal stresses at a point in a body with two-dimensional state of stress are given by

$$\begin{aligned}\sigma_{p1} = \sigma_{\max} &= \frac{\sigma_{11}+\sigma_{22}}{2} + \sqrt{\left(\frac{\sigma_{11}-\sigma_{22}}{2}\right)^2 + \sigma_{12}^2}\,,\\ \sigma_{p2} = \sigma_{\min} &= \frac{\sigma_{11}+\sigma_{22}}{2} - \sqrt{\left(\frac{\sigma_{11}-\sigma_{22}}{2}\right)^2 + \sigma_{12}^2}\,,\end{aligned} \tag{2}$$

and that the orientation of the principal planes is given by

$$\theta_p = \pm\frac{1}{2}\tan^{-1}\left[\frac{2\sigma_{12}}{\sigma_{11}-\sigma_{22}}\right], \tag{3}$$

and (b) the maximum shear stress is given by

$$(\sigma_s)_{\max} = \pm\frac{\sigma_{p1}-\sigma_{p2}}{2}\,. \tag{4}$$

Also, determine the plane on which the maximum shear stress occurs.

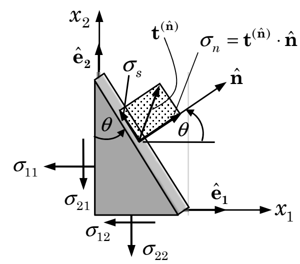


Fig. P4.12

4.13 through **4.16** Determine the normal and shear stress components on the plane indicated in Figs. P4.13–4.16.

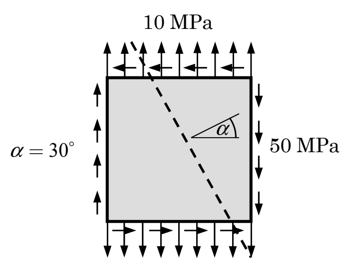


Fig. P4.13

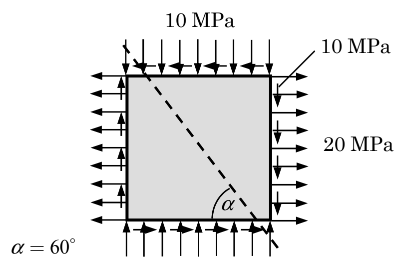


Fig. P4.14

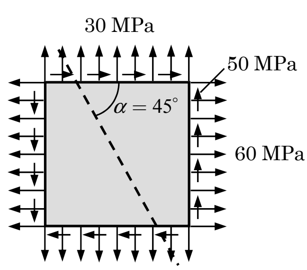


Fig. P4.15

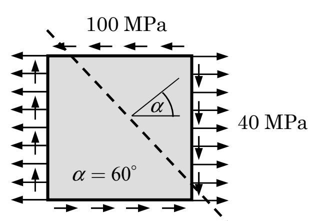


Fig. P4.16

4.17 Find the values of σ_s and σ_{22} for the state of stress shown in Fig. P4.17.

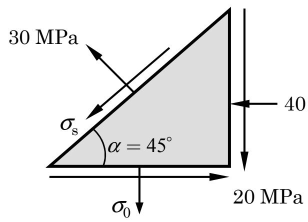


Fig. P4.17

Principal Stresses and Principal Directions

4.18 For the stress state given in Problem **4.4**, determine

(a) the principal stresses and principal directions of stress at this point, and

(b) the maximum shear stress at the point.

4.19 Find the maximum and minimum normal stresses and the orientations of the principal planes for the state of stress shown in Fig. P4.15. What is the maximum shear stress at the point?

4.20 Find the maximum and minimum normal stresses and the orientations of the principal planes for the state of stress shown in Fig. P4.16. What is the maximum shear stress at the point?

4.21 Find the maximum principal stress, maximum shear stress and their orientations for the state of stress given.

$$\text{(a)}\quad [\sigma] = \begin{bmatrix} 12 & 9 & 0 \\ 9 & -12 & 0 \\ 0 & 0 & 6 \end{bmatrix} \text{ MPa.} \qquad \text{(b)}\quad [\sigma] = \begin{bmatrix} 3 & 5 & 8 \\ 5 & 1 & 0 \\ 8 & 0 & 2 \end{bmatrix} \text{ MPa.}$$

4.22 (*Spherical and deviatoric stress tensors*) The stress tensor can be expressed as the sum of *spherical* or *hydrostatic* stress tensor $\tilde{\sigma}$ and *deviatoric* stress tensor σ'

$$\boldsymbol{\sigma} = \tilde{\sigma}\mathbf{I} + \boldsymbol{\sigma}', \quad \tilde{\sigma} = \frac{1}{3}\mathrm{tr}\,\sigma = \frac{1}{3}I_1, \quad \boldsymbol{\sigma}' = \boldsymbol{\sigma} - \frac{1}{3}I_1\mathbf{I}.$$

For the state of stress shown in Fig. P4.16, compute the spherical and deviatoric components of the stress tensor.

4.23 Determine the invariants $I_i^{'}$ and the principal deviator stresses for the following state of stress

$$[\sigma] = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix} \text{ MPa.}$$

4.24 Given the following state of stress at a point in a continuum,

$$[\sigma] = \begin{bmatrix} 7 & 0 & 14 \\ 0 & 8 & 0 \\ 14 & 0 & -4 \end{bmatrix} \text{ MPa,}$$

determine the principal stresses and principal directions.

4.25 Given the following state of stress ($\sigma_{ij} = \sigma_{ji}$),

$$\sigma_{11} = -2x_1^2, \quad \sigma_{12} = -7 + 4x_1x_2 + x_3, \quad \sigma_{13} = 1 + x_1 - 3x_2,$$
$$\sigma_{22} = 3x_1^2 - 2x_2^2 + 5x_3, \quad \sigma_{23} = 0, \quad \sigma_{33} = -5 + x_1 + 3x_2 + 3x_3,$$

determine

(a) the stress vector at point (x_1, x_2, x_3) on the plane $x_1 + x_2 + x_3 =$ constant,

(b) the normal and shearing components of the stress vector at point (1, 1, 3), and

(c) the principal stresses and their orientation at point (1, 2, 1).

4.26 The components of a stress tensor at a point P, referred to the (x_1, x_2, x_3) system, are

$$\begin{bmatrix} 57 & 0 & 24 \\ 0 & 50 & 0 \\ 24 & 0 & 43 \end{bmatrix} \text{MPa.}$$

Determine the principal stresses and principal directions at point P. What is the maximum shear stress at the point?

4.27 Given the following state of stress at a point in a continuum,

$$[\sigma] = \begin{bmatrix} 0 & 0 & Ax_2 \\ 0 & 0 & -Bx_3 \\ Ax_2 & -Bx_3 & 0 \end{bmatrix} \text{MPa,}$$

where A and B are constants. Determine

(a) the body force vector such that the stress tensor corresponds to an equilibrium state,

(b) the three principal invariants of $\boldsymbol{\sigma}$ at the point $\mathbf{x} = B\hat{\mathbf{e}}_2 + A\hat{\mathbf{e}}_3$,

(c) the principal stress components and the associated planes at the point $\mathbf{x} = B\hat{\mathbf{e}}_2 + A\hat{\mathbf{e}}_3$, and

(d) the maximum shear stress and associated plane at the point $\mathbf{x} = B\hat{\mathbf{e}}_2 + A\hat{\mathbf{e}}_3$.

Equilibrium Equations (associated with infinitesimal deformations)

4.28 Derive the stress equilibrium equations in cylindrical coordinates by considering the equilibrium of a typical volume element shown in Fig. P4.28. Assume that the body force components are (not shown in the figure) $\rho_0 f_r$, $\rho_0 f_\theta$, and $\rho_0 f_z$ along the r, θ, and z coordinates, respectively.

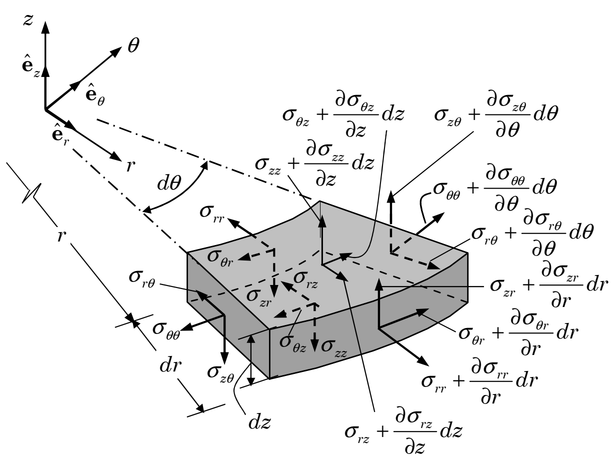


Fig. P4.28

4.29 Given the following state of stress at a point in a continuum,

$$[\sigma] = \begin{bmatrix} 1 & 0 & 2x_2 \\ 0 & 1 & 4x_1 \\ 2x_2 & 4x_1 & 1 \end{bmatrix} \text{MPa,}$$

determine the body force vector such that the stress tensor corresponds to an equilibrium state.

4.30 Given the following state of stress at a point in a continuum,

$$[\sigma] = \begin{bmatrix} 5x_2x_3 & 3x_2^2 & 0 \\ 3x_2^2 & 0 & -x_1 \\ 0 & -x_1 & 0 \end{bmatrix} \text{ MPa},$$

determine the body force vector such that the stress tensor corresponds to an equilibrium state.

4.31 Given the following state of stress at a point in a continuum,

$$[\sigma] = \begin{bmatrix} A(x_1 - x_2) & Bx_1^2x_2 & 0 \\ Bx_1^2x_2 & -A(x_1 - x_2) & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ MPa},$$

determine the constants A and B such that the stress tensor corresponds to an equilibrium state in the absence of body forces.

4.32 Given the following state of stress at a point in a continuum,

$$[\sigma] = \begin{bmatrix} Ax_1^2x_2 & A(B^2 - x_2^2)x_1 & 0 \\ A(B^2 - x_2^2)x_1 & C(x_2^2 - 3B^2)x_2 & 0 \\ 0 & 0 & 2Bx_3^2 \end{bmatrix} \text{ MPa},$$

where A, B, and $C = A/3$ are constants, determine the body force components necessary for the body to be in equilibrium.

4.33 Given the following Cauchy stress components ($\sigma_{ij} = \sigma_{ji}$),

$$\sigma_{11} = -2x_1^2, \quad \sigma_{12} = -7 + 4x_1x_2 + x_3, \quad \sigma_{13} = 1 + x_1 - 3x_2,$$
$$\sigma_{22} = 3x_1^2 - 2x_2^2 + 5x_3, \quad \sigma_{23} = 0, \quad \sigma_{33} = -5 + x_1 + 3x_2 + 3x_3,$$

determine the body force components for which the stress field describes a state of equilibrium.

4.34 Given the following stress field, expressed in terms of its components referred to a rectangular Cartesian basis,

$$\sigma_{11} = x_1^2x_2, \quad \sigma_{12} = (c^2 - x_2^2)x_1, \quad \sigma_{13} = 0,$$
$$\sigma_{22} = \frac{1}{3}\left(x_2^3 - 3c^2x_2\right), \quad \sigma_{23} = 0, \quad \sigma_{33} = 2cx_3^2,$$

where c is a constant, find the body-force field necessary for the stress field to be in equilibrium.

4.35 The equilibrium configuration of a deformed body is described by the mapping

$$\boldsymbol{\chi}(\mathbf{X}) = AX_1\,\hat{\mathbf{e}}_1 - BX_3\,\hat{\mathbf{e}}_2 + CX_2\,\hat{\mathbf{e}}_3,$$

where A, B, and C are constants. If the Cauchy stress tensor for this body is

$$[\sigma] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_0 \end{bmatrix} \text{ MPa},$$

where σ_0 is a constant, determine

(a) the deformation tensor and its inverse in matrix form,

(b) the matrices of the first and second Piola–Kirchhoff stress tensors, and

(c) the pseudo stress vectors associated with the first and second Piola–Kirchhoff stress tensors on the $\hat{\mathbf{e}}_3$-plane in the deformed configuration.

4.36 A body experiences deformation characterized by the mapping

$$\boldsymbol{\chi}(\mathbf{X}, t) = \mathbf{x} = AX_2\,\hat{\mathbf{e}}_1 + BX_1\,\hat{\mathbf{e}}_2 + CX_3\,\hat{\mathbf{e}}_3,$$

where A, B, and C are constants. The Cauchy stress tensor components at certain point of the body are given by

$$[\sigma] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sigma_0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ MPa},$$

where σ_0 is a constant. Determine the Cauchy stress vector $\mathbf{t}$ and the first Piola–Kirchhoff stress vector $\mathbf{T}$ on a plane whose normal in the current configuration is $\hat{\mathbf{n}} = \hat{\mathbf{e}}_2$.

4.37 Express the stress equilibrium equations in Eq. (4.5.6) in terms of the stress components and body force components in the (a) cylindrical and (b) spherical coordinate systems.

4.38 Equation (4.2.7) can also be written

$$\mathbf{t} = (\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_i)\mathbf{t}_i = \hat{\mathbf{n}} \cdot (\hat{\mathbf{e}}_1\, \mathbf{t}_1 + \hat{\mathbf{e}}_2\, \mathbf{t}_2 + \hat{\mathbf{e}}_3\, \mathbf{t}_3)\,. \tag{1}$$

The terms in the parenthesis can be defined as the *stress dyadic* or *stress tensor* $\mathbf{T}$:

$$\mathbf{T} \equiv \hat{\mathbf{e}}_1\, \mathbf{t}_1 + \hat{\mathbf{e}}_2\, \mathbf{t}_2 + \hat{\mathbf{e}}_3\, \mathbf{t}_3 = \hat{\mathbf{e}}_i\, \mathbf{t}_i. \tag{2}$$

Show that $\mathbf{T}$ is the transpose of $\boldsymbol{\sigma}$ defined in Eq. (4.2.13).

4.39 Show that the material time derivative of the Cauchy stress tensor is not objective, unless the superposed rigid-body rotation is time-independent (that is, $\mathbf{Q}$ is not a function of time); that is, show

$$\dot{\boldsymbol{\sigma}}^* \neq \mathbf{Q} \cdot \dot{\boldsymbol{\sigma}} \cdot \mathbf{Q}^{\mathrm{T}},$$

unless $\mathbf{Q}$ is independent of time.

4.40 Prove that if the stress tensor is real and symmetric, $\sigma_{ij} = \sigma_{ji}$, then its eigenvalues are real. Also, prove that the eigenvectors of a real and symmetric σ_{ij} are orthogonal.