5
CONSERVATION AND BALANCE LAWS

Although to penetrate into the intimate mysteries of nature and thence to learn the true causes of phenomena is not allowed to us, nevertheless it can happen that a certain fictive hypothesis may suffice for explaining many phenomena. — Leonard Euler (1707–1783)

Nothing is too wonderful to be true if it be consistent with the laws of nature.
— Michael Faraday (1791–1867)

5.1 Introduction

Virtually every phenomenon in nature can be described in terms of mathematical relations among certain quantities that are responsible for the phenomenon. Most mathematical models of physical phenomena are based on fundamental scientific laws of physics that are extracted from centuries of observations and research on the behavior of mechanical systems subjected to the action of natural forces. The most exciting thing about the laws of physics, which are also termed *principles of mechanics*, is that they govern biological systems as well (because of mass and energy transports). However, biological systems may require additional laws, yet to be discovered, from biology and chemistry to reasonably complete their descriptions.

This chapter is devoted to the study of fundamental laws of physics and resulting mathematical models as applied to mechanical systems. The laws of physics are expressed in analytical form with the aid of the concepts and quantities introduced in the previous chapters. The principles of mechanics to be studied are (1) the principle of conservation of mass, (2) the principle of balance of linear momentum, (3) the principle of balance of angular momentum, and (4) the principle of balance of energy. These principles allow us to write mathematical relationships – algebraic, differential, or integral type – between quantities such as displacements, velocities, temperature, stresses, and strains that arise in mechanical systems. The solution of these equations, in conjunction with the constitutive relations and boundary and initial conditions, represents the response of the system. The equations developed here not only are used not only in the later chapters of this book, but they are also useful in other engineering and applied science courses. In addition, the equations developed herein form the basis of most mathematical models employed in the study of a variety of phenomena. Thus, *the present chapter is the heart and soul of a course on continuum mechanics and elasticity.*

5.2 Conservation of Mass

5.2.1 Preliminary Discussion

It is common knowledge that the mass of a given system cannot be created or destroyed. For example, the mass flow of the blood entering a section of an artery is equal to the mass flow leaving the artery, provided that no blood is added or lost through the artery walls. Thus, mass of the blood is conserved even when the artery cross section changes along the length.

The principle of conservation of mass states that *the total mass of any part $\partial\mathcal{B}$ of the body $\mathcal{B}$ does not change in any motion.* The mathematical form of this principle is different in spatial and material descriptions of motion. Before we derive the mathematical forms of the principle, certain other identities need to be established.

5.2.2 Material Time Derivative

As discussed in Chapter 3 [see Eqs. (3.2.4) and (3.2.5)], the partial time derivative with the material coordinates $\mathbf{X}$ held constant should be distinguished from the partial time derivative with spatial coordinates $\mathbf{x}$ held constant due to the difference in the descriptions of motion. The *material time derivative*, denoted here by D/Dt, is the time derivative with the material coordinates held constant. Thus, the time derivative of a function ϕ in material description (i.e., $\phi = \phi(\mathbf{X}, t)$) with $\mathbf{X}$ held constant is nothing but the partial derivative with respect to time [see Eq. (3.2.4)]:

$$\frac{D\phi}{Dt} \equiv \left(\frac{\partial\phi}{\partial t}\right)_{\mathbf{X}=\text{const}} = \frac{\partial\phi}{\partial t}\,. \tag{5.2.1}$$

In particular, we have

$$\frac{D\mathbf{x}}{Dt} = \left(\frac{\partial\mathbf{x}}{\partial t}\right)_{\mathbf{X}=\text{const}} = \left(\frac{\partial\mathbf{x}}{\partial t}\right) \equiv \mathbf{v}, \tag{5.2.2}$$

where $\mathbf{v}$ is the velocity vector. Similarly, the time derivative of $\mathbf{v}$ is

$$\frac{D\mathbf{v}}{Dt} = \left(\frac{\partial\mathbf{v}}{\partial t}\right)_{\mathbf{X}=\text{const}} = \left(\frac{\partial\mathbf{v}}{\partial t}\right) \equiv \mathbf{a}, \tag{5.2.3}$$

where $\mathbf{a}$ is the acceleration vector.

In the spatial description, we have $\phi = \phi(\mathbf{x}, t)$ and the partial time derivative

$$\left(\frac{\partial\phi}{\partial t}\right)_{\mathbf{X}=\text{const}} \text{ is different from } \left(\frac{\partial\phi}{\partial t}\right)_{\mathbf{x}=\text{const}}.$$

The time derivative

$$\left(\frac{\partial\phi}{\partial t}\right)_{\mathbf{x}=const}$$

denotes the *local rate of change* of ϕ. If $\phi = \mathbf{v}$, then it is the rate of change of $\mathbf{v}$ read by a velocity meter located at the fixed spatial location $\mathbf{x}$, which is not the same as the acceleration of the particle just passing the place $\mathbf{x}$.

To calculate the material time derivative of a function ϕ of spatial coordinates $\mathbf{x}$, $\phi = \phi(\mathbf{x}, t)$, we assume that there exists differentiable mapping $\boldsymbol{\chi}(\mathbf{X}, t) = \mathbf{x}(\mathbf{X}, t)$ so that we can write $\phi(\mathbf{x}, t) = \phi[\mathbf{x}(\mathbf{X}, t), t]$ and compute the derivative using the chain rule of differentiation:

$$\begin{aligned}\frac{D\phi}{Dt} \equiv \left(\frac{\partial \phi}{\partial t}\right)_{\mathbf{X}=\text{const}} &= \left(\frac{\partial \phi}{\partial t}\right)_{\mathbf{x}=\text{const}} + \left(\frac{\partial x_i}{\partial t}\right)_{\mathbf{X}=\text{const}} \frac{\partial \phi}{\partial x_i} \\ &= \left(\frac{\partial \phi}{\partial t}\right)_{\mathbf{x}=\text{const}} + v_i \frac{\partial \phi}{\partial x_i} \\ &= \left(\frac{\partial \phi}{\partial t}\right)_{\mathbf{x}=\text{const}} + \mathbf{v} \cdot \boldsymbol{\nabla}\phi, \end{aligned} \tag{5.2.4}$$

where Eq. (5.2.2) is used in the second line. Thus, the material derivative operator is given by

$$\frac{D}{Dt} = \left(\frac{\partial}{\partial t}\right)_{\mathbf{x}=\text{const}} + \mathbf{v} \cdot \boldsymbol{\nabla}. \tag{5.2.5}$$

Example 5.2.1 illustrates the calculation of the material time derivative based on the material and spatial descriptions.

Example 5.2.1

Suppose that a motion is described by the one-dimensional mapping, $x = (1 + t)X$, for $t \geq 0$. Determine (a) the velocities and accelerations in the spatial and material descriptions, and (b) the time derivative of a function $\phi(X, t) = Xt^2$ in the spatial and material descriptions.

Solution: The velocity $v \equiv Dx/Dt$ can be expressed in the material and spatial coordinates as

$$v(X,t) = \frac{Dx}{Dt} = \frac{\partial x}{\partial t} = X, \quad v(x,t) = X(x,t) = \frac{x}{1+t}.$$

The acceleration $a \equiv Dv/Dt$ in the two descriptions is

$$\begin{aligned} a &\equiv \frac{Dv(X,t)}{Dt} = \frac{\partial v}{\partial t} = 0, \\ a &\equiv \frac{Dv(x,t)}{Dt} = \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} \\ &= -\frac{x}{(1+t)^2} + \frac{x}{1+t}\frac{1}{1+t} = 0. \end{aligned}$$

The material time derivative of $\phi = \phi(X, t)$ in the material description is simply

$$\frac{D\phi(X,t)}{Dt} = \frac{\partial \phi(X,t)}{\partial t} = 2Xt.$$

The material time derivative of $\phi = \phi(x, t) = X(x, t)t^2 = xt^2/(1 + t)$ in the spatial description is

$$\frac{D\phi}{Dt} = \frac{\partial \phi}{\partial t} + v\frac{\partial \phi}{\partial x} = \frac{2xt}{1+t} - \frac{xt^2}{(1+t)^2} + \left(\frac{x}{1+t}\right)\left(\frac{t^2}{1+t}\right) = \frac{2xt}{1+t},$$

which is the same as that calculated before, except that it is expressed in terms of the current coordinate, x.

5.2.3 Vector and Integral Identities

In the sections of the chapter that follow, we will make use of several vector identities and the gradient and divergence theorems [see Tables 2.4.1 and 2.4.2 and Eqs. (2.4.45)–(2.4.47)]. For a ready reference, some key results are provided here.

5.2.3.1 Vector identities

For any scalar function $F(\mathbf{x})$, vector-valued function $\mathbf{A}(\mathbf{x})$, and tensor-valued functions $\mathbf{S}(\mathbf{x})$, the following identities hold:

$$\boldsymbol{\nabla}\cdot(F\mathbf{A}) = F\,\boldsymbol{\nabla}\cdot\mathbf{A} + \boldsymbol{\nabla}F\cdot\mathbf{A}, \tag{5.2.6a}$$

$$\mathbf{A}\cdot\boldsymbol{\nabla}\mathbf{A} = \tfrac{1}{2}\boldsymbol{\nabla}(\mathbf{A}\cdot\mathbf{A}) - \mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{A})\,, \tag{5.2.6b}$$

$$\boldsymbol{\nabla}\cdot(\mathbf{S}\cdot\mathbf{A}) = (\boldsymbol{\nabla}\cdot\mathbf{S})\cdot\mathbf{A} + \mathbf{S} : \boldsymbol{\nabla}\mathbf{A}, \tag{5.2.7}$$

$$\mathbf{S} : (\boldsymbol{\nabla}\mathbf{A}) = \mathbf{S}^{\text{sym}} : (\boldsymbol{\nabla}\mathbf{A})^{\text{sym}} + \mathbf{S}^{\text{skew}} : (\boldsymbol{\nabla}\mathbf{A})^{\text{skew}}, \tag{5.2.8}$$

where : denotes the double-dot product defined in Eq. (2.5.13), and the superscripts sym and skew denote the symmetric and skew symmetric parts of the enclosed quantity [see Eq. (2.5.25)]. In addition, the del operator $\boldsymbol{\nabla}$ and the divergence of $\mathbf{A}$ and $\mathbf{S}$ in the rectangular Cartesian, cylindrical, and spherical coordinates [see Figs. 2.4.4(b) and 2.4.5 for the coordinate systems] have the forms given here (see Chapter 2 for details).

Cartesian coordinates $[\mathbf{x} = (x_1, x_2, x_3)]$

$$\boldsymbol{\nabla} = \hat{\mathbf{e}}_i\frac{\partial}{\partial x_i}\,, \quad \boldsymbol{\nabla}\cdot\mathbf{A} = \frac{\partial A_i}{\partial x_i}, \quad \boldsymbol{\nabla}\cdot\mathbf{S} = \frac{\partial S_{ij}}{\partial x_i}\hat{\mathbf{e}}_j. \tag{5.2.9}$$

Cylindrical coordinates $[\mathbf{x} = (r, \theta, z)]$

$$\boldsymbol{\nabla} = \hat{\mathbf{e}}_r\frac{\partial}{\partial r} + \frac{1}{r}\hat{\mathbf{e}}_\theta\frac{\partial}{\partial\theta} + \hat{\mathbf{e}}_z\frac{\partial}{\partial z}\,, \tag{5.2.10}$$

$$\boldsymbol{\nabla}\cdot\mathbf{A} = \frac{1}{r}\left[\frac{\partial(rA_r)}{\partial r} + \frac{\partial A_\theta}{\partial\theta} + r\frac{\partial A_z}{\partial z}\right], \tag{5.2.11}$$

$$\begin{aligned}\boldsymbol{\nabla}\cdot\mathbf{S} = &\left[\frac{\partial S_{rr}}{\partial r} + \frac{1}{r}\frac{\partial S_{\theta r}}{\partial\theta} + \frac{\partial S_{zr}}{\partial z} + \frac{1}{r}\left(S_{rr} - S_{\theta\theta}\right)\right]\hat{\mathbf{e}}_r \\ &+ \left[\frac{\partial S_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial S_{\theta\theta}}{\partial\theta} + \frac{\partial S_{z\theta}}{\partial z} + \frac{1}{r}\left(S_{r\theta} + S_{\theta r}\right)\right]\hat{\mathbf{e}}_\theta \\ &+ \left[\frac{\partial S_{rz}}{\partial r} + \frac{1}{r}\frac{\partial S_{\theta z}}{\partial\theta} + \frac{\partial S_{zz}}{\partial z} + \frac{1}{r}S_{rz}\right]\hat{\mathbf{e}}_z.\end{aligned} \tag{5.2.12}$$

Spherical coordinates $[\mathbf{x} = (R, \phi, \theta)]$

$$\boldsymbol{\nabla} = \hat{\mathbf{e}}_R\frac{\partial}{\partial R} + \frac{1}{R}\hat{\mathbf{e}}_\phi\frac{\partial}{\partial\phi} + \frac{1}{R\sin\phi}\hat{\mathbf{e}}_\theta\frac{\partial}{\partial\theta}, \tag{5.2.13}$$

$$\boldsymbol{\nabla}\cdot\mathbf{A} = 2\frac{A_R}{R} + \frac{\partial A_R}{\partial R} + \frac{1}{R\sin\phi}\frac{\partial(A_\phi\sin\phi)}{\partial\phi} + \frac{1}{R\sin\phi}\frac{\partial A_\theta}{\partial\theta}, \tag{5.2.14}$$

$$
\begin{aligned}
\boldsymbol{\nabla}\cdot\mathbf{S} = &\left\{\frac{\partial S_{RR}}{\partial R} + \frac{1}{R}\frac{\partial S_{\phi R}}{\partial \phi} + \frac{1}{R\sin\phi}\frac{\partial S_{\theta R}}{\partial \theta}\right.\\
&\left. + \frac{1}{R}\left[2S_{RR} - S_{\phi\phi} - S_{\theta\theta} + S_{\phi R}\cot\phi\right]\right\}\hat{\mathbf{e}}_R\\
&+\left\{\frac{\partial S_{R\phi}}{\partial R} + \frac{1}{R}\frac{\partial S_{\phi\phi}}{\partial \phi} + \frac{1}{R\sin\phi}\frac{\partial S_{\theta\phi}}{\partial \theta}\right.\\
&\left. + \frac{1}{R}\left[(S_{\phi\phi} - S_{\theta\theta})\cot\phi + S_{\phi R} + 2S_{R\phi}\right]\right\}\hat{\mathbf{e}}_\phi\\
&+\left\{\frac{\partial S_{R\theta}}{\partial R} + \frac{1}{R}\frac{\partial S_{\phi\theta}}{\partial \phi} + \frac{1}{R\sin\phi}\frac{\partial S_{\theta\theta}}{\partial \theta}\right.\\
&\left. + \frac{1}{R}\left[(S_{\phi\theta} + S_{\theta\phi})\cot\phi + 2S_{R\theta} + S_{\theta R}\right]\right\}\hat{\mathbf{e}}_\theta. \qquad (5.2.15)
\end{aligned}
$$

5.2.3.2 Integral identities

The following relations hold for a closed region Ω bounded by surface Γ with outward unit normal vector $\hat{\mathbf{n}}$:

$$\oint_\Gamma \hat{\mathbf{n}}\,\mathbf{S}\,ds = \int_\Omega \boldsymbol{\nabla}\mathbf{S}\,d\mathbf{x} \quad \text{(Gradient theorem)} \qquad (5.2.16)$$

$$\oint_\Gamma \phi\,\hat{\mathbf{n}}\cdot\mathbf{S}\,ds = \int_\Omega \boldsymbol{\nabla}\cdot(\phi\mathbf{S})\,d\mathbf{x} \quad \text{(Divergence theorem)} \qquad (5.2.17)$$

$$\oint_\Gamma \mathbf{x}\times(\hat{\mathbf{n}}\cdot\mathbf{S})\,ds = \int_\Omega \left[\mathbf{x}\times(\boldsymbol{\nabla}\cdot\mathbf{S}) + \mathcal{E}:\mathbf{S}\right]d\mathbf{x}, \qquad (5.2.18)$$

where $\mathbf{x}$ is the position vector, $\mathbf{S}$ is a tensor-valued function of position, $\mathcal{E}$ is the third-order permutation tensor [see Eq. (2.5.23)], ϕ is a scalar-valued function, $d\mathbf{x}$ denotes a volume element, and ds is an area element on the surface.

5.2.4 Continuity Equation in the Spatial Description

Let an arbitrary region in a continuous medium $\mathcal{B}$ be denoted by Ω, and the bounding closed surface of this region be continuous and denoted by Γ. Let each point on the bounding surface move with velocity $\mathbf{v}_s$. It can be shown that the time derivative of the volume integral of some continuous function $\phi(\mathbf{x},t)$ is given by

$$
\begin{aligned}
\frac{d}{dt}\int_\Omega \phi(\mathbf{x},t)\,d\mathbf{x} &\equiv \frac{\partial}{\partial t}\int_\Omega \phi\,d\mathbf{x} + \oint_\Gamma \phi\,\hat{\mathbf{n}}\cdot\mathbf{v}_s\,ds,\\
&= \int_\Omega \frac{\partial \phi}{\partial t}\,d\mathbf{x} + \oint_\Gamma \phi\,\hat{\mathbf{n}}\cdot\mathbf{v}_s\,ds. \qquad (5.2.19)
\end{aligned}
$$

This expression for the differentiation of a volume integral with variable limits is sometimes known as the three-dimensional *Leibnitz rule*.

Let each element of mass in the medium move with the velocity $\mathbf{v}(\mathbf{x},t)$ and consider a special region Ω such that the bounding surface Γ is attached to a fixed set of material elements. Then each point of this surface moves with the material velocity, that is, $\mathbf{v}_s = \mathbf{v}$, and the region Ω thus contains a fixed total amount of mass because no mass crosses the boundary surface Γ. To distinguish the time rate of change of an integral over this material region, we replace d/dt by D/Dt and write

$$\frac{D}{Dt}\int_\Omega \phi(\mathbf{x},t)\,d\mathbf{x} \equiv \int_\Omega \frac{\partial \phi}{\partial t}\,d\mathbf{x} + \oint_\Gamma \phi\,\hat{\mathbf{n}}\cdot\mathbf{v}\,ds, \tag{5.2.20}$$

which holds for a material region, that is, a region of fixed total mass. In some books, Eq. (5.2.20) is referred to as the *Reynolds transport theorem.* The relation between the time derivative following an arbitrary region and the time derivative following a material region (fixed total mass) is

$$\frac{d}{dt}\int_\Omega \phi(\mathbf{x},t)\,d\mathbf{x} \equiv \frac{D}{Dt}\int_\Omega \phi(\mathbf{x},t)\,d\mathbf{x} + \oint_\Gamma \phi\,\hat{\mathbf{n}}\cdot(\mathbf{v}_s - \mathbf{v})\,ds. \tag{5.2.21}$$

The velocity difference $\mathbf{v} - \mathbf{v}_s$ is the velocity of the material measured relative to the velocity of the surface. The surface integral

$$\oint_\Gamma \phi\,\hat{\mathbf{n}}\cdot(\mathbf{v} - \mathbf{v}_s)\,ds,$$

thus measures the total *outflow* of the property ϕ from the region Ω.

Let $\rho(\mathbf{x},t)$ denote the mass density of a continuous region. Then the principle of conservation of mass for a fixed *material* region Ω requires that

$$\frac{D}{Dt}\int_\Omega \rho\,d\mathbf{x} = 0. \tag{5.2.22}$$

Then from Eq. (5.2.21), with $\phi = \rho$, it follows that for a fixed *spatial region* Ω (i.e., $\mathbf{v}_s = 0$) the principle of conservation of mass can also be stated as

$$\frac{d}{dt}\int_\Omega \rho\,d\mathbf{x} = -\oint_\Gamma \rho\,\hat{\mathbf{n}}\cdot\mathbf{v}\,ds. \tag{5.2.23}$$

Thus, the time rate of change of mass inside a region Ω is equal to the mass inflow (because of the negative sign) through the surface into the region. Equation (5.2.23) is known as the control volume formulation of the conservation of mass principle. In Eq. (5.2.23), Ω denotes the *control volume* (cv) and Γ the *control surface* (cs) enclosing Ω.

Using Eq. (5.2.19) with $\phi = \rho$, Eq. (5.2.23) can be expressed as

$$\int_\Omega \frac{\partial \rho}{\partial t}\,d\mathbf{x} = -\oint_\Gamma \rho\,\hat{\mathbf{n}}\cdot\mathbf{v}\,ds.$$

Converting the surface integral to a volume integral by means of the divergence theorem (5.2.17), we obtain

$$\int_\Omega \left[\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\,\mathbf{v})\right] d\mathbf{x} = 0.$$

Since this integral vanishes for any continuous medium occupying an arbitrary region Ω, we deduce that this is true only if the integrand itself vanishes identically, giving the following local form of the principle of conservation of mass:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\, \mathbf{v}) = 0. \tag{5.2.24}$$

This equation, also known as the *continuity equation*, expresses local conservation of mass at any point in a continuous medium.

An alternative derivation of Eq. (5.2.24) that is found in fluid mechanics books is presented next. Consider an arbitrary control volume Ω in space where flow occurs into and out of the control volume. Conservation of mass in this case means that the time rate of change of mass in Ω is equal to the mass inflow through the control surface Γ into the control volume Ω. Consider an elemental area ds with unit normal $\hat{\mathbf{n}}$ around a point P on the control surface, as shown in Fig. 5.2.1. Let $\mathbf{v}$ and ρ be the velocity and mass density, respectively, at point P. The mass *outflow* (slug/s or kg/s) through the elemental surface is $\rho\, \mathbf{v} \cdot d\mathbf{s}$, where $d\mathbf{s} = \hat{\mathbf{n}}\, ds$. The total mass *inflow* through the entire surface of the control volume is

$$\oint_\Gamma (-\rho\, v_n)\, ds = -\oint_\Gamma \rho\, \hat{\mathbf{n}} \cdot \mathbf{v}\, ds = -\int_\Omega \nabla \cdot (\rho\, \mathbf{v})\, d\mathbf{x}, \tag{5.2.25}$$

where the divergence theorem (5.2.17) is used in arriving at the last expression in Eq. (5.2.25). If a continuous medium of mass density ρ fills the region Ω at time t, the total mass in Ω is $M = \int_\Omega \rho(\mathbf{x}, t)\, d\mathbf{x}$. The rate of increase of mass in the fixed region Ω is

$$\frac{\partial M}{\partial t} = \int_\Omega \frac{\partial \rho}{\partial t}\, d\mathbf{x}. \tag{5.2.26}$$

Equating Eqs. (5.2.25) and (5.2.26), we obtain

$$\int_\Omega \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\, \mathbf{v}) \right] d\mathbf{x} = 0,$$

which results in the same equation as the one in Eq. (5.2.24).

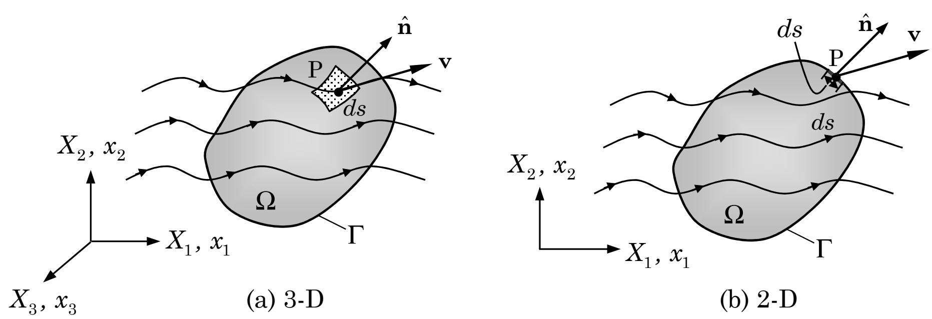

Fig. 5.2.1: A control volume for the derivation of the continuity equation.

Equation (5.2.24) can be written in an alternative form as follows [see Eq. (5.2.6a)]:

$$0 = \frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho\,\mathbf{v}) = \frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} \rho + \rho\,\boldsymbol{\nabla} \cdot \mathbf{v} = \frac{D\rho}{Dt} + \rho\,\boldsymbol{\nabla} \cdot \mathbf{v}, \tag{5.2.27}$$

where the definition of *material time derivative*, Eq. (5.2.5), is used in arriving at the final result.

The one-dimensional version of the local form of the continuity equation (5.2.24) can be obtained by considering flow along the x-axis (see Fig. 5.2.2). The amount of mass entering (i.e., mass flow) per unit time at the left section of the elemental volume is:

$$\text{density} \times \text{cross-sectional area} \times \text{velocity of the flow} = (\rho\, A v_x)_x.$$

The mass leaving at the right section of the elemental volume is $(\rho\, A v_x)_{x+\Delta x}$, where v_x is the velocity along the x-direction. The subscript denotes the distance at which the enclosed quantity is evaluated. It is assumed that the cross-sectional area A is a function of position x but not of time t. The net mass flow *into* the elemental volume is

$$(A\rho\, v_x)_x - (A\rho\, v_x)_{x+\Delta x}.$$

On the other hand, the time rate of *increase* of the total mass inside the elemental volume is

$$\bar{A}\Delta x \frac{(\bar{\rho})_{t+\Delta t} - (\bar{\rho})_t}{\Delta t},$$

where $\bar{\rho}$ and $\bar{A}$ are the average values of the density and cross-sectional area, respectively, inside the elemental volume.

If no mass is created or destroyed inside the elemental volume, the rate of increase of mass should be equal to the mass inflow:

$$\bar{A}\Delta x \frac{(\bar{\rho})_{t+\Delta t} - (\bar{\rho})_t}{\Delta t} = (A\rho\, v_x)_x - (A\rho\, v_x)_{x+\Delta x}.$$

Dividing throughout by Δx and taking the limits $\Delta t \to 0$ and $\Delta x \to 0$, we obtain

$$\lim_{\Delta t,\, \Delta x \to 0} \bar{A}\frac{(\rho)_{t+\Delta t} - (\rho)_t}{\Delta t} + \frac{(A\rho\, v_x)_{x+\Delta x} - (A\rho\, v_x)_x}{\Delta x} = 0,$$

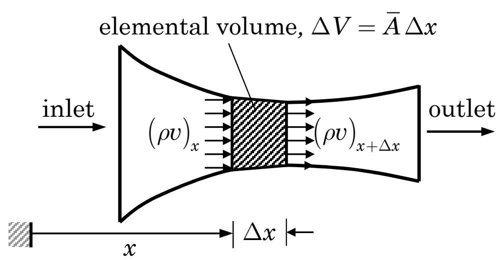

Fig. 5.2.2: Derivation of the local form of the continuity equation in one dimension.

or ($\bar{\rho} \to \rho$ and $\bar{A} \to A$ as $\Delta x \to 0$)

$$A\frac{\partial \rho}{\partial t} + \frac{\partial(A\rho\, v_x)}{\partial x} = 0. \tag{5.2.28}$$

Equation (5.2.28) is the same as Eq. (5.2.24) when $\mathbf{v}$ is replaced with $\mathbf{v} = v_x\hat{\mathbf{e}}_x$ and A is a constant. Note that for the steady-state case, Eq. (5.2.28) reduces to

$$\frac{\partial(A\rho\, v_x)}{\partial x} = 0 \;\to\; A\rho\, v_x = \text{constant} \Rightarrow (A\rho\, v_x)_1 = (A\rho\, v_x)_2 = \cdots = (A\rho\, v_x)_i, \tag{5.2.29}$$

where the subscript i refers to ith section along the direction of the (one-dimensional) flow. The quantity $Q = Av_x$ is called the *flow*, whereas $\rho\, Av_x$ is called the *mass flow*.

The continuity equation in Eq. (5.2.24) can also be expressed in orthogonal curvilinear coordinate systems as [see Eqs. (5.2.10)–(5.2.15); Problems **5.4–5.6** are designed to obtain these results]

Cylindrical coordinate system (r, θ, z)

$$0 = \frac{\partial \rho}{\partial t} + \frac{1}{r}\left[\frac{\partial(r\rho\, v_r)}{\partial r} + \frac{\partial(\rho\, v_\theta)}{\partial \theta} + r\frac{\partial(\rho\, v_z)}{\partial z}\right]. \tag{5.2.30}$$

Spherical coordinate system (R, ϕ, θ)

$$0 = \frac{\partial \rho}{\partial t} + \frac{1}{R^2}\frac{\partial(\rho\, R^2 v_R)}{\partial R} + \frac{1}{R\sin\phi}\frac{\partial(\rho\, v_\phi \sin\phi)}{\partial \theta} + \frac{1}{R\sin\phi}\frac{\partial(\rho\, v_\theta)}{\partial \theta}. \tag{5.2.31}$$

For steady state, we set the time derivative terms in Eqs. (5.2.24), (5.2.30), and (5.2.31) to zero. The invariant form of continuity equation for steady-state flows is (so-called divergence-free velocity field)

$$\boldsymbol{\nabla} \cdot (\rho\, \mathbf{v}) = 0. \tag{5.2.32}$$

For materials with constant density, we set $D\rho/Dt = 0$ and obtain (the so-called divergence-free condition on the velocity field)

$$\rho\, \boldsymbol{\nabla} \cdot \mathbf{v} = 0 \quad \text{or} \quad \boldsymbol{\nabla} \cdot \mathbf{v} = 0. \tag{5.2.33}$$

Thus the motion is isochoric, and the velocity field is said to be *solenoidal.*

Next, we consider two examples of application of the principle of conservation of mass in spatial description.

Example 5.2.2

Consider a water hose with a conical-shaped nozzle at its end, as shown in Fig. 5.2.3(a). (a) Determine the pumping capacity required for the velocity of the water (assuming incompressible for the present case) exiting the nozzle to be 25 m/s. (b) If the hose is connected to a rotating sprinkler through its base, as shown in Fig. 5.2.3(b), determine the average speed of the water leaving the sprinkler nozzle.

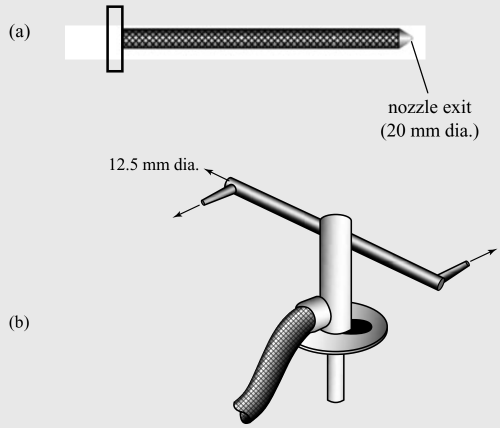

Fig. 5.2.3: (a) Water hose with a conical head. (b) Water hose connected to a sprinkler.

Solution: (**a**) The principle of conservation of mass for steady one-dimensional flow requires

$$\rho_1 A_1 v_1 = \rho_2 A_2 v_2.$$

If the exit of the nozzle is taken as the section 2 and the inlet is taken as the section 1 [see Fig. 5.2.3(a)], we can write (for an incompressible fluid, $\rho_1 = \rho_2$)

$$Q_1 = A_1 v_1 = A_2 v_2 = \frac{\pi(20 \times 10^{-3})^2}{4} 25 = 0.0025\pi \text{ m}^3/\text{s}.$$

(**b**) The average speed of the water leaving the sprinkler nozzle can be calculated using the principle of conservation of mass for steady one-dimensional flow. We obtain

$$Q_1 = 2A_2 v_2 \quad \rightarrow \quad v_2 = \frac{2Q_1}{\pi d^2} = \frac{0.005}{(12.5 \times 10^{-3})^2} = 32 \text{ m/s}.$$

Example 5.2.3

A syringe used to inoculate large animals has a cylinder, plunger, and needle combination, as shown in Fig. 5.2.4. Let the internal diameter of the cylinder be d and the plunger face area be A_p. If the liquid in the syringe is to be injected at a steady rate of Q_0, determine the speed of the plunger. Assume that the leakage rate past the plunger is 10% of the volume flow rate out of the needle.

Solution: In this problem, the control volume (shown in dotted lines in Fig. 5.2.4) is not constant. Even though there is a leakage, the plunger surface area can be taken as equal to the open cross-sectional area of the cylinder, $A_p = \pi d^2/4$. Let us consider Section 1 to be the plunger face and Section 2 to be the needle exit to apply the continuity equation.

Assuming that the flow through the needle and leakage are steady, application of the global form of the continuity equation, Eq. (5.2.23), to the control volume gives

$$\begin{aligned} 0 &= \frac{d}{dt}\int_\Omega \rho\, d\mathbf{x} + \oint_\Gamma \rho\, \hat{\mathbf{n}} \cdot \mathbf{v}\, ds \\ &= \frac{d}{dt}\int_\Omega \rho\, d\mathbf{x} + \rho\, Q_0 + \rho\, Q_{\text{leak}}. \end{aligned} \tag{1}$$

The integral in the above equation can be evaluated as follows:

$$\frac{d}{dt}\int_\Omega \rho\, d\mathbf{x} = \frac{d}{dt}\big(\rho\, x A_p + \rho\, V_n\big) = \rho\, A_p \frac{dx}{dt} = -\rho\, A_p v_p, \tag{2}$$

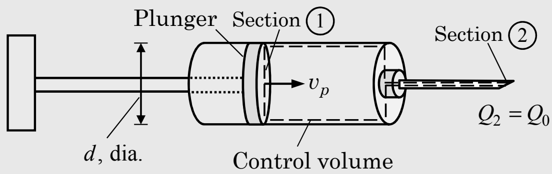

Fig. 5.2.4: The syringe discussed in Example 5.2.3.

where x is the distance between the plunger face and the end of the cylinder, V_n is the volume of the needle opening, and $v_p = -dx/dt$ is the speed of the plunger that we are after. Noting that $Q_{\rm leak} = 0.1Q_0$, we can write the continuity equation (1) as

$$-\rho\, A_p v_p + 1.1\rho\, Q_0 = 0,$$

from which we obtain

$$v_p = 1.1\frac{Q_0}{A_p} = \frac{4.4Q_0}{\pi d^2}\,. \tag{3}$$

For $Q_0 = 250\,\text{cm}^3/\text{min}$ and $d = 25\,\text{mm}$, we obtain

$$v_p = \frac{4.4 \times (250 \times 10^3)}{\pi(25 \times 25)} = 560\ \text{mm/min}.$$

5.2.5 Continuity Equation in the Material Description

Under the assumption that mass is neither created nor destroyed during motion, we require that the total mass of any material volume be the same at any instant during the motion. To express this in analytical terms, we consider a material body $\mathcal{B}$ that occupies configuration κ_0 with density ρ_0 and volume Ω_0 at time $t = 0$. The same material body occupies the configuration κ with volume Ω at time $t > 0$, and it has a density ρ. As per the principle of conservation of mass, we have

$$\int_{\Omega_0} \rho_0\, d\mathbf{X} = \int_{\Omega} \rho\, d\mathbf{x}. \tag{5.2.34}$$

Using the relation between $d\mathbf{X}$ and $d\mathbf{x}$, $d\mathbf{x} = J\, d\mathbf{X}$, where J is the determinant of the deformation gradient tensor $\mathbf{F}$, we arrive at

$$\int_{\Omega_0} (\rho_0 - J\,\rho)\, d\mathbf{X} = 0. \tag{5.2.35}$$

This is the *global form* of the continuity equation. Since the material volume Ω_0 we selected is arbitrarily small, we can shrink the volume to a point and obtain the *local form* of the continuity equation

$$\rho_0 = J\,\rho. \tag{5.2.36}$$

Example 5.2.4 illustrates the use of the material time derivative in computing velocities and use of the continuity equation to compute the density in the current configuration.

Example 5.2.4

Consider the motion of a body $\mathcal{B}$ described by the mapping

$$x_1 = \frac{X_1}{1 + tX_1}\,, \quad x_2 = X_2, \quad x_3 = X_3.$$

Determine the material density as a function of position $\mathbf{x}$ and time t.

Solution: The inverse mapping is given by

$$X_1 = \frac{x_1}{1 - tx_1}\,, \quad X_2 = x_2, \quad X_3 = x_3. \tag{1}$$

We then compute the velocity components

$$\mathbf{v} = \frac{D\mathbf{x}}{Dt} = \left(\frac{\partial \mathbf{x}}{\partial t}\right)_{\mathbf{X}=\text{fixed}}; \quad v_i = \frac{Dx_i}{Dt} = \left(\frac{\partial x_i}{\partial t}\right)_{\mathbf{X}=\text{fixed}}. \tag{2}$$

Therefore, we have

$$v_1 = -\frac{X_1^2}{(1+tX_1)^2} = -x_1^2, \quad v_2 = 0, \quad v_3 = 0. \tag{3}$$

Next, we compute $D\rho/Dt$ from the continuity equation (5.2.27)

$$\frac{D\rho}{Dt} = -\rho\, \boldsymbol{\nabla}\cdot\mathbf{v} = -\rho\left(\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}\right) = 2\rho\, x_1, \tag{4}$$

and in the material coordinates

$$\frac{D\rho}{Dt} = 2\rho\,\frac{X_1}{1 + tX_1}. \tag{5}$$

Integrating the above equation (for fixed X_1), we obtain

$$\int \frac{1}{\rho}\, D\rho = 2\int \frac{X_1}{1+tX_1}\, Dt \;\Rightarrow\; \ln\rho = 2\ln(1 + tX_1) + \ln c,$$

where c is the constant of integration. If $\rho = \rho_0$ at time $t = 0$, we have $\ln c = \ln\rho_0$. Thus, the material density in the current configuration is

$$\rho = \rho_0\left(1 + tX_1\right)^2 = \frac{\rho_0}{(1 - tx_1)^2}\,. \tag{6}$$

It can be verified that the material time derivative of ρ gives the same result as in Eq. (4),

$$\begin{aligned}\frac{D\rho}{Dt} &= \frac{\partial\rho}{\partial t} + v_1\frac{\partial\rho}{\partial x_1}\\ &= \frac{2\rho_0\, x_1}{(1-tx_1)^3} + (-x_1^2)\frac{2\rho_0 t}{(1-tx_1)^3} = \frac{2\rho_0 x_1}{(1-tx_1)^2} = 2\rho x_1.\end{aligned}$$

The mass density in the current configuration can also be computed using the continuity equation in the material description, $\rho_0 = \rho\, J$. Noting that

$$dx_1 = \frac{1}{(1+tX_1)}\,dX_1 - \frac{tX_1}{(1+tX_1)^2}\,dX_1 = \frac{1}{(1+tX_1)^2}\,dX_1, \quad J = \frac{dx_1}{dX_1} = \frac{1}{(1+tX_1)^2}\,,$$

we obtain

$$\rho = \frac{1}{J}\,\rho_0 = \rho_0(1 + tX_1)^2\,.$$

5.2.6 Reynolds Transport Theorem

The material derivative operator D/Dt corresponds to changes with respect to a fixed mass, that is, $\rho\, d\mathbf{x}$ is constant with respect to this operator. Therefore, from Eq. (5.2.20) by substituting for $\phi = \rho\, F(\mathbf{x}, t)$, where F is an arbitrary function, we obtain the result

$$\frac{D}{Dt}\int_\Omega \rho\, F(\mathbf{x}, t)\, d\mathbf{x} = \frac{\partial}{\partial t}\int_\Omega \rho\, F\, d\mathbf{x} + \oint_\Gamma \rho\, F\, \hat{\mathbf{n}}\cdot \mathbf{v}\, ds, \tag{5.2.38}$$

or

$$\begin{aligned}\frac{D}{Dt}\int_\Omega \rho\, F(\mathbf{x}, t)\, d\mathbf{x} &= \int_\Omega \left[\rho\, \frac{\partial F}{\partial t} + F\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\, F\mathbf{v})\right] d\mathbf{x}\\ &= \int_\Omega \left[\rho\, \left(\frac{\partial F}{\partial t} + \mathbf{v}\cdot \boldsymbol{\nabla} F\right) + F\left(\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\, \mathbf{v})\right)\right] d\mathbf{x}.\end{aligned} \tag{5.2.39}$$

Now using the continuity equation (5.2.24) and the definition of the material time derivative, we arrive at the result

$$\frac{D}{Dt}\int_\Omega \rho\, F\, d\mathbf{x} = \int_\Omega \rho\, \frac{DF}{Dt}\, d\mathbf{x}. \tag{5.2.40}$$

Equation (5.2.40) is known as the *Reynolds transport theorem.* Equation (5.2.40) also holds when F is a vector- on tensor-valued function.

5.3 Balance of Linear and Angular Momentum

5.3.1 Principle of Balance of Linear Momentum

The principle of balance of linear momentum, also known as Newton's second law of motion, applied to a set of particles (or rigid body) can be stated as follows: *The time rate of change of (linear) momentum of a collection of particles equals the net force exerted on the collection.* Written in vector form, the principle implies

$$\frac{d}{dt}(m\mathbf{v}) = \mathbf{F}, \tag{5.3.1}$$

where m is the total mass, $\mathbf{v}$ is the velocity, and $\mathbf{F}$ is the resultant force on the collection of particles. For constant mass, Eq. (5.3.1) becomes

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt} = m\mathbf{a}, \tag{5.3.2}$$

which is the familiar form of Newton's second law.

Newton's second law for a control volume Ω can be expressed as

$$\mathbf{F} = \frac{\partial}{\partial t}\int_\Omega \mathbf{v}(\rho\, d\mathbf{x}) + \oint_\Gamma \mathbf{v}(\rho\, \mathbf{v}\cdot d\mathbf{s}), \tag{5.3.3}$$

where $\mathbf{F}$ is the resultant force and $d\mathbf{s}$ denotes the vector representing a surface area element of the outflow. Several simple examples that illustrate the use of Eq. (5.3.3) are presented next.

Example 5.3.1

Suppose that a jet of fluid with cross-sectional area A and mass density ρ issues from a nozzle with a velocity v and impinges against a smooth inclined flat plate, as shown in Fig. 5.3.1. Assuming that there is no frictional resistance between the jet and the plate, determine the distribution of the flow and the force required to keep the plate in position.

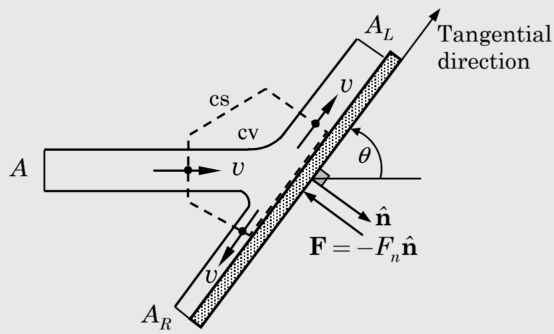

Fig. 5.3.1: Jet of fluid impinging on an inclined plate.

Solution: Since there is no change in pressure or elevation before and after impact, the velocity of the fluid remains the same before and after impact, but the flow to the left and right would be different. Let the amount of flow to the left be Q_L and to the right be Q_R. Then the total flow $Q = vA$ of the jet is equal to the sum (by the continuity equation):

$$Q = Q_L + Q_R. \tag{1}$$

Next, we use the principle of balance of linear momentum to relate Q_L and Q_R. Applying Eq. (5.3.3) to the positive tangential direction to the plate, and noting that the resultant force is zero and the first term on the right-hand side of Eq. (5.3.3) is zero by virtue of the steady-state condition, we obtain (note that the control surface has three segments that have nonzero flow across the boundary)

$$0 = \oint_{\text{cs}} v_t\, \rho\, \mathbf{v}\cdot d\mathbf{s} = v\cos\theta(-\rho\, vA) + v(\rho\, vA_L) + (-v)(\rho\, vA_R), \tag{2}$$

where the minus sign in the first term on the right side of the equality is due to the fact that the mass flow is into the control volume, and the minus sign in the third term is due to the fact that the velocity is in the opposite direction to the tangent direction (but the mass flow is out of the control volume, that is, positive). With $A_L v = Q_L$, $A_R v = Q_R$, and $Av = Q$, we obtain

$$Q_L - Q_R = Q\cos\theta.$$

Solving the two equations for Q_L and Q_R, we obtain

$$Q_L = \tfrac{1}{2}(1+\cos\theta)\,Q, \quad Q_R = \tfrac{1}{2}(1-\cos\theta)\,Q. \tag{3}$$

Thus, the total flow Q is divided into the left flow of Q_L and right flow of Q_R, as given above.

The force exerted on the plate is normal to the plate. By applying the balance of linear momentum in the normal direction (hence, the flow along the plate has zero component normal to the plate), we obtain

$$\mathbf{F}\cdot\hat{\mathbf{n}} = \oint_{\text{cs}} (\mathbf{v}\cdot\hat{\mathbf{n}})(\rho\,\mathbf{v}\cdot d\mathbf{s}) = (v\sin\theta)(-\rho\, vA),$$

or ($v_n = v\sin\theta$)

$$-F_n = \oint_{\text{cs}} v_n(\rho\,\mathbf{v}\cdot d\mathbf{s}) = (v\sin\theta)(-\rho\, vA) \;\to\; F_n = \rho\, Qv\sin\theta = \rho\, Av^2\sin\theta. \tag{4}$$

Example 5.3.2

When a free jet of fluid impinges on a smooth (frictionless) curved vane with a velocity v, the jet is deflected in a tangential direction as shown in Fig. 5.3.2, changing the momentum and exerting a force on the vane. Assuming that the velocity is uniform throughout the jet and there is no change in the pressure, determine the force exerted on a fixed vane.

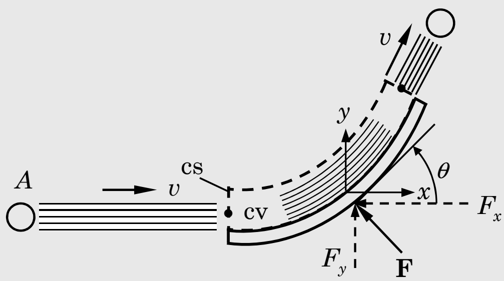

Fig. 5.3.2: Jet of fluid deflected by a curved vane.

Solution: For the steady-state condition, applying Eq. (5.3.3), we obtain

$$\mathbf{F} = \oint_{\mathrm{cs}} \mathbf{v}(\rho\,\mathbf{v}\cdot d\mathbf{s}),$$

$$-F_x\,\hat{\mathbf{e}}_x + F_y\,\hat{\mathbf{e}}_y = v\hat{\mathbf{e}}_x\,(-\rho\,vA) + v\,(\cos\theta\,\hat{\mathbf{e}}_x + \sin\theta\,\hat{\mathbf{e}}_y)\,(\rho\,vA), \tag{1}$$

or

$$F_x = \rho\,v^2 A\,(1-\cos\theta)\,,\quad F_y = \rho\,v^2 A\sin\theta. \tag{2}$$

When a jet of water ($\rho = 10^3$ kg/m^3) discharging 80 L/s at a velocity of 60 m/s is deflected through an angle of $\theta = 60°$, we obtain ($Q = vA$)

$$F_x = 10^3 \times 0.08 \times 60\,(1-\cos 60°) = 2.4\ \text{kN},$$
$$F_y = 10^3 \times 0.08 \times 60 \sin 60° = 4.157\ \text{kN}.$$

When the vane moves with a horizontal velocity of $v_0 < v$, Eq. (5.3.3) becomes

$$\mathbf{F} = \oint_{\mathrm{cs}} (\mathbf{v}-\mathbf{v}_0)[\rho(\mathbf{v}-\mathbf{v}_0)\cdot d\mathbf{s}],$$

$$-F_x\,\hat{\mathbf{e}}_x + F_y\,\hat{\mathbf{e}}_y = (v-v_0)\,[-\rho(v-v_0)A\,\hat{\mathbf{e}}_x] + (v-v_0)\,(\cos\theta\,\hat{\mathbf{e}}_x + \sin\theta\,\hat{\mathbf{e}}_y)\,\rho(v-v_0)A,$$

from which we obtain

$$F_x = \rho(v-v_0)^2 A\,(1-\cos\theta)\,,\quad F_y = \rho(v-v_0)^2 A\sin\theta. \tag{3}$$

Example 5.3.3

A chain of total length L and mass ρ per unit length slides down from the edge of a smooth table with an initial overhang x_0 to initiate motion, as shown in Fig. 5.3.3. Assuming that the chain is rigid, find the equation of motion governing the chain and the tension in the chain.

Solution: Let x be the amount of chain sliding down the table at any instant t. By considering the entire chain as the control volume, the linear momentum of the chain is

$$\rho(L-x)\cdot\dot{x}\,\hat{\mathbf{e}}_x - \rho x\cdot\dot{x}\,\hat{\mathbf{e}}_y.$$

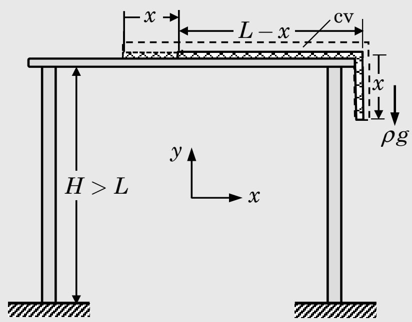

Fig. 5.3.3: Chain sliding down a table.

The resultant force in the chain is $-\rho g\, x\, \hat{\mathbf{e}}_y$. The principle of balance of linear momentum gives

$$-\rho\, gx\, \hat{\mathbf{e}}_y = \frac{d}{dt}\left[\rho(L-x)\,\dot{x}\,\hat{\mathbf{e}}_x - \rho\, x\, \dot{x}\,\hat{\mathbf{e}}_y\right], \tag{1}$$

or

$$0 = (L-x)\ddot{x} - \dot{x}^2, \quad -gx = -x\ddot{x} - \dot{x}^2.$$

Eliminating $\dot{x}^2$ from the two equations, we arrive at the equation of motion:

$$\ddot{x} - \frac{g}{L}x = 0. \tag{2}$$

The solution of this second-order differential equation is

$$x(t) = A\cosh\lambda t + B\sinh\lambda t, \quad \text{where} \quad \lambda = \sqrt{\frac{g}{L}}\,.$$

The constants of integration A and B are determined from the initial conditions

$$x(0) = x_0, \quad \dot{x}(0) = 0,$$

where x_0 denotes the initial overhang of the chain. We obtain

$$A = x_0, \quad B = 0,$$

and the solution becomes

$$x(t) = x_0\cosh\lambda t, \quad \lambda = \sqrt{\frac{g}{L}}\,. \tag{3}$$

The tension T in the chain can be computed by using the principle of balance of linear momentum applied to the control volume of the chain on the table as well as hanging

$$\begin{aligned} T &= \frac{\partial}{\partial t}\int_{\Omega}\mathbf{v}(\rho\, d\mathbf{x}) + \oint_{\Gamma}\mathbf{v}(\rho\,\mathbf{v}\cdot d\mathbf{s}) \\ &= \frac{d}{dt}\left[\rho(L-x)\dot{x}\right] + \rho\,\dot{x}\dot{x} \\ &= \rho(L-x)\ddot{x} = \frac{\rho\, g}{L}(L-x)x, \end{aligned} \tag{4}$$

where Eq. (2) is used in arriving at the last step.

Example 5.3.4

Consider a chain of length L and mass density ρ per unit length that is piled on a stationary table, as shown in Fig. 5.3.4. Determine the force F required to lift the chain with a constant velocity v.

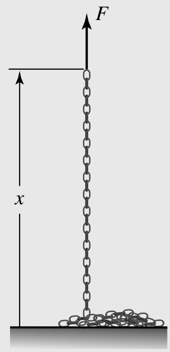

Fig. 5.3.4: Lifting of a chain piled on a table.

Solution: Let x be the height of the chain lifted off the table. Taking the control volume to be that enclosing the lifted chain and using Eq. (5.3.3) at a point, we obtain

$$\begin{aligned} F - \rho\, gx &= \frac{\partial}{\partial t}\int_\Omega \mathbf{v}(\rho\, d\mathbf{x}) + \oint_\Gamma \mathbf{v}(\rho\, \mathbf{v}\cdot d\mathbf{s}) \\ &= \frac{\partial}{\partial t}(\rho\, v) + \rho\, vv = 0 + \rho\, v^2, \end{aligned} \tag{1}$$

or

$$F = \rho\left(gx + v^2\right).$$

The same result can be obtained using Newton's second law of motion:

$$F - \rho\, gx = \frac{d}{dt}(mv) = m\dot{v} + \dot{m}v = 0 + \dot{m}v, \tag{2}$$

where the rate of increase of mass $m = \rho\, x$ is $\dot{m} = \rho\, \dot{x} = \rho\, v$.

5.3.1.1 Equations of motion in the spatial description

To derive the equation of motion applied to an arbitrarily fixed region in space through which material flows (i.e., control volume), we must identify the forces acting on it. Forces acting on a volume element can be classified as *internal* and *external*. The internal forces resist the tendency of one part of the region/body to be separated from another part. The internal force per unit area is termed stress, as defined in Eq. (4.2.1). The external forces are those transmitted by the body. The external forces can be further classified as *body* (*or volume*) *forces* and *surface forces*.

Body forces act on the distribution of mass inside the body. Examples of body forces are provided by the gravitational and electromagnetic forces. Body forces are usually measured per unit mass or unit volume of the body. Let $\mathbf{f}$ denote the body force per unit mass. Consider an elemental volume $d\mathbf{x}$ inside

Ω. The body force of the elemental volume is equal to $\rho\, d\mathbf{x}\, \mathbf{f}$. Hence, the total body force of the control volume is

$$\int_{\Omega} \rho\, \mathbf{f}\, d\mathbf{x}. \tag{5.3.4}$$

Surface forces are contact forces acting on the boundary surface of the body, and they are measured per unit area. If $\mathbf{t}$ is the surface force per unit area, the surface force on an elemental surface area ds is $\mathbf{t}\, ds$. The total surface force acting on the closed surface of the region Ω is

$$\oint_{\Gamma} \mathbf{t}\; ds. \tag{5.3.5}$$

The principle of balance of linear momentum applied to a given mass of a medium $\mathcal{B}$, instantaneously occupying a region Ω with bounding surface Γ, and acted upon by external surface force $\mathbf{t}$ per unit area and body force $\mathbf{f}$ per unit mass, can be expressed as

$$\frac{D}{Dt} \int_{\Omega} \rho\, \mathbf{v}\, d\mathbf{x} = \oint_{\Gamma} \mathbf{t}\, ds + \int_{\Omega} \rho\, \mathbf{f}\, d\mathbf{x}, \tag{5.3.6}$$

where $\rho\, \mathbf{v}\, d\mathbf{x}$ denotes the linear momentum associated with elemental volume $d\mathbf{x}$, $\mathbf{v}$ being the velocity vector.

Since the stress vector $\mathbf{t}$ on the surface is related to the (internal) stress tensor $\boldsymbol{\sigma}$ by Cauchy's formula $\mathbf{t} = \boldsymbol{\sigma} \cdot \hat{\mathbf{n}}$, [see Eq. (4.2.10)], where $\hat{\mathbf{n}}$ denotes the unit normal to the surface, we can express the surface force as

$$\oint_{\Gamma} \boldsymbol{\sigma} \cdot \hat{\mathbf{n}}\, ds = \oint_{\Gamma} \hat{\mathbf{n}} \cdot \boldsymbol{\sigma}^{\mathrm{T}}\, ds = \int_{\Omega} \boldsymbol{\nabla} \cdot \boldsymbol{\sigma}^{\mathrm{T}}\, d\mathbf{x}, \tag{5.3.7}$$

where the divergence theorem (5.2.17) is used to convert the surface integral into volume integral. Thus Eq. (5.3.6) takes the form

$$\frac{D}{Dt} \int_{\Omega} \rho\, \mathbf{v}\, d\mathbf{x} = \int_{\Omega} \left(\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}^{\mathrm{T}} + \rho\, \mathbf{f}\right) d\mathbf{x}. \tag{5.3.8}$$

Using the Reynolds transport theorem, Eq. (5.2.40), we arrive at

$$0 = \int_{\Omega} \left[\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}^{\mathrm{T}} + \rho\, \mathbf{f} - \rho\, \frac{D\mathbf{v}}{Dt}\right] d\mathbf{x}, \tag{5.3.9}$$

which is the global form of the equation of motion. The local form is given by

$$\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}^{\mathrm{T}} + \rho\, \mathbf{f} = \rho\, \frac{D\mathbf{v}}{Dt}\,; \quad \sigma_{ij,j} + \rho\, f_i = \rho\, \frac{Dv_i}{Dt}\,, \tag{5.3.10}$$

or

$$\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}^{\mathrm{T}} + \rho\, \mathbf{f} = \rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} \mathbf{v}\right). \tag{5.3.11}$$

Equation (5.3.11) is known as *Cauchy's equation of motion.* In Cartesian rectangular system, we have

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \rho\, f_i = \rho \left(\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} \right). \tag{5.3.12}$$

In steady–state conditions, Eq. (5.3.11) reduces to

$$\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}^{\mathrm{T}} + \rho\, \mathbf{f} = \rho\, \mathbf{v} \cdot \boldsymbol{\nabla} \mathbf{v}\,; \quad \frac{\partial \sigma_{ij}}{\partial x_j} + \rho\, f_i = \rho\, v_j \frac{\partial v_i}{\partial x_j}\,. \tag{5.3.13}$$

When the state of stress in the medium is of the form $\boldsymbol{\sigma} = -p\mathbf{I}$ (i.e., hydrostatic state of stress), the equation of motion (5.3.10) reduces to

$$-\boldsymbol{\nabla} p + \rho\, \mathbf{f} = \rho\, \frac{D\mathbf{v}}{Dt}\,. \tag{5.3.14}$$

5.3.1.2 Equations of motion in the material description

To derive the equation of motion applied to an arbitrarily fixed material of density ρ_0, occupying region Ω_0 in the reference configuration, we express Eqs. (5.3.4), (5.3.7), and (5.3.8) in terms of quantities referred to the reference configuration. We have ($d\mathbf{x} = J\, d\mathbf{X}$, and $\rho_0 = \rho\, J$)

$$\begin{aligned}
\int_{\Omega} \rho\, \mathbf{f}(\mathbf{x})\, d\mathbf{x} &= \int_{\Omega_0} \rho_0\, \mathbf{f}(\mathbf{X})\, d\mathbf{X}, \\
\oint_{\Gamma} \boldsymbol{\sigma} \cdot \hat{\mathbf{n}}\, ds &= \oint_{\Gamma_0} \mathbf{P} \cdot \hat{\mathbf{N}}\, dS = \int_{\Omega_0} \boldsymbol{\nabla}_0 \cdot \mathbf{P}^{\mathrm{T}}\, d\mathbf{X}, \\
\frac{D}{Dt} \int_{\Omega} \rho\, \mathbf{v}\, d\mathbf{x} &= \frac{\partial}{\partial t} \int_{\Omega_0} \rho_0\, \frac{\partial \mathbf{u}}{\partial t}\, d\mathbf{X} = \int_{\Omega_0} \rho_0\, \frac{\partial^2 \mathbf{u}}{\partial t^2}\, d\mathbf{X}.
\end{aligned} \tag{5.3.15}$$

where $\mathbf{P}$ is the first Piola–Kirchhoff stress tensor. In arriving at the above results we have made use of Eqs. (4.4.7) and (3.3.20):

$$\boldsymbol{\sigma} \cdot d\mathbf{a} = \mathbf{P} \cdot d\mathbf{A}, \quad dv = J\, dV \quad (\text{or } d\mathbf{x} = J\, d\mathbf{X}). \tag{5.3.16}$$

Then the principle of balance of linear momentum yields

$$\boldsymbol{\nabla}_0 \cdot \mathbf{P}^{\mathrm{T}} + \rho_0\, \mathbf{f} = \rho_0\, \frac{\partial^2 \mathbf{u}}{\partial t^2}. \tag{5.3.17}$$

Then using Eq. (4.4.11), namely, $\mathbf{S} = \mathbf{F}^{-1} \cdot \mathbf{P}$ or $\mathbf{P} = \mathbf{F} \cdot \mathbf{S}$, we can express the equation of motion in terms of the second Piola–Kirchhoff stress tensor $\mathbf{S}$

$$\boldsymbol{\nabla}_0 \cdot \left(\mathbf{S}^{\mathrm{T}} \cdot \mathbf{F}^{\mathrm{T}} \right) + \rho_0\, \mathbf{f} = \rho_0\, \frac{\partial^2 \mathbf{u}}{\partial t^2}. \tag{5.3.18}$$

Expressing the deformation tensor $\mathbf{F}$ in terms of the displacement vector $\mathbf{u}$ [see Eq. (3.3.8)], we obtain

$$\boldsymbol{\nabla}_0 \cdot \left[\mathbf{S}^{\mathrm{T}} \cdot (\mathbf{I} + \boldsymbol{\nabla}_0 \mathbf{u}) \right] + \rho_0\, \mathbf{f} = \rho_0\, \frac{\partial^2 \mathbf{u}}{\partial t^2}. \tag{5.3.19}$$

In rectangular Cartesian component form, we have

$$\frac{\partial}{\partial X_J}\left[S_{KJ}\left(\delta_{KI}+\frac{\partial u_I}{\partial X_K}\right)\right]+\rho_0\,\mathbf{f}_I=\rho_0\,\frac{\partial^2 u_I}{\partial t^2},\quad I=1,2,3, \tag{5.3.20}$$

where $v_I = (\partial u_I/\partial t)$. Clearly, the equations of motion expressed in terms of the second Piola–Kirchhoff stress tensor are nonlinear, because of the term $S_{KJ}(\partial u_I/\partial X_K)$, and this nonlinearity is in addition to the nonlinearity in the strain-displacement relations (see Chapter 3) and constitutive relations (to be discussed in Chapter 6).

For kinematically infinitesimal deformation, no distinction is made between $\mathbf{X}$ and $\mathbf{x}$ and between the second Piola–Kirchhoff stress tensor $\mathbf{S}$ and the Cauchy stress tensor $\boldsymbol{\sigma}$, that is, $\mathbf{X} \approx \mathbf{x}$ and $\mathbf{S} \approx \boldsymbol{\sigma}$. In this case, Eq. (5.3.20) reduces to

$$\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}^{\mathrm{T}}+\rho_0\,\mathbf{f}=\rho_0\frac{\partial^2\mathbf{u}}{\partial t^2}\,;\quad \frac{\partial\sigma_{ij}}{\partial x_j}+\rho_0\,f_i=\rho_0\frac{\partial^2 u_i}{\partial t^2}\,. \tag{5.3.21}$$

For bodies in static equilibrium, Eq. (5.3.21) reduces to [see Eq. (4.5.6)]

$$\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}^{\mathrm{T}}+\rho_0\,\mathbf{f}=\mathbf{0}\,;\quad \frac{\partial\sigma_{ij}}{\partial x_j}+\rho_0\,f_i=0. \tag{5.3.22}$$

Applications of the stress equilibrium equation, Eq. (5.3.22), for kinematically infinitesimal deformation were presented in Examples 4.5.1 and 4.5.2. Here we reconsider an example of application of Eq. (5.3.20).

Example 5.3.5

Given the following state of stress ($S_{IJ} = S_{JI}$),

$$\begin{aligned}&S_{11}=-2X_1^2,\quad S_{12}=-7+4X_1X_2+X_3,\quad S_{13}=1+X_1-3X_2,\\ &S_{22}=3X_1^2-2X_2^2+5X_3,\quad S_{23}=0,\quad S_{33}=-5+X_1+3X_2+3X_3,\end{aligned}$$

and displacement field,

$$u_1 = AX_2,\quad u_2 = BX_1,\quad u_3 = 0,$$

where A and B are arbitrary constants, determine the body force components for which the stress field describes a state of equilibrium.

Solution: Using Eq. (5.3.20), the body force components are

$$\rho_0\,f_I=-\frac{\partial S_{IJ}}{\partial X_J}-\frac{\partial}{\partial X_J}\left(S_{1J}\frac{\partial u_I}{\partial X_1}+S_{2J}\frac{\partial u_I}{\partial X_2}+S_{3J}\frac{\partial u_I}{\partial X_3}\right),\quad I=1,2,3.$$

We have

$$\begin{aligned}\rho_0\,f_1=&-\left(\frac{\partial S_{11}}{\partial X_1}+\frac{\partial S_{12}}{\partial X_2}+\frac{\partial S_{13}}{\partial X_3}\right)\\ &-\frac{\partial}{\partial X_1}\left(S_{11}\frac{\partial u_1}{\partial X_1}+S_{21}\frac{\partial u_1}{\partial X_2}+S_{31}\frac{\partial u_1}{\partial X_3}\right)\\ &-\frac{\partial}{\partial X_2}\left(S_{12}\frac{\partial u_1}{\partial X_1}+S_{22}\frac{\partial u_1}{\partial X_2}+S_{32}\frac{\partial u_1}{\partial X_3}\right)\\ &-\frac{\partial}{\partial X_3}\left(S_{13}\frac{\partial u_1}{\partial X_1}+S_{23}\frac{\partial u_1}{\partial X_2}+S_{33}\frac{\partial u_1}{\partial X_3}\right)\\ =&-[(-4X_1)+(4X_1)+0]-A[(4X_2)+(-4X_2)+0]=0,\end{aligned}$$

$$
\begin{aligned}
\rho_0 f_2 = &-\left(\frac{\partial S_{21}}{\partial X_1} + \frac{\partial S_{22}}{\partial X_2} + \frac{\partial S_{23}}{\partial X_3}\right) \\
&- \frac{\partial}{\partial X_1}\left(S_{11}\frac{\partial u_2}{\partial X_1} + S_{21}\frac{\partial u_2}{\partial X_2} + S_{31}\frac{\partial u_2}{\partial X_3}\right) \\
&- \frac{\partial}{\partial X_2}\left(S_{12}\frac{\partial u_2}{\partial X_1} + S_{22}\frac{\partial u_2}{\partial X_2} + S_{32}\frac{\partial u_2}{\partial X_3}\right) \\
&- \frac{\partial}{\partial X_3}\left(S_{13}\frac{\partial u_2}{\partial X_1} + S_{23}\frac{\partial u_2}{\partial X_2} + S_{33}\frac{\partial u_2}{\partial X_3}\right) \\
= &-[(4X_2) + (-4X_2) + 0] - B[(-4X_1) + (4X_1) + 0] = 0,
\end{aligned}
$$

$$
\begin{aligned}
\rho_0 f_3 = &-\left(\frac{\partial S_{31}}{\partial X_1} + \frac{\partial S_{32}}{\partial X_2} + \frac{\partial S_{33}}{\partial X_3}\right) \\
&- \frac{\partial}{\partial X_1}\left(S_{11}\frac{\partial u_3}{\partial X_1} + S_{21}\frac{\partial u_3}{\partial X_2} + S_{31}\frac{\partial u_3}{\partial X_3}\right) \\
&- \frac{\partial}{\partial X_2}\left(S_{12}\frac{\partial u_3}{\partial X_1} + S_{22}\frac{\partial u_3}{\partial X_2} + S_{32}\frac{\partial u_3}{\partial X_3}\right) \\
&- \frac{\partial}{\partial X_3}\left(S_{13}\frac{\partial u_3}{\partial X_1} + S_{23}\frac{\partial u_3}{\partial X_2} + S_{33}\frac{\partial u_3}{\partial X_3}\right) \\
= &-[1 + 0 + 3] + 0 = -4.
\end{aligned}
$$

Thus, the body is in equilibrium for the body force components $\rho_0 f_1 = 0$, $\rho_0 f_2 = 0$, and $\rho_0 f_3 = -4$.

5.3.2 Spatial Equations of Motion in Cylindrical and Spherical Coordinates

Here we express the equations of motion in the spatial description, Eq. (5.3.11), in terms of the components in the cylindrical and spherical coordinate systems (see Figure 5.3.5). The equations are also valid for kinematically infinitesimal deformations in the material description, with the density ρ replaced with ρ_0 (also, contributions from the term $\mathbf{v} \cdot \boldsymbol{\nabla}\mathbf{v}$ should be omitted).

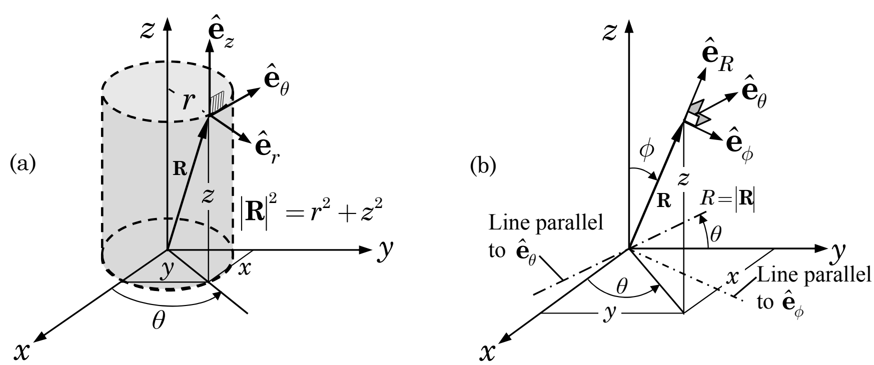

Fig. 5.3.5: (a) Cylindrical and (b) spherical coordinate systems.

5.3.2.1 Cylindrical coordinates

To express the equations of motion (5.3.11) in terms of the components in the cylindrical coordinate system, the operator $\boldsymbol{\nabla}$, velocity vector $\mathbf{v}$, body force vector $\mathbf{f}$, and stress tensor $\boldsymbol{\sigma}$ are written in the cylindrical coordinates (r, θ, z) as

$$\begin{aligned}
\boldsymbol{\nabla} &= \hat{\mathbf{e}}_r \frac{\partial}{\partial r} + \frac{1}{r}\hat{\mathbf{e}}_\theta \frac{\partial}{\partial \theta} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z},\\
\mathbf{v} &= \hat{\mathbf{e}}_r v_r + \hat{\mathbf{e}}_\theta v_\theta + \hat{\mathbf{e}}_z v_z,\\
\mathbf{f} &= \hat{\mathbf{e}}_r f_r + \hat{\mathbf{e}}_\theta f_\theta + \hat{\mathbf{e}}_z f_z,\\
\boldsymbol{\sigma} &= \sigma_{rr}\,\hat{\mathbf{e}}_r\hat{\mathbf{e}}_r + \sigma_{r\theta}\,\hat{\mathbf{e}}_r\hat{\mathbf{e}}_\theta + \sigma_{rz}\,\hat{\mathbf{e}}_r\hat{\mathbf{e}}_z\\
&\quad + \sigma_{\theta r}\,\hat{\mathbf{e}}_\theta\hat{\mathbf{e}}_r + \sigma_{\theta\theta}\,\hat{\mathbf{e}}_\theta\hat{\mathbf{e}}_\theta + \sigma_{\theta z}\,\hat{\mathbf{e}}_\theta\hat{\mathbf{e}}_z\\
&\quad + \sigma_{zr}\,\hat{\mathbf{e}}_z\hat{\mathbf{e}}_r + \sigma_{z\theta}\,\hat{\mathbf{e}}_z\hat{\mathbf{e}}_\theta + \sigma_{zz}\,\hat{\mathbf{e}}_z\hat{\mathbf{e}}_z.
\end{aligned} \tag{5.3.23}$$

Substituting these expressions into Eq. (5.3.11), we arrive at the following equations of motion in the cylindrical coordinate system (from the solutions of Problems **2.49** and **4.28**; see also Table 2.4.2 for the gradient of a vector):

$$\begin{aligned}
&\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) + \rho f_r\\
&\qquad = \rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z\frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r}\right),\\
&\frac{\partial \sigma_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{\sigma_{\theta r} + \sigma_{r\theta}}{r} + \rho f_\theta\\
&\qquad = \rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta v_r}{r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z\frac{\partial v_\theta}{\partial z}\right),\\
&\frac{\partial \sigma_{zr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{z\theta}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{zr}}{r} + \rho f_z\\
&\qquad = \rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right).
\end{aligned} \tag{5.3.24}$$

5.3.2.2 Spherical coordinates

In the spherical coordinate system (R, ϕ, θ), we write

$$\begin{aligned}
\boldsymbol{\nabla} &= \hat{\mathbf{e}}_R \frac{\partial}{\partial R} + \frac{1}{R}\hat{\mathbf{e}}_\phi \frac{\partial}{\partial \phi} + \frac{1}{R\sin\phi}\hat{\mathbf{e}}_\theta \frac{\partial}{\partial \theta},\\
\mathbf{v} &= \hat{\mathbf{e}}_R v_R + \hat{\mathbf{e}}_\phi v_\phi + \hat{\mathbf{e}}_\theta v_\theta,\\
\mathbf{f} &= \hat{\mathbf{e}}_R f_R + \hat{\mathbf{e}}_\phi f_\phi + \hat{\mathbf{e}}_\theta f_\theta,\\
\boldsymbol{\sigma} &= \sigma_{RR}\,\hat{\mathbf{e}}_R\hat{\mathbf{e}}_R + \sigma_{R\phi}\,\hat{\mathbf{e}}_R\hat{\mathbf{e}}_\phi + \sigma_{R\theta}\,\hat{\mathbf{e}}_R\hat{\mathbf{e}}_\theta\\
&\quad + \sigma_{\phi R}\,\hat{\mathbf{e}}_\phi\hat{\mathbf{e}}_R + \sigma_{\phi\phi}\,\hat{\mathbf{e}}_\phi\hat{\mathbf{e}}_\phi + \sigma_{\phi\theta}\,\hat{\mathbf{e}}_\phi\hat{\mathbf{e}}_\theta\\
&\quad + \sigma_{\theta R}\,\hat{\mathbf{e}}_\phi\hat{\mathbf{e}}_\theta + \sigma_{\theta\phi}\,\hat{\mathbf{e}}_\theta\hat{\mathbf{e}}_\phi + \sigma_{\theta\theta}\hat{\mathbf{e}}_\theta\hat{\mathbf{e}}_\theta.
\end{aligned} \tag{5.3.25}$$

Substituting these expressions into Eq. (5.3.11), we arrive at the following equations of motion in the spherical coordinate system (from the solution to Problem **2.51** and Table 2.4.2):

$$\begin{aligned}
&\frac{\partial \sigma_{RR}}{\partial R} + \frac{1}{R}\frac{\partial \sigma_{R\phi}}{\partial \phi} + \frac{1}{R\sin\phi}\frac{\partial \sigma_{R\theta}}{\partial \theta} + \frac{1}{R}\left(2\sigma_{RR} - \sigma_{\phi\phi} - \sigma_{\theta\theta} + \sigma_{R\phi}\cot\phi\right) + \rho f_R \\
&\quad = \rho\left(\frac{\partial v_R}{\partial t} + v_R\frac{\partial v_R}{\partial R} + \frac{v_\phi}{R}\frac{\partial v_R}{\partial \phi} + \frac{v_\theta}{R\sin\phi}\frac{\partial v_R}{\partial \theta} - \frac{v_\phi^2 + v_\theta^2}{R}\right), \\
&\frac{\partial \sigma_{\phi R}}{\partial R} + \frac{1}{R}\frac{\partial \sigma_{\phi\phi}}{\partial \phi} + \frac{1}{R\sin\phi}\frac{\partial \sigma_{\phi\theta}}{\partial \theta} + \frac{1}{R}\left[(\sigma_{\phi\phi} - \sigma_{\theta\theta})\cot\phi + \sigma_{R\phi} + 2\sigma_{\phi R}\right] + \rho f_\phi \\
&\quad = \rho\left[\frac{\partial v_\phi}{\partial t} + v_R\frac{\partial v_\phi}{\partial R} + \frac{v_\phi}{R}\left(\frac{\partial v_\phi}{\partial \phi} + v_R\right) + \frac{v_\theta}{R\sin\phi}\left(\frac{\partial v_\phi}{\partial \theta} - v_\theta\cos\phi\right)\right], \\
&\frac{\partial \sigma_{\theta R}}{\partial R} + \frac{1}{R}\frac{\partial \sigma_{\theta\phi}}{\partial \phi} + \frac{1}{R\sin\phi}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{1}{R}\left[(\sigma_{\phi\theta} + \sigma_{\theta\phi})\cot\phi + \sigma_{R\theta}\right] + \rho f_\theta \\
&\quad = \rho\left[\frac{\partial v_\theta}{\partial t} + v_R\frac{\partial v_\theta}{\partial R} + \frac{v_\phi}{R}\frac{\partial v_\theta}{\partial \phi} + \frac{v_\theta}{R\sin\phi}\left(\frac{\partial v_\theta}{\partial \theta} + v_\phi\cos\phi\right) + \frac{v_\theta v_R}{R}\right].
\end{aligned} \tag{5.3.26}$$

5.3.3 Principle of Balance of Angular Momentum

5.3.3.1 Monopolar case

This book is concerned primarily with *monopolar continuum mechanics*, where the topological features of the arrangement of matter at a micro scale, such as the distributed couples and couple stresses present at the molecular level, are overlooked. The monopolar continuum mechanics describes only macroscopic features of motion, which is sufficient in a vast majority of problems of mechanics.

The principle of balance of angular momentum for the monopolar case can be stated as follows: *The time rate of change of the total moment of momentum for a continuum is equal to the vector sum of the moments of external forces acting on the continuum.* The principle as applied to a control volume Ω with a control surface Γ can be expressed as

$$\text{moment of external forces} = \frac{\partial}{\partial t}\int_{\text{cv}} \rho\mathbf{r}\times\mathbf{v}\; d\mathbf{x} + \int_{\text{cs}} \rho\mathbf{r}\times\mathbf{v}\,(\mathbf{v}\cdot d\mathbf{s}), \tag{5.3.27}$$

where cv and cs denote the control volume and control surface, respectively. An application of the principle is presented in Example 5.3.6.

Example 5.3.6

Consider the top view of a sprinkler as shown in Fig. 5.3.6. The sprinkler discharges water outward in a horizontal plane (which is in the plane of the paper). The sprinkler exits are oriented at an angle of θ from the tangent line to the circle formed by rotating the sprinkler about its vertical centerline. The sprinkler has a constant cross-sectional flow area of A and discharges a flow rate of Q when $\omega = 0$ at time $t = 0$. Hence, the radial velocity is equal to $v_r = Q/2A$. Determine ω (counterclockwise) as a function of time.

Solution: Suppose that the moment of inertia of the empty sprinkler head is I_z and the resisting torque due to friction (from bearings and seals) is T (clockwise). The control volume is taken to be the cylinder of unit height (into the plane of the page) and radius R, formed by the rotating sprinkler head. The inflow, being along the axis, has no moment of momentum. Thus the time rate of change of the moment of momentum of the sprinkler head plus the net efflux of the moment of momentum from the control surface is equal to the torque T:

$$-T\hat{\mathbf{e}}_z = \left[2\frac{d}{dt}\int_0^R A\,\rho\,\omega r^2\, dr + I_z\frac{d\omega}{dt} + 2R\left(\rho\frac{Q}{2}\right)(\omega R - v_r\cos\theta)\right]\hat{\mathbf{e}}_z,$$

where the first term represents the time rate of change of the moment of momentum [moment arm times mass of a differential length dr times the velocity: $r \times (\rho A\, dr)(\omega r)$], the second term is the time rate of change of the angular momentum, and the last term represents the efflux of the moment of momentum at the control surface (i.e., exit of the sprinkler nozzles). Simplifying the equation, we arrive at

$$\left(I_z + \tfrac{2}{3}\rho A\, R^3\right)\frac{d\omega}{dt} + \rho Q R^2\omega = \rho Q R v_r\cos\theta - T.$$

The above equation indicates that for rotation to start $\rho Q R v_r\cos\theta - T > 0$. The final value of ω is obtained when the sprinkler motion reaches the steady state, that is, $d\omega/dt = 0$. Thus, at steady state, we have

$$\omega_f = \frac{v_r}{R}\cos\theta - \frac{T}{\rho Q R^2}\,.$$

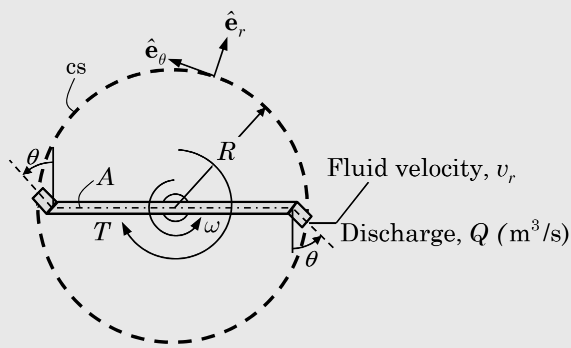

Fig. 5.3.6: A rotating sprinkler system.

The mathematical statement of the principle of balance of angular momentum as applied to a continuum is

$$\oint_\Gamma (\mathbf{x}\times\mathbf{t})ds + \int_\Omega (\mathbf{x}\times\rho\,\mathbf{f})d\mathbf{x} = \frac{D}{Dt}\int_\Omega (\mathbf{x}\times\rho\,\mathbf{v})\,d\mathbf{x}, \qquad (5.3.28)$$

where $\mathbf{t}$ denotes the stress vector and $\mathbf{f}$ denotes the body force vector (measured per unit mass). Equation (5.3.28) can be simplified with the help of the index notation in rectangular Cartesian coordinates (but the result will hold in any coordinate system). In index notation (kth component) Eq. (5.3.28) takes the form

$$\oint_\Gamma e_{ijk}\,x_i\,t_j\,ds + \int_\Omega (\rho\,e_{ijk}\,x_i\,f_j)\,d\mathbf{x} = \frac{D}{Dt}\int_\Omega \rho\,e_{ijk}\,x_i\,v_j\,d\mathbf{x}. \qquad (5.3.29)$$

We use several steps to simplify the expression. First replace t_j with $t_j = n_p\sigma_{pj}$ (Cauchy's formula). Then transform the surface integral to a volume integral and use the Reynolds transport theorem, Eq. (5.2.40), for the material time derivative of a volume integral to obtain

$$\int_\Omega e_{ijk} \left(x_i \, \sigma_{jp}\right)_{,p} \, d\mathbf{x} + \int_\Omega \left(\rho e_{ijk} \, x_i \, f_j\right) d\mathbf{x} = \int_\Omega \rho \, e_{ijk} \frac{D}{Dt} \left(x_i \, v_j\right) d\mathbf{x},$$

where $(\cdot)_{,p} = \partial(\cdot)/\partial x_p$. Carrying out the indicated differentiations and noting $Dx_i/Dt = v_i$, we obtain

$$\int_\Omega e_{ijk} \left(x_i \, \sigma_{jp,p} + \delta_{ip} \, \sigma_{jp} + \rho \, x_i \, f_j\right) d\mathbf{x} = \int_\Omega \rho \, e_{ijk} \left(v_i \, v_j + x_i \, \frac{Dv_j}{Dt}\right) d\mathbf{x},$$

or (noting that $e_{ijk} \, v_i \, v_j = 0$):

$$\int_\Omega \left\{ e_{ijk} \left[x_i \left(\sigma_{jp,p} + \rho \, f_j - \rho \frac{Dv_j}{Dt} \right) \right] + e_{ijk} \, \sigma_{ji} \right\} d\mathbf{x} = 0.$$

which, in view of the equations of motion (5.3.10), yields

$$e_{ijk} \, \sigma_{ji} = 0 \quad \text{or} \quad \boldsymbol{\mathcal{E}} : \boldsymbol{\sigma}^{\mathrm{T}} = \mathbf{0}. \tag{5.3.30}$$

Thus, in the absence of body couples, we have

$$\boldsymbol{\mathcal{E}} : \boldsymbol{\sigma}^{\mathrm{T}} = \mathbf{0} \ \Rightarrow \ \boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{T}} \quad (\text{or } \sigma_{ij} = \sigma_{ji}). \tag{5.3.31}$$

From Eq. (4.4.12) it follows that the second Piola–Kirchhoff stress tensor is also symmetric, $\mathbf{S} = \mathbf{S}^{\mathrm{T}}$, whenever $\boldsymbol{\sigma}$ is symmetric. Also, when $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{T}}$, it follows from Eq. (4.4.8) that

$$\mathbf{P} \cdot \mathbf{F}^{\mathrm{T}} = \mathbf{F} \cdot \mathbf{P}^{\mathrm{T}} \quad (\text{i.e., } \mathbf{P} \cdot \mathbf{F}^{\mathrm{T}} \text{ is symmetric}). \tag{5.3.32}$$

5.3.3.2 Multipolar case

In *multipolar continuum mechanics*, a molecule may be represented by a "deformable" particle, undergoing its own internal strains (so-called "microstrains"), which represent certain types of collective behavior of the sub-particles of a molecule[1]. A multipolar continuum also contains an angular momentum vector $\rho\mathbf{p}$ and a body couple vector $\rho\mathbf{c}$ (both measured per unit mass) inside the body, in the same way as the linear momentum vector $\rho\mathbf{v}$ and the body force vector $\rho\mathbf{f}$, and a couple traction vector $\mathbf{m}$ on the boundary, in the same way as the traction vector $\mathbf{t}$. Analogous to Cauchy's formula, the couple traction vector $\mathbf{m}$ is related to the *couple stress tensor* $\mathbf{M}$ according to

$$\mathbf{m} = \mathbf{M} \cdot \hat{\mathbf{n}} \quad (\text{or } m_i = M_{ij} \, n_j). \tag{5.3.33}$$

By the principle of balance of linear momentum, body and surface couples do not enter the calculation and therefore the equation of motion (5.3.10) is unaffected. However, for a multipolar case the principle leads to additional equations. For the sake of completeness, we present these additional equations, although their use is not illustrated in this book.

[1]See Jaunzemis (1967) and Eringen and Hanson (2002) for further discussion.

The principle of balance of angular momentum for the multipolar case can be stated as follows: *The time rate of change of the total moment of momentum for a continuum is equal to vector sum of couples and the moments of external forces acting on the continuum.* The principle as applied to a control volume Ω with a control surface Γ can be expressed as [see Eq. (5.3.28)]

$$\oint_\Gamma (\mathbf{x}\times\mathbf{t}+\mathbf{m})\,ds+\int_\Omega(\mathbf{x}\times\rho\mathbf{f}+\rho\,\mathbf{c})\,d\mathbf{x}=\frac{D}{Dt}\Big[\int_v(\mathbf{x}\times\rho\mathbf{v}+\rho\,\mathbf{p})\,d\mathbf{x}\Big]. \quad (5.3.34)$$

In index notation (kth component of), Eq. (5.3.34) takes the form

$$\oint_\Gamma (e_{ijk}\,x_i\,t_j+m_k)\;ds+\int_\Omega(\rho\,e_{ijk}\,x_i\,f_j+\rho\,c_k)\;d\mathbf{x} = \frac{D}{Dt}\int_\Omega(\rho\,e_{ijk}\,x_i\,v_j+\rho\,p_k)\;d\mathbf{x}. \quad (5.3.35)$$

Using Cauchy's formulas, $t_j = \sigma_{jp}n_p$ and $m_k = M_{kp}n_p$, and the divergence theorem, we arrive at the result

$$\int_\Omega\Big[e_{ijk}\,(x_i\,\sigma_{jp})_{,p}+M_{kp,p}+\rho\,e_{ijk}\,x_i\,f_j+\rho\,c_k\Big]\,d\mathbf{x} = \int_\Omega\rho\Big[e_{ijk}\,\frac{D}{Dt}\,(x_i\,v_j)+\frac{Dp_k}{Dt}\Big]d\mathbf{x}. \quad (5.3.36)$$

After simplification using $Dx_i/Dt = v_i$ and $e_{ijk}\,v_i\,v_j = 0$, we obtain

$$\int_\Omega\Big[e_{ijk}\,x_i\Big(\sigma_{jm,m}+\rho\,f_j-\rho\frac{Dv_j}{Dt}\Big)+M_{ki,i}+e_{ijk}\,\sigma_{ji}+\rho\,c_k-\rho\frac{Dp_k}{Dt}\Big]\,d\mathbf{x}=0,$$

or in vector form

$$\int_\Omega\Big[\mathbf{x}\times\Big(\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}^{\mathrm{T}}+\rho\,\mathbf{f}-\rho\frac{D\mathbf{v}}{Dt}\Big)+\boldsymbol{\nabla}\cdot\mathbf{M}^{\mathrm{T}}+\boldsymbol{\mathcal{E}}:\boldsymbol{\sigma}^{\mathrm{T}}+\rho\,\mathbf{c}-\rho\frac{D\mathbf{p}}{Dt}\Big]\,d\mathbf{x}=0, \quad (5.3.37)$$

where $\boldsymbol{\mathcal{E}}$ is the third-order permutation tensor. Using the equation of motion (5.3.10), we deduce the following local form, known as *Cosserat's equation*:

$$\boldsymbol{\nabla}\cdot\mathbf{M}^{\mathrm{T}}+\boldsymbol{\mathcal{E}}:\boldsymbol{\sigma}^{\mathrm{T}}+\rho\,\mathbf{c}=\rho\frac{D\mathbf{p}}{Dt}. \quad (5.3.38)$$

Thus, for a multipolar continuum, the stress tensor is no longer symmetric.

5.4 Thermodynamic Principles

5.4.1 Introduction

The first law of thermodynamics is commonly known as the principle of balance of energy, and it can be regarded as a statement of the interconvertibility of heat and work, while the total energy remains constant. The law does not place any restrictions on the direction of the process. For instance, in the study of mechanics of particles and rigid bodies, the kinetic energy and potential energy can be fully transformed from one to the other in the absence of friction and other dissipative mechanisms. From our experience, we know that mechanical energy that is converted into heat cannot all be converted back into mechanical energy. For example, the motion (kinetic energy) of a flywheel can all be converted into heat (internal energy) by means of a friction brake; if the whole system is insulated, the internal energy causes the temperature of the system to rise. Although the first law does not restrict the reversal process, namely the conversion of heat to internal energy and internal energy to motion (of the flywheel), such a reversal cannot occur because the frictional dissipation is an *irreversible process.* The second law of thermodynamics provides the restriction on the interconvertibility of energies.

5.4.2 Balance of Energy

The first law of thermodynamics states that *the time rate of change of the total energy is equal to the sum of the rate of work done by the external forces and the change of heat content per unit time.* The total energy is the sum of the kinetic energy and the internal energy. The principle of the balance of energy can be expressed as

$$\frac{D}{Dt}(K+U) = W + Q_h. \tag{5.4.1}$$

Here, K denotes the kinetic energy, U is the internal energy, W is the power input, and Q_h is the heat input to the system.

5.4.2.1 Energy equation in the spatial description

The kinetic energy of the system is given by

$$K = \tfrac{1}{2}\int_\Omega \rho\,\mathbf{v}\cdot\mathbf{v}\,d\mathbf{x}, \tag{5.4.2}$$

where $\mathbf{v}$ is the velocity vector. If e is the energy per unit mass (or *specific internal energy*), the total internal energy of the system is given by

$$U = \int_\Omega \rho\, e\, d\mathbf{x}. \tag{5.4.3}$$

The kinetic energy (K) is the energy associated with the macroscopically observable velocity of the continuum. The kinetic energy associated with the (microscopic) motions of molecules of the continuum is a part of the internal energy; the elastic strain energy and other forms of energy are also parts of the internal energy, U.

Here we consider only the nonpolar case, that is, body couples are zero and the stress tensor is symmetric. The power input consists of the rate of work done by external surface tractions $\mathbf{t}$ per unit area and body forces $\mathbf{f}$ per unit volume of the region Ω bounded by Γ:

$$W = \oint_\Gamma \mathbf{t}\cdot\mathbf{v}\,ds + \int_\Omega \rho\,\mathbf{f}\cdot\mathbf{v}\,d\mathbf{x}. \tag{5.4.4}$$

The rate of heat input consists of conduction through the boundary Γ and heat generation inside the region Ω (possibly from a radiation field or transmission of electric current). Let $\mathbf{q}$ be the heat flux vector and r_h be the internal heat generation per unit mass. Then the heat inflow across the surface element ds is $-\mathbf{q}\cdot\hat{\mathbf{n}}\,ds$, and internal heat generation in volume element $d\mathbf{x}$ is $\rho\, r_h d\mathbf{x}$. Hence, the total heat input is

$$Q_h = -\oint_\Gamma \hat{\mathbf{n}}\cdot\mathbf{q}\,ds + \int_\Omega \rho\, r_h\, d\mathbf{x}. \tag{5.4.5}$$

Substituting expressions for K, U, W, and Q_h from Eqs. (5.4.2)–(5.4.5) into Eq. (5.4.1), we obtain

$$\begin{aligned}\frac{D}{Dt}\int_\Omega \rho\left(\frac{1}{2}\mathbf{v}\cdot\mathbf{v}+e\right)d\mathbf{x} &= \oint_\Gamma \mathbf{t}\cdot\mathbf{v}\,ds + \int_\Omega \rho\mathbf{f}\cdot\mathbf{v}\,d\mathbf{x}\\ &\quad - \oint_\Gamma \hat{\mathbf{n}}\cdot\mathbf{q}\,ds + \int_\Omega \rho\, r_h\, d\mathbf{x}.\end{aligned} \tag{5.4.6}$$

Equation (5.4.6) can be simplified using a number of previously derived equations and identities, as explained next.

We begin with the expression for W (symmetry of $\boldsymbol{\sigma}$ allows us to write $\boldsymbol{\sigma}\cdot\hat{\mathbf{n}} = \hat{\mathbf{n}}\cdot\boldsymbol{\sigma}$)

$$\begin{aligned}W &= \oint_\Gamma \mathbf{t}\cdot\mathbf{v}\,ds + \int_\Omega \rho\,\mathbf{f}\cdot\mathbf{v}\,d\mathbf{x} = \oint_\Gamma (\hat{\mathbf{n}}\cdot\boldsymbol{\sigma})\cdot\mathbf{v}\,ds + \int_\Omega \rho\mathbf{f}\cdot\mathbf{v}\,d\mathbf{x}\\ &= \int_\Omega \left[\boldsymbol{\nabla}\cdot(\boldsymbol{\sigma}\cdot\mathbf{v}) + \rho\,\mathbf{f}\cdot\mathbf{v}\right]d\mathbf{x} = \int_\Omega \left[(\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}+\rho\,\mathbf{f})\cdot\mathbf{v} + \boldsymbol{\sigma}:\boldsymbol{\nabla}\mathbf{v}\right]d\mathbf{x}\\ &= \int_\Omega \left(\rho\frac{D\mathbf{v}}{Dt}\cdot\mathbf{v} + \boldsymbol{\sigma}:\boldsymbol{\nabla}\mathbf{v}\right)d\mathbf{x},\end{aligned} \tag{5.4.7}$$

where : denotes the "double-dot product" $\mathbf{S}:\mathbf{T} = S_{ij}T_{ij}$ [see Eq. (2.5.13)]. The Cauchy formula, vector identity in Eq. (5.2.7), and the equation of motion (5.3.10) are used in arriving at the last step. We note that [see Eq. (5.2.8)]

$$\boldsymbol{\sigma}:\boldsymbol{\nabla}\mathbf{v} = \boldsymbol{\sigma}^{\text{sym}}:\mathbf{D} - \boldsymbol{\sigma}^{\text{skew}}:\mathbf{W},$$

where $\mathbf{D}$ is the symmetric part, called the rate of deformation tensor, and $\mathbf{W}$ is the skew symmetric part, called the vorticity (or spin) tensor, of $(\boldsymbol{\nabla}\mathbf{v})^{\mathrm{T}}$ [see Eq. (3.6.2)],

$$\mathbf{D} = \tfrac{1}{2}\left[\boldsymbol{\nabla}\mathbf{v} + (\boldsymbol{\nabla}\mathbf{v})^{\mathrm{T}}\right], \quad \mathbf{W} = \tfrac{1}{2}\left[(\boldsymbol{\nabla}\mathbf{v})^{\mathrm{T}} - \boldsymbol{\nabla}\mathbf{v}\right], \tag{5.4.8}$$

and $\boldsymbol{\sigma}^{\text{sym}}$ and $\boldsymbol{\sigma}^{\text{skew}}$ are the symmetric and skew symmetric parts of $\boldsymbol{\sigma}$. When $\boldsymbol{\sigma}$ is symmetric, we have $\boldsymbol{\sigma}^{\text{sym}} = \boldsymbol{\sigma}$ and $\boldsymbol{\sigma}^{\text{skew}} = \mathbf{0}$. Hence, Eq. (5.4.7) becomes

$$W = \tfrac{1}{2}\int_\Omega \rho\frac{D}{Dt}(\mathbf{v}\cdot\mathbf{v})\,d\mathbf{x} + \int_\Omega \boldsymbol{\sigma}:\mathbf{D}\,d\mathbf{x} = \tfrac{1}{2}\frac{D}{Dt}\int_\Omega \rho\,\mathbf{v}\cdot\mathbf{v}\,d\mathbf{x} + \int_\Omega \boldsymbol{\sigma}:\mathbf{D}\,d\mathbf{x},$$

where the Reynolds transport theorem, Eq. (5.2.40), is used to write the final expression. Next, Q_h can be expressed as

$$Q_h = -\oint_\Gamma \hat{\mathbf{n}}\cdot\mathbf{q}\,ds + \int_\Omega \rho\, r_h\, d\mathbf{x} = \int_\Omega \left(-\boldsymbol{\nabla}\cdot\mathbf{q} + \rho\, r_h\right)d\mathbf{x}. \tag{5.4.9}$$

With the new expressions for W and Q_h, Eq. (5.4.6) can be written as

$$0 = \int_\Omega \left(\rho \frac{De}{Dt} - \boldsymbol{\sigma} : \mathbf{D} + \boldsymbol{\nabla} \cdot \mathbf{q} - \rho\, r_h \right) d\mathbf{x}, \tag{5.4.10}$$

which is the global form of the energy equation. The local form of the energy equation is given by

$$\rho \frac{De}{Dt} = \boldsymbol{\sigma} : \mathbf{D} - \boldsymbol{\nabla} \cdot \mathbf{q} + \rho\, r_h, \tag{5.4.11}$$

which is known as the *thermodynamic form* of the energy equation for a continuum. The term $\boldsymbol{\sigma} : \mathbf{D}$ is known as the *stress power*, which can be regarded as the internal work production.

5.4.2.2 Energy equation in the material description

To derive the energy equation in the material description, we write K, U, W, and Q_h in the material description:

$$K = \tfrac{1}{2} \int_\Omega \rho \mathbf{v} \cdot \mathbf{v}\, d\mathbf{x} = \tfrac{1}{2} \int_{\Omega_0} \rho_0\, \mathbf{v} \cdot \mathbf{v}\, d\mathbf{X}, \tag{5.4.12}$$

$$U = \int_\Omega \rho\, e\, d\mathbf{x} = \int_{\Omega_0} \rho_0\, e\, d\mathbf{X}, \tag{5.4.13}$$

$$\begin{aligned} W &= \oint_\Gamma \mathbf{t} \cdot \mathbf{v}\, ds + \int_\Omega \rho\, \mathbf{f} \cdot \mathbf{v}\, d\mathbf{x} = \oint_{\Gamma_0} \mathbf{T} \cdot \mathbf{v}\, dS + \int_{\Omega_0} \rho_0\, \mathbf{f} \cdot \mathbf{v}\, d\mathbf{X} \\ &= \oint_{\Gamma_0} (\hat{\mathbf{N}} \cdot \mathbf{P}^{\mathrm{T}}) \cdot \mathbf{v}\, dS + \int_{\Omega_0} \rho_0\, \mathbf{f} \cdot \mathbf{v}\, d\mathbf{X} = \int_{\Omega_0} \big[\boldsymbol{\nabla}_0 \cdot (\mathbf{P}^{\mathrm{T}} \cdot \mathbf{v}) + \rho_0\, \mathbf{f} \cdot \mathbf{v} \big]\, d\mathbf{X} \\ &= \int_{\Omega_0} \big[\big(\boldsymbol{\nabla}_0 \cdot \mathbf{P}^{\mathrm{T}} + \rho_0\, \mathbf{f} \big) \cdot \mathbf{v} + \mathbf{P}^{\mathrm{T}} : \boldsymbol{\nabla}_0 \mathbf{v} \big]\, d\mathbf{X} \\ &= \int_\Omega \left[\tfrac{1}{2} \rho_0 \frac{\partial}{\partial t} (\mathbf{v} \cdot \mathbf{v}) + \mathbf{P}^{\mathrm{T}} : \boldsymbol{\nabla}_0 \mathbf{v} \right] d\mathbf{X}, \end{aligned} \tag{5.4.14}$$

$$\begin{aligned} Q_h &= - \oint_\Gamma \hat{\mathbf{n}} \cdot \mathbf{q}\, ds + \int_\Omega \rho\, r_h\, d\mathbf{x} = - \oint_{\Gamma_0} \hat{\mathbf{N}} \cdot \mathbf{q}_0\, dS + \int_{\Omega_0} \rho_0\, r_h\, d\mathbf{X} \\ &= \int_{\Omega_0} [-\boldsymbol{\nabla}_0 \cdot \mathbf{q}_0 + \rho_0\, r_h]\, d\mathbf{X}, \end{aligned} \tag{5.4.15}$$

where Eqs. (5.3.15) and (5.3.17), and the following relations are used to write the final expressions for K, W, and Q_h:

$$\hat{\mathbf{n}}\, ds = J\, \mathbf{F}^{-\mathrm{T}} \cdot \hat{\mathbf{N}}\, dS = J\, \hat{\mathbf{N}} \cdot \mathbf{F}^{-1}\, dS, \quad \rho\, d\mathbf{x} = \rho_0\, d\mathbf{X}, \tag{5.4.16}$$

and all variables are now function of $\mathbf{X}$, the material coordinates.

Substitution of expressions from Eqs. (5.4.12)–(5.4.15) into Eq. (5.4.1), we obtain the following local form of the energy equation in the material description:

$$\rho_0 \frac{\partial e}{\partial t} = \mathbf{P}^{\mathrm{T}} : \boldsymbol{\nabla}_0 \mathbf{v} - \boldsymbol{\nabla}_0 \cdot \mathbf{q}_0 + \rho_0\, r_h. \tag{5.4.17}$$

In terms of the second Piola–Krichhoff stress tensor, we have

$$\rho_0 \frac{\partial e}{\partial t} = (\mathbf{S} \cdot \mathbf{F}^{\mathrm{T}}) : \boldsymbol{\nabla}_0 \mathbf{v} - \boldsymbol{\nabla}_0 \cdot \mathbf{q}_0 + \rho_0\, r_h. \tag{5.4.18}$$

5.4.3 Entropy Inequality

The concept of entropy is a difficult one to explain in simple terms; it has its roots in statistical physics and thermodynamics and is generally considered as a measure of the tendency of the atoms toward a disorder. For example, carbon has a lower entropy in the form of diamond, a hard crystal with atoms closely bound in a highly ordered array.

Temperature cannot be decreased below a certain absolute minimum. We introduce θ as the absolute temperature whose greatest lower bound is zero. We also recall that in an admissible deformation, the deformation gradient tensor $\mathbf{F}$ should be nonsingular, that is, $J = |\mathbf{F}| \neq 0$. Thus each thermodynamic process should satisfy the conditions

$$\theta \geq 0, \quad |\mathbf{F}| \neq 0.$$

5.4.3.1 Homogeneous processes

Let us denote total entropy by the symbol H, and define *internal dissipation*, $\mathcal{D}$, as

$$\mathcal{D} = \theta \, \dot{H} - Q_h, \tag{5.4.19}$$

where Q_h denotes the rate of heat supply to the body; $\theta \dot{H}$ is interpreted as the time rate of change of the heat content of the body. The ratio of $\mathcal{D}$ to θ is called the *internal entropy production*,

$$\Gamma \equiv \frac{\mathcal{D}}{\theta} = \dot{H} - \frac{Q_h}{\theta} \, . \tag{5.4.20}$$

The second law of thermodynamics states that the internal entropy production is always positive, which is known as the *Clausius–Duhem inequality*, and it is expressed, for homogeneous processes, as

$$\mathcal{D} = \theta \, \dot{H} - Q_h \geq 0. \tag{5.4.21}$$

If $\mathcal{D} = 0$, then the process is said to be *reversible*, and we have $\dot{H} = Q_h/\theta$; otherwise, the process is said to be *irreversible*. The processes in which $Q_h = 0$, hence $\dot{H} \geq 0$, are said to be *adiabatic*. Processes in which $\dot{H} = 0$ (i.e., $Q_h \leq 0$) are called *isentropic*. The second law of thermodynamics essentially states that the time rate of change of the heat content $\theta \, \dot{H}$ of a body can never be less than the rate of heat supply Q_h.

5.4.3.2 Nonhomogeneous processes

To derive the Clausius–Duhem inequality for nonhomogeneous processes (that is, processes that depend not only on time but also on position), let us introduce the entropy density per unit mass, η, so that

$$H = \int_{\Omega} \rho \eta \, d\mathbf{x}. \tag{5.4.22}$$

We define the entropy production as [in the same form as Eq. (5.4.20) for the homogeneous case]

$$\Gamma = \frac{D}{Dt}\int_\Omega \rho\eta\, d\mathbf{x} - \left[-\oint_\Gamma \frac{1}{\theta}\mathbf{q}\cdot\hat{\mathbf{n}}\, ds + \int_\Omega \frac{\rho r_h}{\theta}\, d\mathbf{x}\right]$$
$$= \int_\Omega \rho\frac{D\eta}{Dt}\, d\mathbf{x} + \int_\Omega \left[\boldsymbol{\nabla}\cdot\left(\frac{\mathbf{q}}{\theta}\right) - \frac{\rho\, r_h}{\theta}\right] d\mathbf{x}. \tag{5.4.23}$$

Then, the second law of thermodynamics requires $\Gamma \geq 0$, giving

$$\int_\Omega \rho\frac{D\eta}{Dt}\, d\mathbf{x} \geq \int_\Omega \left[\left(\frac{\rho r_h}{\theta}\right) - \boldsymbol{\nabla}\cdot\left(\frac{\mathbf{q}}{\theta}\right)\right] d\mathbf{x}. \tag{5.4.24}$$

The local form of the Clausius–Duhem inequality, or *entropy inequality* is

$$\frac{D\eta}{Dt} \geq \frac{r_h}{\theta} - \frac{1}{\rho}\boldsymbol{\nabla}\cdot\left(\frac{\mathbf{q}}{\theta}\right) \quad \text{or} \quad \rho\theta\frac{D\eta}{Dt} - \rho\, r_h + \boldsymbol{\nabla}\cdot\mathbf{q} - \frac{1}{\theta}\mathbf{q}\cdot\boldsymbol{\nabla}\theta \geq 0. \tag{5.4.25}$$

The quantity $\mathbf{q}/\theta$ is known as the *entropy flux* and r_h/θ is the *entropy supply density.*

The sum of internal energy (e) and irreversible heat energy ($-\theta\eta$) is known as the *Helmhotz free energy density*:

$$\Psi = e - \theta\eta. \tag{5.4.26}$$

Substituting Eq. (5.4.26) into Eq. (5.4.11), we obtain

$$\rho\,\frac{D\Psi}{Dt} = \boldsymbol{\sigma} : \mathbf{D} - \rho\,\frac{D\theta}{Dt}\eta - \mathcal{D}, \tag{5.4.27}$$

where $\mathbf{D}$ is the symmetric part of the velocity gradient tensor [see Eq. (5.4.8)], and $\mathcal{D}$ is the *internal dissipation*

$$\mathcal{D} = \rho\,\theta\frac{D\eta}{Dt} + \boldsymbol{\nabla}\cdot\mathbf{q} - \rho\, r_h. \tag{5.4.28}$$

In view of Eq. (5.4.25) we can write

$$\mathcal{D} - \frac{1}{\theta}\mathbf{q}\cdot\boldsymbol{\nabla}\theta \geq 0. \tag{5.4.29}$$

We have $\mathcal{D} > 0$ for an irreversible process, and $\mathcal{D} = 0$ for a reversible process. Expressing the entropy inequality (5.4.25) in terms of the Helmholtz free energy density, we obtain

$$\boldsymbol{\sigma} : \mathbf{L} - \rho\dot{\theta}\eta - \rho\dot{\Psi} - \frac{1}{\theta}\mathbf{q}\cdot\boldsymbol{\nabla}\theta \geq 0, \tag{5.4.30}$$

where the superposed dot denotes the material time derivative, and $\mathbf{L}$ is the velocity gradient tensor, $\mathbf{L} = (\boldsymbol{\nabla}\mathbf{v})^{\mathrm{T}} = \mathbf{D} + \mathbf{W}$, $\mathbf{W}$ being the skew symmetric spin tensor in Eq. (5.4.8). Note that when $\boldsymbol{\sigma}$ is symmetric, we have $\boldsymbol{\sigma} : \mathbf{L} = \boldsymbol{\sigma} : \mathbf{D}$.

5.5 Summary

5.5.1 Preliminary Comments

This chapter was devoted to the derivation of the field equations governing a continuous medium using the principle of conservation of mass and balance of momenta and energy, and therefore constitutes the heart of the book. The equations are derived in invariant (i.e., vector and tensor) form so that they can be expressed in any chosen coordinate system (e.g., rectangular, cylindrical, spherical, or curvilinear system). The principle of conservation of mass results in the continuity equation; the principle of balance of linear momentum, which is equivalent to Newton's second law of motion, leads to the equations of motion in terms of the Cauchy stress tensor; the principle of balance of angular momentum yields, in the absence of body couples, the symmetry of the Cauchy stress tensor; and the principles of thermodynamics – the first and second laws of thermodynamics – give rise to the energy equation and Clausius–Duhem inequality.

In closing this chapter, we summarize the invariant form of the equations resulting from the application of conservation principles to a continuum. The variables appearing in the equations were already defined and are not repeated here. In this study, we will be concerned only with monopolar media, where no body couples and body moments are accounted for. This amounts to assuming the symmetry of the stress tensors:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{T}}, \quad \mathbf{S} = \mathbf{S}^{\mathrm{T}}, \quad \mathbf{P} \cdot \mathbf{F}^{\mathrm{T}} = \mathbf{F} \cdot \mathbf{P}^{\mathrm{T}}.$$

5.5.2 Conservation and Balance Equations in the Spatial Description

Conservation of mass

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{v}) = 0 \tag{5.5.1}$$

Balance of linear momentum

$$\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \rho \, \mathbf{f} = \rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} \mathbf{v} \right) \tag{5.5.2}$$

Balance of angular momentum

$$\boldsymbol{\sigma}^{\mathrm{T}} = \boldsymbol{\sigma} \tag{5.5.3}$$

Balance of energy

$$\rho \frac{De}{Dt} = \boldsymbol{\sigma} : \mathbf{D} - \boldsymbol{\nabla} \cdot \mathbf{q} + \rho \, r_h \tag{5.5.4}$$

Entropy inequality

$$\rho \theta \frac{D\eta}{Dt} - \rho \, r_h + \boldsymbol{\nabla} \cdot \mathbf{q} - \frac{1}{\theta} \mathbf{q} \cdot \boldsymbol{\nabla} \theta \geq 0 \tag{5.5.5}$$

5.5.3 Conservation and Balance Equations in the Material Description

Conservation of mass

$$\rho_0 = \rho\, J \tag{5.5.6}$$

Balance of linear momentum

$$\boldsymbol{\nabla}_0 \cdot (\mathbf{S} \cdot \mathbf{F}^{\mathrm{T}}) + \rho_0\, \mathbf{f} = \rho_0\, \frac{\partial^2 \mathbf{u}}{\partial t^2} \tag{5.5.7}$$

Balance of angular momentum

$$\mathbf{S}^{\mathrm{T}} = \mathbf{S} \tag{5.5.8}$$

Balance of energy

$$\rho_0 \frac{\partial e}{\partial t} = (\mathbf{S} \cdot \mathbf{F}^{\mathrm{T}}) : \boldsymbol{\nabla}_0 \mathbf{v} - \boldsymbol{\nabla}_0 \cdot \mathbf{q}_0 + \rho_0\, r_0 \tag{5.5.9}$$

Entropy inequality

$$\rho_0 \theta \frac{D\eta}{Dt} - \rho_0\, r_0 + \boldsymbol{\nabla}_0 \cdot \mathbf{q}_0 - \frac{1}{\theta} \mathbf{q}_0 \cdot \boldsymbol{\nabla}_0 \theta \geq 0 \tag{5.5.10}$$

We shall return to these equations in the subsequent chapters as needed. These equations may be supplemented by other field equations, such as Maxwell's equations governing electromagnetics, depending on the field of study.

5.5.4 Closing Comments

The subject of continuum mechanics is concerned primarily with the determination of the behavior (e.g., $\mathbf{F}$, θ, $\boldsymbol{\nabla}\theta = \mathbf{g}$, etc.) of a body under externally applied causes (e.g., $\mathbf{f}$, r_h, and so on). After introducing suitable constitutive relations for $\boldsymbol{\sigma}$, e, η, and $\mathbf{q}$ (to be discussed in Chapter 6), this task involves solving the initial-boundary-value problem described by partial differential equations (5.5.1)–(5.5.4) under specified initial and boundary conditions. The role of the entropy inequality in formulating the problem is to make sure that the behavior of a body is consistent with the inequality (5.5.5). Often, the constitutive relations developed are required to be consistent with the second law of thermodynamics (i.e., satisfy the entropy inequality). The *entropy principle* states that constitutive relations be such that the entropy inequality is satisfied identically for any thermodynamic process.

An examination of the conservation principles presented in this chapter shows that all of the mathematical statements resulting from the principles share a common mathematical structure. These all can be expressed in terms of a general balance (or conservation) equation in the spatial description as

$$\frac{D}{Dt} \int_\Omega \phi(\mathbf{x}, t)\, d\mathbf{x} = \oint_\Gamma \psi(\mathbf{x}, t, \hat{\mathbf{n}})\, ds + \int_\Omega f(\mathbf{x}, t)\, d\mathbf{x}, \tag{5.5.11}$$

where ϕ is a tensor field of order n measured per unit volume; $\psi(\mathbf{x}, t, \hat{\mathbf{n}})$ is a surface tensor of order $n-1$, measured per unit current area and depends on the surface orientation $\hat{\mathbf{n}}$; and $f(\mathbf{x}, t)$ is a source tensor of order n, also measured per unit volume ($n = 0, 1$). The variables ϕ, ψ, and f associated with the balance equations resulting from the principles of conservation of mass, balance of linear and angular momentum, and the first and second laws of thermodynamics are presented in Table 5.5.1.

Table 5.5.1: Expressions for variables ϕ, ψ, and f in Eq. (5.5.11) for the four conservation principles.

No.	$\phi(\mathbf{x}, t)$	$\psi(\mathbf{x}, t, \hat{\mathbf{n}})$	$f(\mathbf{x}, t)$	number
1.	ρ	0	0	(5.2.22)
2.	$\rho\,\mathbf{v}$	$\mathbf{t}$	$\rho\,\mathbf{f}$	(5.3.6)
3.	$\mathbf{x} \times \rho\,\mathbf{v}$	$\mathbf{x} \times \mathbf{t}$	$\rho\,\mathbf{x} \times \mathbf{f}$	(5.3.28)
4.	$\rho\,(v^2/2 + e)$	$\mathbf{t} \cdot \mathbf{v} - \hat{\mathbf{n}} \cdot \mathbf{q}$	$\rho\,\mathbf{f} \cdot \mathbf{v} + \rho\,r$	(5.4.6)
5.	$\rho\,\eta$	$-\frac{\mathbf{q}\cdot\hat{\mathbf{n}}}{\theta}$	$\frac{\rho\,r}{\theta}$	(5.4.23)

To complete the mathematical description of the behavior of a continuous medium, the conservation equations derived in this chapter must be supplemented with the constitutive equations that relate $\boldsymbol{\sigma}$, e, η, and $\mathbf{q}$ to $\mathbf{F}$, θ, and $\mathbf{g} \equiv \boldsymbol{\nabla}\theta$. The strain (or strain rate) measures ($\mathbf{e}$, $\mathbf{D}$, $\mathbf{E}$, $\mathbf{F}$, $\mathbf{C}$) introduced in Chapter 3 and the stress measures ($\boldsymbol{\sigma}$, $\mathbf{P}$, $\mathbf{S}$) introduced in Chapter 4 are objective and, therefore, they are suitable candidates for the description of material response, which should be independent of the observer. Chapter 6 is dedicated to the discussion of the material constitutive relations. Applications of the governing equations to linearized elasticity problems and fluid mechanics and heat transfer problems are discussed in Chapters 7 and 8, respectively.

Problems

Conservation of Mass

5.1 The acceleration of a material element in a continuum is described by

$$\frac{D\mathbf{v}}{Dt} \equiv \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla}\mathbf{v}, \tag{1}$$

where $\mathbf{v}$ is the velocity vector. Show by means of vector identities that the acceleration can also be written as

$$\frac{D\mathbf{v}}{Dt} \equiv \frac{\partial \mathbf{v}}{\partial t} + \boldsymbol{\nabla}\left(\frac{v^2}{2}\right) - \mathbf{v} \times \boldsymbol{\nabla} \times \mathbf{v}, \quad v^2 = \mathbf{v} \cdot \mathbf{v}. \tag{2}$$

5.2 Show that the local form of the principle of conservation of mass, Eq. (5.2.22), can be expressed as

$$\frac{D}{Dt}(\rho J) = 0.$$

5.3 Use the equation

$$\frac{D}{Dt}(\rho J) = 0,$$

to derive the continuity equation

$$\frac{D\rho}{Dt} + \rho\,\boldsymbol{\nabla} \cdot \mathbf{v} = 0.$$

5.4 Derive the continuity equation in the cylindrical coordinate system by considering a differential volume element shown in Fig. P5.4.

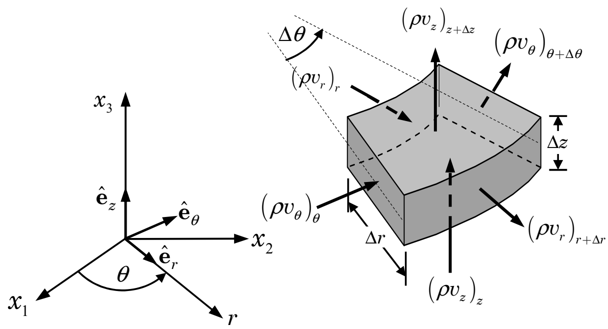

Fig. P5.4

5.5 Express the continuity equation (5.2.24) in the cylindrical coordinate system (see Table 2.4.2 for various operators). The result should match the one in Eq. (5.2.30).

5.6 Express the continuity equation (5.2.24) in the spherical coordinate system (see Table 2.4.2 for various operators). The result should match the one in Eq. (5.2.31).

5.7 Determine if the following velocity fields for an incompressible flow satisfy the continuity equation:

$$\text{(a)} \quad v_1(x_1,x_2) = -\frac{x_1}{r^2}\,, \quad v_2(x_1,x_2) = -\frac{x_2}{r^2} \quad \text{where} \quad r^2 = x_1^2 + x_2^2.$$

$$\text{(b)} \quad v_r = 0,\ v_\theta = 0,\ v_z = c\left(1 - \frac{r^2}{R^2}\right)$$

where c and R are constants.

5.8 The velocity distribution between two parallel plates separated by distance b is

$$v_x(y) = \frac{y}{b}v_0 - c\frac{y}{b}\left(1 - \frac{y}{b}\right), \quad v_y = 0, \quad v_z = 0, \quad 0 < y < b,$$

where y is measured from and normal to the bottom plate, x is taken along the plates, v_x is the velocity component parallel to the plates, v_0 is the velocity of the top plate in the x direction, and c is a constant. Determine if the velocity field satisfies the continuity equation and find the volume rate of flow and the average velocity.

Balance of Linear Momentum

5.9 Calculate the force exerted by a water ($\rho = 10^3$ kg/m^3) jet of diameter $d = 8$ mm and velocity $v = 12$ m/s that impinges against a smooth inclined flat plate at an angle of 45° to the axis of the jet, as shown in Fig. P5.9.

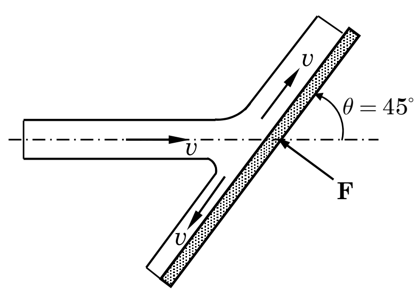

Fig. P5.9

5.10 Calculate the force exerted by a water ($\rho = 10^3$ kg/m^3) jet of diameter $d = 60$ mm and velocity $v = 6$ m/s that impinges against a smooth inclined flat plate at an angle of 60° to the axis of the jet. Also calculate the volume flow rates Q_L and Q_R.

5.11 A jet of air ($\rho = 1.206$ kg/m^3) impinges on a smooth vane with a velocity $v = 50$ m/s at the rate of $Q = 0.4$ m^3/s. Determine the force required to hold the plate in position for the two different vane configurations shown in Fig. P5.11. Assume that the vane splits the jet into two equal streams, and neglect any energy loss in the streams.

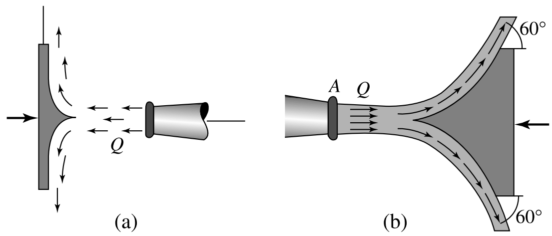

Fig. P5.11

5.12 In Example 5.3.3, determine (a) the velocity and accelerations as functions of x, and (b) the velocity as the chain leaves the table.

5.13 Using the definition of $\boldsymbol{\nabla}$, vector forms of the velocity vector, body force vector, and the dyadic form of $\boldsymbol{\sigma}$ [see Eq. (5.3.23)], express the equation of motion (5.3.11) in the cylindrical coordinate system as given in Eq. (5.3.24).

5.14 Using the definition of $\boldsymbol{\nabla}$, vector forms of the velocity vector, body force vector, and the dyadic form of $\boldsymbol{\sigma}$ [see Eq. (5.3.25)], express the equation of motion (5.3.11) in the spherical coordinate system as given in Eq. (5.3.26).

5.15 Use the continuity equation and the equation of motion to obtain the so-called *conservation form* of the linear momentum equation

$$\frac{\partial}{\partial t}(\rho\mathbf{v}) + \text{div}\left(\rho\mathbf{v}\mathbf{v} - \boldsymbol{\sigma}^{\mathrm{T}}\right) = \rho\mathbf{f}.$$

5.16 Show that

$$\rho\frac{D}{Dt}\left(\frac{v^2}{2}\right) = \mathbf{v}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}^{\mathrm{T}} + \rho\mathbf{v}\cdot\mathbf{f} \qquad (v = |\mathbf{v}|).$$

5.17 Deduce that

$$\boldsymbol{\nabla}\times\left(\frac{D\mathbf{v}}{Dt}\right) \equiv \frac{D\mathbf{w}}{Dt} + \mathbf{w}\boldsymbol{\nabla}\cdot\mathbf{v} - \mathbf{w}\cdot\boldsymbol{\nabla}\mathbf{v}, \qquad \text{(a)}$$

where $\mathbf{w} \equiv \frac{1}{2}\boldsymbol{\nabla}\times\mathbf{v}$ is the vorticity vector. *Hint:* Use the result of Problem **5.1** and the identity (you need to prove it)

$$\boldsymbol{\nabla}\times(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot\boldsymbol{\nabla}\mathbf{A} - \mathbf{A}\cdot\boldsymbol{\nabla}\mathbf{B} + \mathbf{A}\,\boldsymbol{\nabla}\cdot\mathbf{B} - \mathbf{B}\,\boldsymbol{\nabla}\cdot\mathbf{A}. \qquad (2)$$

5.18 If the stress field $\boldsymbol{\sigma}$ in a continuum has the following components in a rectangular Cartesian coordinate system

$$[\sigma] = a\begin{bmatrix} x_1^2x_2 & (b^2 - x_2^2)x_1 & 0 \\ (b^2 - x_2^2)x_1 & \frac{1}{3}(x_2^2 - 3b^2)x_2 & 0 \\ 0 & 0 & 2bx_3^2 \end{bmatrix},$$

where a and b are constants, determine the body force components necessary for the body to be in equilibrium.

5.19 If the stress field $\boldsymbol{\sigma}$ in a continuum has the following components in a rectangular Cartesian coordinate system

$$[\sigma] = \begin{bmatrix} x_1x_2 & x_1^2 & -x_2 \\ x_1^2 & 0 & 0 \\ -x_2 & 0 & x_1^2 + x_2^2 \end{bmatrix},$$

determine the body force components necessary for the body to be in equilibrium.

5.20 A two-dimensional state of stress $\boldsymbol{\sigma}$ exists in a continuum with no body forces. The following components of stress tensor are given ($\sigma_{21} = \sigma_{12}$):

$$\sigma_{11} = c_1 x_2^3 + c_2 x_1^2 x_2 - c_3 x_1, \quad \sigma_{22} = c_4 x_2^3 - c_5, \quad \sigma_{12} = c_6 x_1 x_2^2 + c_7 x_1^2 x_2 - c_8,$$

where c_i are constants. Determine the conditions on the constants so that the stress field is in equilibrium.

5.21 Given the following stress field with respect to the cylindrical coordinate system in a body that is in equilibrium ($\sigma_{\theta r} = \sigma_{r\theta}$):

$$\sigma_{rr} = 2A\left(r + \frac{B}{r^3} - \frac{C}{r}\right)\sin\theta,$$
$$\sigma_{\theta\theta} = 2A\left(3r - \frac{B}{r^3} - \frac{C}{r}\right)\sin\theta,$$
$$\sigma_{r\theta} = -2A\left(r + \frac{B}{r^3} - \frac{C}{r}\right)\cos\theta,$$

where A, B, and C are constants, determine if the stress field satisfies the equilibrium equations when the body forces are zero. Assume that all other stress components are zero.

5.22 Given the following stress field with respect to the spherical coordinate system in a body that is in equilibrium:

$$\sigma_{RR} = -\left(A + \frac{B}{R^3}\right), \quad \sigma_{\phi\phi} = \sigma_{\theta\theta} = -\left(A + \frac{C}{R^3}\right),$$

where A, B, and C are constants, determine if the stress field satisfies the equilibrium equations when the body forces are zero and all other stress components are zero.

5.23 For a cantilevered beam bent by a point load at the free end, for kinematically infinitesimal deformations, the bending moment M_3 about the x_3-axis is given by $M_3 = -Px_1$ (see Fig. P5.23). The bending stress σ_{11} is given by

$$\sigma_{11} = \frac{M_3 x_2}{I_3} = -\frac{P x_1 x_2}{I_3},$$

where I_3 is the moment of inertia of the cross section about the x_3-axis. Starting with this equation, use the two-dimensional equilibrium equations to determine stresses σ_{22} and σ_{12} as functions of x_1 and x_2.

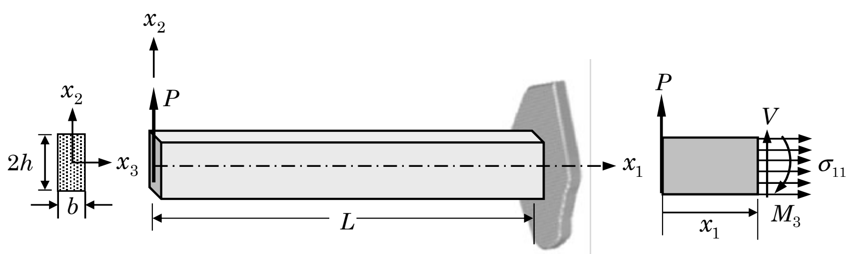

Fig. P5.23

5.24 For a cantilevered beam bent by a uniformly distributed load (see Fig. P5.24), for kinematically infinitesimal deformations, the bending stress σ_{11} is given by [because $M_3 = -q_0 x_1^2/2$]

$$\sigma_{11} = \frac{M_3 x_2}{I_3} = -\frac{q_0 x_1^2 x_2}{2I_3},$$

where I_3 is the moment of inertia of the cross section about the x_3-axis. Starting with this equation, use the two-dimensional equilibrium equations to determine the stresses σ_{22} and σ_{12} as functions of x_1 and x_2.

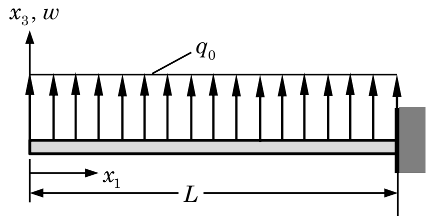

Fig. P5.24

5.25 Given the following components of the second Piola–Kirchhoff stress tensor **S** and displacement vector **u** in a body without body forces:

$$S_{11} = c_1 X_2^3 + c_2 X_1^2 X_2 - c_3 X_1, \quad S_{22} = c_4 X_2^3 - c_5, \quad S_{12} = c_6 X_1 X_2^2 + c_7 X_1^2 X_2 - c_8,$$

$$S_{13} = S_{23} = S_{33} = 0, \quad u_1 = c_9 X_2, \quad u_2 = c_{10} X_1, \quad u_3 = 0,$$

where c_i are constants, determine the conditions on the constants so that the stress field is in equilibrium.

5.26 Given the following components of the second Piola–Kirchhoff stress tensor **S** and displacement vector **u** in a body without body forces (expressed in the cylindrical coordinate system):

$$S_{rr} = -c_1 \frac{\cos\theta}{r}, \quad S_{r\theta} = S_{\theta\theta} = 0,$$

$$u_r = c_2 \log\left(\frac{r}{a}\right) \cos\theta + c_3\, \theta \sin\theta, \quad u_\theta = -c_2 \log\left(\frac{r}{a}\right) \sin\theta + c_3\, \theta \cos\theta - c_4 \sin\theta,$$

where c_i are constants, determine the conditions on the constants so that the stress field is in equilibrium for (a) the linear (i.e., infinitesimal deformations) case and (b) the finite deformation case. Assume a two-dimensional state of stress and deformation in r and θ coordinates.

Conservation of Angular Momentum

5.27 A sprinkler with four nozzles, each nozzle having an exit area of $A = 0.25$ cm^2, rotates at a constant angular velocity of $\omega = 20$ rad/s and distributes water ($\rho = 10^3$ kg/m^3) at the rate of $Q = 0.5$ L/s (see Fig. P5.27). Determine

(a) the torque T required on the shaft of the sprinkler to maintain the given motion and

(b) the angular velocity ω_0 at which the sprinkler rotates when no external torque is applied. Take $r = 0.1$ m.

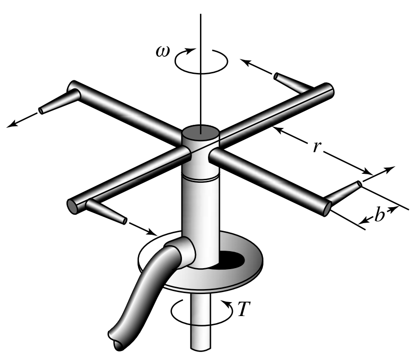

Fig. P5.27

5.28 Consider an unsymmetric sprinkler head shown in Fig. P5.28. If the discharge is $Q = 0.5$ L/s through each nozzle, determine the angular velocity of the sprinkler. Assume that no external torque is exerted on the system. Take $A = 10^{-4}$ m^2.

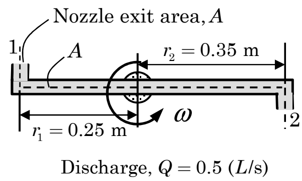

Fig. P5.28

Balance of Energy

5.29 Show that for a multipolar continuum the Clausius–Duhem inequality (5.4.24) remains unchanged.

5.30 Establish the following alternative form of the energy equation ($\boldsymbol{\sigma}^{\mathrm{T}} = \boldsymbol{\sigma}$):

$$\rho \frac{D}{Dt}\left(e + \frac{v^2}{2}\right) = \boldsymbol{\nabla} \cdot (\boldsymbol{\sigma} \cdot \mathbf{v}) + \rho \mathbf{f} \cdot \mathbf{v} + \rho\, r_h - \boldsymbol{\nabla} \cdot \mathbf{q}.$$

5.31 Establish the following *thermodynamic form* of the energy equation ($\boldsymbol{\sigma}^{\mathrm{T}} = \boldsymbol{\sigma}$):

$$\rho \frac{De}{Dt} = \boldsymbol{\nabla} \cdot (\boldsymbol{\sigma} \cdot \mathbf{v}) - \mathbf{v} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \rho\, r_h - \boldsymbol{\nabla} \cdot \mathbf{q}.$$

5.32 The total rate of work done by the surface stresses per unit volume is given by $\boldsymbol{\nabla} \cdot (\boldsymbol{\sigma} \cdot \mathbf{v})$. The rate of work done by the resultant of the surface stresses per unit volume is given by $\mathbf{v} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\sigma}$. The difference between these two terms yields the rate of work done by the surface stresses in deforming the material particle, per unit volume. Show that this difference can be written as $\boldsymbol{\sigma} : \mathbf{D}$, where $\mathbf{D}$ is the strain rate tensor defined in Eq. (5.4.8).

5.33 The rate of internal work done (power) in a continuous medium in the current configuration can be expressed as

$$W = \tfrac{1}{2} \int_{\Omega} \boldsymbol{\sigma} : \mathbf{D}\, d\mathbf{x}, \tag{1}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress tensor and $\mathbf{D}$ is the strain rate tensor (i.e., symmetric part of the velocity gradient tensor)

$$\mathbf{D} = \tfrac{1}{2}\left[(\boldsymbol{\nabla}\mathbf{v})^{\mathrm{T}} + \boldsymbol{\nabla}\mathbf{v}\right], \quad \mathbf{v} = \frac{d\mathbf{x}}{dt}. \tag{2}$$

The pair $(\boldsymbol{\sigma}, \mathbf{D})$ is said to be *energetically conjugate* because it produces the (strain) energy stored in a deformable medium. Show that

(a) the first Piola–Kirchhoff stress tensor $\mathbf{P}$ is energetically conjugate to the rate of deformation gradient $\dot{\mathbf{F}}$, and

(b) the second Piola–Kirchhoff stress tensor $\mathbf{S}$ is energetically conjugate to the rate of Green strain tensor $\dot{\mathbf{E}}$.

Hints: Note the following identities:

$$d\mathbf{x} = J\, d\mathbf{X}, \quad \mathbf{L} \equiv (\boldsymbol{\nabla}\mathbf{v})^{\mathrm{T}} = \dot{\mathbf{F}} \cdot \mathbf{F}^{-1}, \quad \mathbf{P} = J\mathbf{F}^{-1} \cdot \boldsymbol{\sigma}, \quad \boldsymbol{\sigma} = \frac{1}{J}\mathbf{F} \cdot \mathbf{S} \cdot \mathbf{F}^{\mathrm{T}}.$$