6

CONSTITUTIVE EQUATIONS

The truth is, the science of Nature has been already too long made only a work of the brain and the fancy. It is now high time that it should return to the plainness and soundness of observations on material and obvious things. —— Robert Hooke (1635–1703)

There are two possible outcomes: If the result confirms the hypothesis, then you've made a measurement. If the result is contrary to the hypothesis, then you've made a discovery.
—— Enrico Fermi (1901–1954)

6.1 Introduction

6.1.1 General Comments

The kinematic relations developed in Chapter 3, and the principles of conservation of mass, balance of momenta, and thermodynamic principles discussed in Chapter 5, are applicable to any continuum irrespective of its physical constitution. The kinematic variables such as strains and temperature gradient, and kinetic variables such as stresses and heat flux were introduced independently of each other. *Constitutive equations* are those relations that connect the *primary* field variables (e.g., ρ, θ, $\boldsymbol{\nabla}\theta$, $\mathbf{u}$, $\boldsymbol{\nabla}\mathbf{u}$, $\mathbf{v}$, and $\boldsymbol{\nabla}\mathbf{v}$) to the *secondary* field variables (e.g., e, η, $\mathbf{q}$, and $\boldsymbol{\sigma}$), and they involve the intrinsic physical properties of a continuum. Constitutive equations are *not* derived from any physical principles, although they are subject to obeying certain rules and the entropy inequality. In essence, constitutive equations are mathematical models of the real behavior of materials that are validated against experimental results. The differences between theoretical predictions and experimental findings are often attributed to an inaccurate mathematical representation of the constitutive behavior. *Fluid mechanics*, which deals with liquids and gases, and *solid mechanics*, which deals with metals, fiber-reinforced composites, rubber, ceramics, and so on, share the same field equations (developed in Chapter 5), but their constitutive equations differ considerably.

The main objective of this chapter is to study the most commonly known phenomenological constitutive equations that describe the macroscopic nature of the material response of idealized continua. Constitutive equations from solid mechanics, fluid mechanics, and heat transfer are discussed. We begin with certain terminologies that can be found in introductory texts on mechanics of materials and fluid mechanics.

- A continuum is said to be *homogeneous* if the material properties are the same throughout the continuum (i.e., material properties are independent of position). In a *heterogeneous* continuum, the material properties are a function of position.

- An *anisotropic* continuum is one that has different values of a material property in different directions at a point, that is, material properties are direction dependent.

- An *isotropic* material is one for which a material property is the same in all directions at a point.

An isotropic or anisotropic material can be nonhomogeneous or homogeneous.

6.1.2 General Principles of Constitutive Theory

Constitutive equations are often postulated based on experimental observations. Although experiments are necessary in the determination of various parameters (e.g., elastic constants, thermal conductivity, thermal coefficient of expansion, and coefficients of viscosity) appearing in the constitutive equations, the formulation of the constitutive equations for a given material is guided by certain rules. The approach typically involves assuming the form of the constitutive equation and then restricting the form to a specific one by appealing to certain physical requirements, which are summarized here.

(1) *Consistency (or physical admissibility).* All constitutive equations should be consistent with the conservation of mass, balance of momenta and energy, and the entropy inequality.

(2) *Coordinate frame invariance.* The constitutive equations should not depend on any particular choice of coordinate frame. Although they may have different forms in different coordinate systems, the actual response should be independent of the chosen coordinate system.

(3) *Material frame indifference.* The constitutive equations must be invariant with respect to observer transformations (see Section 3.8). That is, the form of the constitutive function should not change if the material is studied in a different frame of reference. The consequences of this are more apparent in the three-dimensional setting.

(4) *Material symmetry.* The constitutive equations must be form-invariant with respect to a group of unimodular transformations of the material frame of reference. That is, the constitutive equations should reflect material symmetries such as isotropy (infinite number of planes of symmetry) and orthotropy (three mutually perpendicular planes of symmetry).

(5) *Equipresence.* A quantity appearing as an independent variable in one constitutive equation should appear in all constitutive equations, unless

the appearance contradicts the balance laws or the second law of thermodynamics. That is, all dependent variables should be functions of the same list of independent variables; one should not *a priori* omit any independent variable.

(6) *Determinism.* The values of the constitutive variables (e.g., stress, heat flux, entropy, and internal energy) at a material point at any time are determined by the histories of motion and temperature of all points of the continuum.

(7) *Local action.* The constitutive variables at a point $\mathbf{x}$ are not appreciably affected by the values of the dependent variables (e.g., displacements, strains, temperature, pressure, etc.) at points distant from $\mathbf{x}$.

(8) *Dimensionality.* The constitutive functionals should be dimensionally consistent in the sense that all terms appearing on either side of the constitutive equations should be the same.

(9) *Fading memory.* The current values of the constitutive variables are not appreciably affected by their values at past times. This is the time domain counterpart of the axiom of local action.

(10) *Causality.* The variables entering the description of motion of a continuum and temperature are considered as the self-evident observable effects in every thermomechanical behavior of a continuum. The remaining quantities (i.e., those derivable from the motion and temperature) that enter the expression of entropy production are "causes" or dependent constitutive variables.

These principles/axioms ensure that the initial value problems resulting from the conservation principles, constitutive equations, and physically meaningful boundary and initial conditions are well-posed in the sense that the solution exists and it is unique.

In a continuum theory of constitutive equations, one begins with a general form of functional constitutive equations; seeks to determine if certain variables should be included in the constitutive equation based on some general rules, such as those listed above; and then specializes the equations to certain type, as dictated by the material response being studied. This kind of formal approach enables one to account properly for all possible coupling effects (e.g., thermomechanical, electromechanical, electromagnetics, and so on). An extensive discussion of this formal approach is beyond the scope of this introductory book on continuum mechanics and elasticity [see Truesdell and Knoll (1965) for a comprehensive treatment].

6.1.3 Material Frame Indifference

The effect of superposed rigid-body motion $\mathbf{x}^* = \mathbf{c}(t) + \mathbf{Q} \cdot \mathbf{x}$, where $\mathbf{c}$ denotes rigid-body translation and $\mathbf{Q}$ is a proper orthogonal tensor that represents rigid-body rotation, and the importance of frame indifference in calculating/measuring various quantities introduced in the kinematic and kinetic descriptions of a continuum were discussed in Chapters 3 and 4 [see Eq. (3.8.21)]. In summary, the following relations were established to show that the displacement vector $\mathbf{u}$, deformation gradient $\mathbf{F}$, the right Cauchy–Green deformation tensor $\mathbf{C}$, the Green–Lagrange strain tensor $\mathbf{E}$, the rate of deformation tensor $\mathbf{D}$, the Cauchy stress tensor $\boldsymbol{\sigma}$, the first Piola–Kirchhoff stress tensor $\mathbf{P}$, and the second Piola–Kirchhoff stress tensor $\mathbf{S}$ are objective:

$$\mathbf{u}^* = \mathbf{Q}\cdot\mathbf{u}, \quad \mathbf{F}^* = \mathbf{Q}\cdot\mathbf{F}, \quad \mathbf{C}^* = \mathbf{C}, \quad \mathbf{E}^* = \mathbf{E}, \quad \mathbf{D}^* = \mathbf{Q}\cdot\mathbf{D}\cdot\mathbf{Q}^{\mathrm{T}}$$
$$\boldsymbol{\sigma}^* = \mathbf{Q}\cdot\boldsymbol{\sigma}\cdot\mathbf{Q}^{\mathrm{T}}, \quad \mathbf{P}^* = \mathbf{Q}\cdot\mathbf{P}, \quad \mathbf{S}^* = \mathbf{S},$$

where quantities with an asterisk (*) are those with superposed rigid-body motion; that is, they are the quantities observed in a frame of reference that is undergoing a rigid-body motion with respect to a stationary frame of reference in which quantities without an asterisk are observed. The central idea of the (principle of) material frame indifference is that the constitutive equations relating the quantities introduced in the kinematic description to those appearing in the kinetic description must be independent of the frame of reference (i.e., invariant under observer transformations). *One must make sure that the quantities entering any constitutive equation must be the same type – objective or not – on both sides of the equation.*

6.1.4 Restrictions Placed by the Entropy Inequality

To ensure thermodynamic equilibrium of the processes under consideration, its constitutive equations must be derived using the entropy inequality; if derived in other ways, the constitutive equations must satisfy the conditions resulting from the entropy inequality. Following the axioms discussed in Section 6.1.2 and by examining the momentum and energy equations and the entropy inequality, it can be concluded that the stress tensor $\boldsymbol{\sigma}$, Helmholtz free energy density Ψ, specific entropy η, and heat flux vector $\mathbf{q}$ must be the dependent variables in the constitutive models for a homogeneous and isotropic material. The arguments of these variables depend on the physics of the process. For isotropic and homogeneous materials, $\mathbf{F}$ is a measure of the deformation, the temperature gradient vector $\mathbf{g}$ is needed because of $\mathbf{q}$, and the temperature θ is an obvious choice as an argument for thermoelastic solids. Hence, we have the following functional forms based on the principle of equipresence[1]:

$$\begin{aligned}
\boldsymbol{\sigma}(\mathbf{x},t) &= \boldsymbol{\mathcal{F}}_\sigma[\mathbf{F},\theta,\mathbf{g}],\\
\Psi(\mathbf{x},t) &= \boldsymbol{\mathcal{F}}_\Psi[\mathbf{F},\theta,\mathbf{g}],\\
\eta(\mathbf{x},t) &= \boldsymbol{\mathcal{F}}_\eta[\mathbf{F},\theta,\mathbf{g}],\\
\mathbf{q}(\mathbf{x},t) &= \boldsymbol{\mathcal{F}}_q[\mathbf{F},\theta,\mathbf{g}],
\end{aligned} \tag{6.1.1}$$

where $\boldsymbol{\mathcal{F}}$ denotes the functional mapping, referred to as the *response function*, whose actual form will become apparent in the following discussion. Any of

---

[1] One can use either Ψ or e, as they are interdependent.

the three arguments $(\mathbf{F}, \theta, \mathbf{g})$ that contradict the constitutive axioms, especially material symmetry, objectivity, or the entropy inequality, will be removed from the argument list in any of the response functions listed in Eq. (6.1.1). The entropy inequality provides guidelines for the form of the constitutive relations.

For example, suppose that Ψ is a function of $\mathbf{F}$, θ, and $\mathbf{g}$, $\Psi = \Psi(\mathbf{F}, \theta, \mathbf{g})$. The entropy inequality from Eq. (5.4.30) is

$$-\rho\dot{\Psi} + \boldsymbol{\sigma} : \mathbf{L} - \rho\dot{\theta}\eta - \frac{1}{\theta}\mathbf{q} \cdot \mathbf{g} \geq 0, \tag{6.1.2}$$

where $\mathbf{g} = \boldsymbol{\nabla}\theta$; $\mathbf{L}$ is the velocity gradient tensor, $\mathbf{L} = (\boldsymbol{\nabla}\mathbf{v})^{\mathrm{T}} = \mathbf{D} + \mathbf{W}$; $\mathbf{D}$ is the symmetric part; and $\mathbf{W}$ is the skew symmetric part of the velocity gradient tensor $\mathbf{L}$. We can write

$$\dot{\Psi} = \frac{\partial\Psi}{\partial\mathbf{F}} : \dot{\mathbf{F}}^{\mathrm{T}} + \frac{\partial\Psi}{\partial\theta}\dot{\theta} + \frac{\partial\Psi}{\partial\mathbf{g}} \cdot \dot{\mathbf{g}}. \tag{6.1.3}$$

Substituting for $\dot{\mathbf{F}}$ from Eq. (3.6.15) into Eq. (6.1.3) and the result into Eq. (6.1.2), we obtain (note that $\boldsymbol{\sigma} : \mathbf{L} = \boldsymbol{\sigma} : \mathbf{L}^{\mathrm{T}}$ when $\boldsymbol{\sigma}$ is symmetric)

$$\left(\boldsymbol{\sigma} - \rho\frac{\partial\Psi}{\partial\mathbf{F}} \cdot \mathbf{F}^{\mathrm{T}}\right) : \mathbf{L}^{\mathrm{T}} - \rho\left(\eta + \frac{\partial\Psi}{\partial\theta}\right)\dot{\theta} - \frac{\partial\Psi}{\partial\mathbf{g}} \cdot \dot{\mathbf{g}} - \frac{1}{\theta}\mathbf{q} \cdot \mathbf{g} \geq 0. \tag{6.1.4}$$

In index notation, we have

$$\left(\sigma_{ij} - \rho\frac{\partial\Psi}{\partial F_{iK}}F_{jK}\right)L_{ij} - \rho\left(\eta + \frac{\partial\Psi}{\partial\theta}\right)\dot{\theta} - \frac{\partial\Psi}{\partial g_i}\dot{g}_i - \frac{1}{\theta}\, q_i\, g_i \geq 0. \tag{6.1.5}$$

Since $\mathbf{L}$, $\dot{\theta}$, and $\dot{\mathbf{g}}$ are linearly independent of each other, it follows that

$$\boldsymbol{\sigma} - \rho\frac{\partial\Psi}{\partial\mathbf{F}} \cdot \mathbf{F}^{\mathrm{T}} = \mathbf{0}, \tag{6.1.6}$$

$$\eta + \frac{\partial\Psi}{\partial\theta} = 0, \tag{6.1.7}$$

$$-\frac{\partial\Psi}{\partial\mathbf{g}} = 0, \tag{6.1.8}$$

$$-\mathbf{q} \cdot \mathbf{g} \geq 0. \tag{6.1.9}$$

Equation (6.1.8) implies that Ψ is not a function of the temperature gradient $\mathbf{g}$. Also, Eq. (6.1.9) implies that

$$\mathbf{q} \cdot \mathbf{g} \leq 0. \tag{6.1.10}$$

Therefore, $\mathbf{q}$ is proportional to the negative of the gradient of the temperature $\mathbf{g} = \boldsymbol{\nabla}\theta$, as we will see shortly from the Fourier heat conduction law. Equation (6.1.7) implies that η can be determined from Ψ and, hence, cannot be a dependent variable in the constitutive model. Furthermore, we conclude from Eq. (6.1.6), because Ψ is a function of only $\mathbf{F}$ and θ, that $\boldsymbol{\sigma}$ can depend only on $\mathbf{F}$ and θ. Thus, Eq. (6.1.1) is modified to read

$$\begin{aligned} \Psi(\mathbf{x}, t) &= \boldsymbol{\mathcal{F}}_{\Psi}[\mathbf{F}, \theta], \\ \eta(\mathbf{x}, t) &= \boldsymbol{\mathcal{F}}_{\eta}[\mathbf{F}, \theta], \\ \mathbf{q}(\mathbf{x}, t) &= \boldsymbol{\mathcal{F}}_{q}[\mathbf{F}, \theta, \mathbf{g}]. \end{aligned} \tag{6.1.11}$$

6.2 Elastic Materials

6.2.1 Cauchy-Elastic Materials

A material is called *Cauchy-elastic* or *elastic* if the stress field at time t depends only on the state of deformation and temperature at that time, and not on the history of these variables. The constitutive relation for an elastic body under isothermal conditions (i.e., no change in the temperature from the reference configuration) relates the Cauchy stress tensor $\boldsymbol{\sigma}(\mathbf{x},t)$ at a point $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X},t)$ and time t to the deformation gradient $\mathbf{F}(\mathbf{X},t)$ [see Eq. (6.1.1)]:

$$\boldsymbol{\sigma}(\mathbf{x},t) = \boldsymbol{\mathcal{F}}[\mathbf{F}(\mathbf{X},t),\mathbf{X}], \tag{6.2.1}$$

where $\boldsymbol{\mathcal{F}}$ is the response function, and $\boldsymbol{\sigma}$ denotes the value of $\boldsymbol{\mathcal{F}}$, which characterizes the material properties of an isothermal Cauchy-elastic material. The requirement that the response function $\boldsymbol{\mathcal{F}}$ be unaffected by superposed rigid-body motions places a restriction on $\boldsymbol{\mathcal{F}}$.

Consider the Cauchy stress tensor after superposed rigid-body motion [see Eq. (6.1.1) for the transformation equations of objective quantities]:

$$\boldsymbol{\sigma}^* = \boldsymbol{\mathcal{F}}(\mathbf{F}^*) = \boldsymbol{\mathcal{F}}(\mathbf{Q}\cdot\mathbf{F}), \tag{6.2.2}$$

but

$$\boldsymbol{\sigma}^* = \mathbf{Q}\cdot\boldsymbol{\sigma}\cdot\mathbf{Q}^{\mathrm{T}} = \mathbf{Q}\cdot\boldsymbol{\mathcal{F}}(\mathbf{F})\cdot\mathbf{Q}^{\mathrm{T}}. \tag{6.2.3}$$

These two relations place the following restriction on $\boldsymbol{\mathcal{F}}$:

$$\boldsymbol{\mathcal{F}}(\mathbf{Q}\cdot\mathbf{F}) = \mathbf{Q}\cdot\boldsymbol{\mathcal{F}}(\mathbf{F})\cdot\mathbf{Q}^{\mathrm{T}}. \tag{6.2.4}$$

Using the right-hand polar decomposition of $\mathbf{F}$, $\mathbf{F} = \mathbf{R}\cdot\mathbf{U}$, in Eq. (6.2.4), we obtain

$$\boldsymbol{\mathcal{F}}(\mathbf{Q}\cdot\mathbf{R}\cdot\mathbf{U}) = \mathbf{Q}\cdot\boldsymbol{\mathcal{F}}(\mathbf{F})\cdot\mathbf{Q}^{\mathrm{T}}. \tag{6.2.5}$$

Since $\mathbf{R}$ is a proper orthogonal rotation matrix, we can take $\mathbf{Q} = \mathbf{R}^{\mathrm{T}}$ and obtain $\boldsymbol{\mathcal{F}}(\mathbf{Q}\cdot\mathbf{Q}^{\mathrm{T}}\cdot\mathbf{U}) = \boldsymbol{\mathcal{F}}(\mathbf{U})$. Thus, from Eq. (6.2.5), we have

$$\boldsymbol{\mathcal{F}}(\mathbf{F}) = \mathbf{R}\cdot\boldsymbol{\mathcal{F}}(\mathbf{U})\cdot\mathbf{R}^{\mathrm{T}}, \tag{6.2.6}$$

which constitutes the restriction on the response function in order that it is objective.

6.2.2 Green-Elastic or Hyperelastic Materials

A hyperelastic material, also known as the *Green-elastic material*, is one for which there exists a *Helmholtz free-energy potential* Ψ (measured per unit volume) whose derivative with respect to a strain gives the corresponding stress and whose derivative with respect to temperature gives the heat flux vector. When Ψ is solely a function of $\mathbf{F}$, $\mathbf{C}$, or some strain tensor, it is called the *strain energy* *density function* and denoted by U_0 (measured per unit mass). For example, if $U_0 = U_0(\mathbf{F})$, we have [see Eq. (6.1.6)]

$$\mathbf{P} = \rho_0 \frac{\partial U_0(\mathbf{F})}{\partial \mathbf{F}} \quad \left(P_{iJ} = \rho_0 \frac{\partial U_0}{\partial F_{iJ}} \right), \tag{6.2.7}$$

$$\boldsymbol{\sigma} = \frac{1}{J} \mathbf{P} \cdot \mathbf{F}^{\mathrm{T}} = \rho \, \frac{\partial U_0(\mathbf{F})}{\partial \mathbf{F}} \cdot \mathbf{F}^{\mathrm{T}}, \tag{6.2.8}$$

$$\mathbf{S} = \mathbf{F}^{-1} \cdot \mathbf{P} = \rho_0 \mathbf{F}^{-1} \cdot \frac{\partial U_0(\mathbf{F})}{\partial \mathbf{F}}. \tag{6.2.9}$$

For an incompressible elastic material (i.e., material for which the volume is preserved and hence $J = 1$), we postulate the existence of a strain energy density function in the form

$$\rho_0 \, \hat{U}_0 = \rho_0 \, U_0(\mathbf{F}) - p\,(J-1), \tag{6.2.10}$$

where p denotes a *hydrostatic pressure*, and $\hat{U}_0(\mathbf{F})$ is the strain energy density for the case $J = 1$. Equation (6.2.10) can be viewed as one in which the strain energy density for the case $J = 1$ is constructed from U_0 by treating $J - 1 = 0$ as a constraint, and using the *Lagrange multiplier method* to include the constraint; it turns out that the *Lagrange multiplier* is $\lambda = -p$. Then the constitutive equation for incompressible hyperelastic material is ($J = 1$):

$$\mathbf{P} = \rho_0 \frac{\partial \hat{U}_0(\mathbf{F})}{\partial \mathbf{F}} = -p\,\mathbf{F}^{-\mathrm{T}} + \rho_0 \frac{\partial U_0(\mathbf{F})}{\partial \mathbf{F}}, \tag{6.2.11}$$

$$\boldsymbol{\sigma} = \mathbf{P} \cdot \mathbf{F}^{\mathrm{T}} = -p\,\mathbf{I} + \rho_0 \, \frac{\partial U_0(\mathbf{F})}{\partial \mathbf{F}} \cdot \mathbf{F}^{\mathrm{T}}, \tag{6.2.12}$$

$$\mathbf{S} = \mathbf{F}^{-1} \cdot \mathbf{P} = -p\,\mathbf{F}^{-1} \cdot \mathbf{F}^{-\mathrm{T}} + \rho_0 \, \mathbf{F}^{-1} \cdot \frac{\partial U_0(\mathbf{F})}{\partial \mathbf{F}}, \tag{6.2.13}$$

where the derivative of J with respect to $\mathbf{F}$ is (the reader is asked to verify this)

$$\frac{\partial J}{\partial \mathbf{F}} = J\,\mathbf{F}^{-\mathrm{T}}. \tag{6.2.14}$$

6.2.3 Linearized Hyperelastic Materials: Infinitesimal Strains

Here we present the constitutive equations for the case of infinitesimal deformation (ie., $|\boldsymbol{\nabla}\mathbf{u}| = \mathrm{O}(\epsilon) << 1$). Hence, we do not distinguish between various measures of stress and strain, and use $\mathbf{S} \approx \boldsymbol{\sigma}$ for the stress tensor and $\mathbf{E} \approx \boldsymbol{\varepsilon}$ for the strain tensor in the material description used in solid mechanics. For such materials, the Helholtz free energy density Ψ is the same as the strain energy density U_0, and it is more meaningful to assume that the strain energy density is a function of the strain, $\boldsymbol{\varepsilon}$, rather than the deformation gradient, although one may also assume that U_0 is a function of the strain invariants.

The constitutive equation to be developed here for stress tensor $\boldsymbol{\sigma}$ does not include creep at constant stress and stress relaxation at constant strain. Thus, the material coefficients that specify the constitutive relationship between the stress and strain components are assumed to be constant during the deformation. This does not automatically imply that we neglect temperature effects on deformation. We account for the thermal expansion of the material, which can produce strains or stresses as large as those produced by the applied mechanical forces.

The constitutive equation for linearized hyperelastic materials can be derived using

$$\boldsymbol{\sigma} = \rho_0 \frac{\partial U_0(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}} \quad \left(\sigma_{ij} = \rho_0 \frac{\partial U_0}{\partial \varepsilon_{ij}}\right). \tag{6.2.15}$$

As indicated earlier, here we assume that U_0 is a function of $\boldsymbol{\varepsilon}$ and expand it in Taylor's series about the strain $\boldsymbol{\varepsilon} = 0$ in the reference configuration,

$$\rho_0 U_0 = C_0 + C_{ij}\,\varepsilon_{ij} + \frac{1}{2!}C_{ijk\ell}\,\varepsilon_{ij}\,\varepsilon_{k\ell} + \frac{1}{3!}C_{ijk\ell mn}\,\varepsilon_{ij}\,\varepsilon_{k\ell}\,\varepsilon_{mn} + \dots, \tag{6.2.16}$$

where C_0, C_{ij}, $C_{ijk\ell}$, and so on are material stiffness coefficients that are independent of the deformation. For linear elastic materials U_0 is a quadratic function of the strain tensor, and for nonlinear elastic materials, U_0 is a cubic function of the strain tensor $\boldsymbol{\varepsilon}$. For linear elastic materials, the mechanical pressure is the same as the negative of the mean normal stress.

This chapter is focused primarily on constitutive relations for Hookean solids (linear elastic solids), Newtonian fluids (fluids with linear relations between stress and strain rate), and Fourier heat conduction law (a linear relation between the heat flux vector and the temperature gradient vector). The constitutive equations presented in Section 6.3 for elastic solids are based on an assumption of small strain. Nonlinear constitutive relations for elastic solids are briefly discussed in Section 6.4. In Section 6.5, constitutive relations for Newtonian fluids are presented, and in Section 6.6 differential and integral generalized Newtonian constitutive relations are reviewed. The Fourier heat conduction law is presented in Section 6.7. Finally, constitutive relations for coupled problems, for example, electromagnetics, electroelasticity, and thermoelasticity , are presented in Section 6.8.

6.3 Hookean Solids

6.3.1 Generalized Hooke's Law

To develop the stress–strain relations for a linear elastic solid, we set up a coordinate system in which the material parameters are measured. This coordinate system is termed the *material coordinate system*, not to be confused with the *material description* of Chapter 5. The coordinate system used to write the equations of motion and strain-displacement equations is called the *problem coordinates* to distinguish it from the material coordinate system. In the remaining discussion of this section, we use the Lagrangian description with coordinates (x, y, z) to describe the kinematics, stress state, and the field equations, and use the material coordinate system (x_1, x_2, x_3) to describe the constitutive response. The material coordinate system is one that is aligned with the *planes of material* *symmetry* (to be defined shortly), so that measurement of material parameters becomes simple. Of course, the constitutive relations have to be transformed to the problem coordinates in order to solve the final boundary-value problem. When no preferred planes of material symmetry exist, the material is called isotropic, and the material coordinates are taken to be the same as the problem coordinates. All tensor quantities measured in (x_1, x_2, x_3) will have integer subscripts, for example, σ_{ij}, ε_{ij}, and so on, whereas those measured in (x, y, z) will have letter subscripts, for example, $\sigma_{xx}, \sigma_{xy}, \cdots$, and $\varepsilon_{xx}, \varepsilon_{xy}, \cdots$, and so on.

We begin with the quadratic form of U_0 in the material coordinate system:

$$\rho_0 U_0 = C_0 + C_{ij}\,\varepsilon_{ij} + \tfrac{1}{2} C_{ijk\ell}\,\varepsilon_{ij}\,\varepsilon_{k\ell}, \tag{6.3.1}$$

where C_0 is a reference value of U_0 from which the strain energy density function is measured. From Eq. (6.2.15), we have

$$\begin{aligned}\sigma_{mn} = \rho_0 \frac{\partial U_0}{\partial \varepsilon_{mn}} &= C_{ij}\,\delta_{mi}\,\delta_{nj} + \tfrac{1}{2} C_{ijk\ell}\left(\varepsilon_{k\ell}\,\delta_{im}\,\delta_{jn} + \varepsilon_{ij}\delta_{km}\delta_{\ell n}\right)\\ &= C_{mn} + \tfrac{1}{2} C_{mnk\ell}\,\varepsilon_{k\ell} + \tfrac{1}{2} C_{ijmn}\,\varepsilon_{ij} = C_{mn} + \tfrac{1}{2}\left(C_{mnij} + C_{ijmn}\right)\varepsilon_{ij}\\ &= C_{mn} + C_{mnij}\,\varepsilon_{ij},\end{aligned} \tag{6.3.2}$$

where

$$C_{mnij} = \tfrac{1}{2}\left(C_{mnij} + C_{ijmn}\right) = \rho_0 \frac{\partial^2 U_0}{\partial \varepsilon_{ij}\partial \varepsilon_{mn}} = \rho_0 \frac{\partial^2 U_0}{\partial \varepsilon_{mn}\partial \varepsilon_{ij}} = C_{ijmn}. \tag{6.3.3}$$

Clearly, C_{mn} have the same units as σ_{mn}, and they represent the *residual stress* components of a solid. We shall assume, without loss of generality, that the body is free of stress prior to the load application so that we may write

$$\boldsymbol{\sigma} = \mathbf{C} : \boldsymbol{\varepsilon} \qquad (\sigma_{ij} = C_{ijk\ell}\,\varepsilon_{k\ell}). \tag{6.3.4}$$

Equation (6.3.4) is known as the generalized Hooke's law. The coefficients $C_{ijk\ell}$ are called elastic *stiffness* coefficients. In general, there are $3^4 = 81$ scalar components of the fourth-order tensor[2] $\mathbf{C}$. The number of coefficients is significantly reduced because (a) the components $C_{ijk\ell}$ satisfy the symmetry conditions implied by Eq. (6.3.3), and (b) the stress and strain tensors are symmetric, requiring Eq. (6.3.4) to be valid when subscripts i and j are interchanged as well as k and ℓ are interchanged. Thus, we have

$$C_{ijk\ell} = C_{k\ell ij}, \quad C_{ijk\ell} = C_{jik\ell}, \quad C_{ij\ell k} = C_{ijk\ell}, \quad C_{ijk\ell} = C_{ji\ell k}, \tag{6.3.5}$$

and the stress–strain relations (6.3.4) take the form

$$\begin{Bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{Bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1113} & C_{1112}\\ & C_{2222} & C_{2233} & C_{2223} & C_{2213} & C_{2212}\\ & & C_{3333} & C_{3323} & C_{3313} & C_{3312}\\ & & & C_{2323} & C_{2313} & C_{2312}\\ & \text{symmetric} & & & C_{1313} & C_{1312}\\ & & & & & C_{1212}\end{bmatrix} \begin{Bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ 2\varepsilon_{23}\\ 2\varepsilon_{13}\\ 2\varepsilon_{12}\end{Bmatrix}. \tag{6.3.6}$$

---

[2] In this chapter $\mathbf{C}$ denotes the fourth-order elasticity tensor $\mathbf{C}$, not the right Cauchy–Green deformation tensor $\mathbf{C}$.

Thus the number of independent coefficients C_{ijmn} is only 6+5+4+3+2+1 = 21. Materials that obey Eq. (6.3.6) are called *triclinic* materials.

We can express Eq. (6.3.4) in an alternative form using a single subscript notation for stresses and strains and a two subscript notation for the material stiffness coefficients:

$$\begin{aligned} &\sigma_1 = \sigma_{11},\ \sigma_2 = \sigma_{22},\ \sigma_3 = \sigma_{33},\ \sigma_4 = \sigma_{23}\ ,\ \sigma_5 = \sigma_{13},\ \sigma_6 = \sigma_{12},\\ &\varepsilon_1 = \varepsilon_{11},\ \varepsilon_2 = \varepsilon_{22},\ \varepsilon_3 = \varepsilon_{33},\ \varepsilon_4 = 2\varepsilon_{23},\ \varepsilon_5 = 2\varepsilon_{13},\ \varepsilon_6 = 2\varepsilon_{12}. \end{aligned} \tag{6.3.7}$$

$$11 \to 1 \quad 22 \to 2 \quad 33 \to 3 \quad 23 \to 4 \quad 13 \to 5 \quad 12 \to 6. \tag{6.3.8}$$

It should be cautioned that the single subscript notation used for stresses and strains and the two-subscript components C_{ij} render them non-tensor components; that is, σ_i, ε_i, and C_{ij} do not transform like the components of a tensor, $\sigma_i \neq \ell_{ij}\sigma_j$. The single subscript notation for stresses and strains is called the *engineering notation* or the *Voigt–Kelvin notation.* Equation (6.3.6) now takes the form

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix}, \tag{6.3.9}$$

or simply

$$\sigma_i = C_{ij}\,\varepsilon_j, \tag{6.3.10}$$

where summation on repeated subscripts is implied (now i and j take values from 1 to 6). Note that the coefficients C_{ij} are symmetric, $C_{ij} = C_{ji}$, a property inherited from Eq. (6.3.6).

We assume that the stress–strain relations (6.3.10) are invertible. Thus, the components of strain are related to the components of stress by

$$\varepsilon_i = S_{ij}\,\sigma_j, \tag{6.3.11}$$

where $S_{ij} = S_{ji}$ are the material *compliance* coefficients with $[S] = [C]^{-1}$ [i.e., the compliance tensor is the inverse of the stiffness tensor: $\mathbf{S} = \mathbf{C}^{-1}$]. In matrix form, Eq. (6.3.11) becomes

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix}. \tag{6.3.12}$$

The strain–stress relations are more suitable in determining the material constants in a laboratory because experiments involve the application of loads and measurement of changes in the geometry (i.e., determine strains from an applied stress state).

6.3.2 Material Symmetry Planes

Further reduction in the number of independent stiffness (or compliance) parameters comes from the so-called material symmetry. When elastic material parameters at a point have *the same values* for every pair of coordinate systems that are mirror images of each other in a certain plane, that plane is called a *material plane of symmetry* (for example, symmetry of internal structure due to crystallographic form, *regular* arrangement of fibers or molecules, and so on). We note that the symmetry under discussion is a directional property and not a positional property. Thus, a material may have a certain elastic symmetry at every point of a material body and the properties may vary from point to point. Positional dependence of material properties is what we called inhomogeneity of the material.

In the following, we discuss various planes of symmetry and forms of associated stress–strain relations. Note that the use of components of stress and strain tensors is necessary in the following discussion because transformation equations are valid only for components of tensors from two different coordinate systems. The components σ_{ij} and ε_{ij} of second-order tensors $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ and the components C_{ijkl} of a fourth-order elasticity tensor $\mathbf{C}$ transform according to the relations

$$\bar{\sigma}_{ij} = \ell_{ip}\,\ell_{jq}\,\sigma_{pq}, \quad \bar{\varepsilon}_{ij} = \ell_{ip}\,\ell_{jq}\,\varepsilon_{pq}, \quad \bar{C}_{ijkl} = \ell_{ip}\,\ell_{jq}\,\ell_{kr}\,\ell_{ls}\,C_{pqrs}, \tag{6.3.13}$$

where ℓ_{ij} are the direction cosines associated with the coordinate systems $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ and (x_1, x_2, x_3), and $\bar{C}_{ijkl}$ and C_{pqrs}, for example, are the components of the fourth-order tensor $\mathbf{C}$ in the barred and unbarred coordinates systems, respectively [see Eqs. (2.2.70), (2.2.71), (3.4.29), and (4.3.3)].

A trivial material symmetry transformation is one in which the barred coordinate system is obtained from the unbarred coordinate system by simply reversing their directions (i.e., mirror reflection): $\bar{x}_1 = -x_1$, $\bar{x}_2 = -x_2$, and $\bar{x}_3 = -x_3$; that is, $\hat{\bar{\mathbf{e}}}_i = -\hat{\mathbf{e}}_i$ and $\ell_{ij} = -\delta_{ij}$ (it does not matter that it is a left-handed coordinate system as it does not affect the discussion), as shown in Fig. 6.3.1(a).

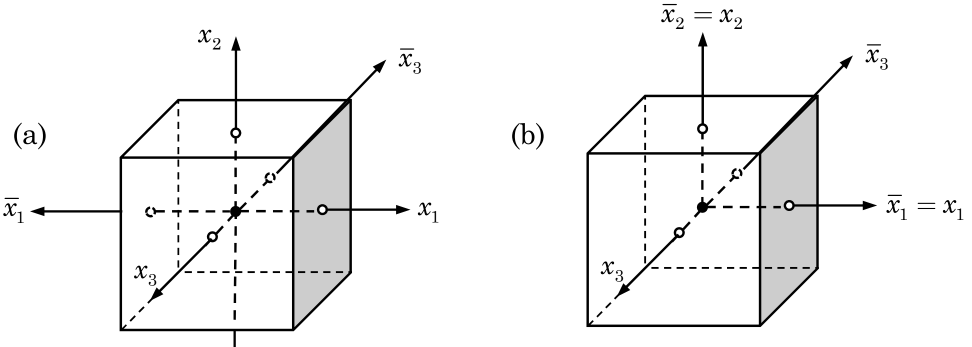

Fig. 6.3.1: (a) Transformation defined by $\hat{\bar{\mathbf{e}}}_i = -\hat{\mathbf{e}}_i$, $i = 1, 2, 3$. (b) Transformation defined by $\hat{\bar{\mathbf{e}}}_\alpha = \hat{\mathbf{e}}_\alpha$, $\alpha = 1, 2$, and $\hat{\bar{\mathbf{e}}}_3 = -\hat{\mathbf{e}}_3$.

Thus, the transformation matrix is

$$[L] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \tag{6.3.14}$$

For this transformation, it follows that

$$\begin{aligned} \bar{\sigma}_{ij} &= (-\delta_{ip})(-\delta_{jq})\sigma_{pq} = \sigma_{ij}; \quad \bar{\varepsilon}_{ij} = (-\delta_{ip})(-\delta_{jq})\varepsilon_{pq} = \varepsilon_{ij}, \\ \bar{C}_{ijk\ell} &= (-1)^4 \delta_{ip}\, \delta_{jq}\, \delta_{kr}\, \delta_{\ell s}\, C_{pqrs} = C_{ijk\ell}. \end{aligned}$$

Thus, the transformation $\ell_{ij} = -\delta_{ij}$ does not alter the constitutive relation (6.3.6) of triclinic materials.

6.3.3 Monoclinic Materials

When the elastic coefficients at a point have the same value for every pair of coordinate systems that are the mirror images of each other with respect to a plane, the material is called *monoclinic* at the point. For example, let (x_1, x_2, x_3) and $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ be two coordinates systems, with the x_1, x_2-plane parallel to the plane of symmetry. Choose the $\bar{x}_3$-axis such that $\bar{x}_3 = -x_3$ so that one system is the mirror image of the other, as shown in Fig. 6.3.1(b). This symmetry transformation can be expressed by the transformation matrix ($\bar{x}_1 = x_1$, $\bar{x}_2 = x_2$, $\bar{x}_3 = -x_3$)

$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \tag{6.3.15}$$

or

$$\ell_{\alpha\beta} = \delta_{\alpha\beta}, \quad \ell_{3\alpha} = 0, \quad \ell_{\alpha 3} = 0, \quad \ell_{33} = -1, \quad \text{for} \quad \alpha, \beta = 1, 2.$$

Then the stress and strain transformation equations

$$[\bar{\sigma}] = [L][\sigma][L]^{\mathrm{T}}, \quad [\bar{\varepsilon}] = [L][\varepsilon][L]^{\mathrm{T}}$$

give the relations

$$\begin{aligned} \bar{\sigma}_{ij} &= \sigma_{ij}, \quad \text{except for} \quad \bar{\sigma}_{13} = -\sigma_{13}, \quad \bar{\sigma}_{23} = -\sigma_{23}, \\ \bar{\varepsilon}_{ij} &= \varepsilon_{ij}, \quad \text{except for} \quad \bar{\varepsilon}_{13} = -\varepsilon_{13}, \quad \bar{\varepsilon}_{23} = -\varepsilon_{23}. \end{aligned}$$

Now consider the stress–strain relations

$$\begin{aligned} \sigma_{11} &= C_{1111}\,\varepsilon_{11} + C_{1122}\,\varepsilon_{22} + C_{1133}\,\varepsilon_{33} + C_{1123}\,\varepsilon_{23} + C_{1113}\,\varepsilon_{13} + C_{1112}\,\varepsilon_{12}, \\ \bar{\sigma}_{11} &= C_{1111}\,\bar{\varepsilon}_{11} + C_{1122}\,\bar{\varepsilon}_{22} + C_{1133}\,\bar{\varepsilon}_{33} + C_{1123}\,\bar{\varepsilon}_{23} + C_{1113}\,\bar{\varepsilon}_{13} + C_{1112}\,\bar{\varepsilon}_{12} \\ &= C_{1111}\,\varepsilon_{11} + C_{1122}\,\varepsilon_{22} + C_{1133}\,\varepsilon_{33} - C_{1123}\,\varepsilon_{23} - C_{1113}\,\varepsilon_{13} + C_{1112}\,\varepsilon_{12}. \end{aligned}$$

Since $\sigma_{11} = \bar{\sigma}_{11}$, from the preceding two relations it follows that

$$C_{1123}\,\varepsilon_{23} + C_{1113}\,\varepsilon_{13} = -C_{1123}\,\varepsilon_{23} - C_{1113}\,\varepsilon_{13},$$

which must hold for any independent set of strain components, ε_{23} and ε_{13}. This implies that $C_{1123} = 0$ and $C_{1113} = 0$. Similarly, from the constitutive relations for σ_{22} and $\bar{\sigma}_{22}$, σ_{33} and $\bar{\sigma}_{33}$, and σ_{12} and $\bar{\sigma}_{12}$ we obtain $C_{2223} = C_{2213} = 0$, $C_{3323} = C_{3313} = 0$, and $C_{1223} = C_{1213} = 0$.

Next consider the constitutive relations for σ_{23} and $\bar{\sigma}_{23}$ (note $C_{ijk\ell} = C_{k\ell ij}$)

$$\begin{aligned}
\sigma_{23} &= C_{2311}\,\varepsilon_{11} + C_{2322}\,\varepsilon_{22} + C_{2333}\,\varepsilon_{33} + C_{2323}\,\varepsilon_{23} + C_{2313}\,\varepsilon_{13} + C_{2312}\,\varepsilon_{12} \\
&= C_{2323}\,\varepsilon_{23} + C_{2313}\,\varepsilon_{13} \\
\bar{\sigma}_{23} &= C_{2311}\,\bar{\varepsilon}_{11} + C_{2322}\,\bar{\varepsilon}_{22} + C_{2333}\,\bar{\varepsilon}_{33} + C_{2323}\,\bar{\varepsilon}_{23} + C_{2313}\,\bar{\varepsilon}_{13} + C_{2312}\,\bar{\varepsilon}_{12} \\
&= -C_{2323}\,\varepsilon_{23} - C_{2313}\,\varepsilon_{13}.
\end{aligned}$$

Since $\sigma_{23} = -\bar{\sigma}_{23}$, these two relations are consistent. In the same way, no new conditions are obtained by considering the constitutive relations for σ_{13} and $\bar{\sigma}_{13}$.

In summary, for monoclinic materials, 8 of the 21 coefficients are zero:

$$C_{1123} = C_{1113} = C_{2223} = C_{2213} = C_{3323} = C_{3313} = C_{1223} = C_{1213} = 0.$$

Therefore, the stress–strain relations of Eq. (6.3.9) become

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ C_{12} & C_{22} & C_{23} & 0 & 0 & C_{26} \\ C_{13} & C_{23} & C_{33} & 0 & 0 & C_{36} \\ 0 & 0 & 0 & C_{44} & C_{45} & 0 \\ 0 & 0 & 0 & C_{45} & C_{55} & 0 \\ C_{16} & C_{26} & C_{36} & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix}, \qquad (6.3.16)$$

which has only 13 independent parameters. We note that monoclinic materials exhibit shear-extensional coupling, that is, a shear strain can produce a normal stress and a normal stress can produce a shear strain.

6.3.4 Orthotropic Materials

When three mutually orthogonal planes of material symmetry exist at a point, the number of elastic coefficients is reduced to nine using arguments similar to those given for a single material symmetry plane. Such materials are called *orthotropic* at the point. The transformation matrices associated with the three planes of symmetry are

$$[L^{(1)}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \; [L^{(2)}] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; [L^{(3)}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \qquad (6.3.17)$$

Under these transformations, we obtain $C_{1112} = C_{16} = 0$, $C_{2212} = C_{26} = 0$, $C_{3312} = C_{36} = 0$, and $C_{2313} = C_{45} = 0$. In view of the aforementioned result, the stress–strain relations for an orthotropic material take the form

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix}. \tag{6.3.18}$$

As stated earlier, in practice we apply stresses and determine the strains. Hence we must write the inverse of the relations in Eq. (6.3.18):

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix}. \tag{6.3.19}$$

The material compliance coefficients S_{ij} are often determined in a laboratory in terms of engineering material parameters such as Young's modulus, shear modulus, and so on. These constants are measured using simple tests such as a uniaxial tension test or a pure shear test. Because of their direct and obvious physical meaning, engineering constants are used in place of the abstract compliance coefficients S_{ij}. Next, we discuss how the compliance coefficients S_{ij} are determined in terms of the engineering parameters.

One of the consequences of linearity (both kinematic and material linearizations) is that the principle of superposition applies. That is, if the applied loads and geometric constraints are independent of deformation, the sum of the displacements (and hence strains) produced by two sets of loads is equal to the displacements (and strains) produced by the sum of the two sets of loads. In particular, the strains of the same kind as produced by the application of individual stress components can be superposed. For example, the extensional strain $\varepsilon_{11}^{(1)}$ in the material coordinate direction x_1 due to the stress σ_{11} in the same direction is σ_{11}/E_1, as shown in Fig. 6.3.2; here E_1 denotes Young's modulus of the material in the x_1 direction. The extensional strain $\varepsilon_{11}^{(2)}$, experienced as a result of the Poisson effect, due to the stress σ_{22} applied in the x_2 direction, is $-\nu_{21}\,(\sigma_{22}/E_2)$, where ν_{21} is Poisson's ratio (note that the first subscript in ν_{ij}, $i \neq j$, corresponds to the load direction and the second subscript refers to the direction of the strain)

$$\varepsilon_{11} = -\nu_{21}\varepsilon_{22} \quad \text{or} \quad \nu_{21} = -\frac{\varepsilon_{11}}{\varepsilon_{22}}\,,$$

and E_2 is Young's modulus of the material in the x_2 direction. Similarly, σ_{33} produces a strain $\varepsilon_{11}^{(3)}$ equal to $-\nu_{31}\,(\sigma_{33}/E_3)$. Therefore, the total strain ε_{11} due to the simultaneous application of all three normal stress components is

$$\begin{aligned} \varepsilon_{11} &= \varepsilon_{11}^{(1)} + \varepsilon_{11}^{(2)} + \varepsilon_{11}^{(3)} = \frac{\sigma_{11}}{E_1} - \nu_{21}\frac{\sigma_{22}}{E_2} - \nu_{31}\frac{\sigma_{33}}{E_3} \\ &= S_{11}\sigma_{11} + S_{12}\sigma_{22} + S_{13}\sigma_{33} \end{aligned} \tag{6.3.20}$$

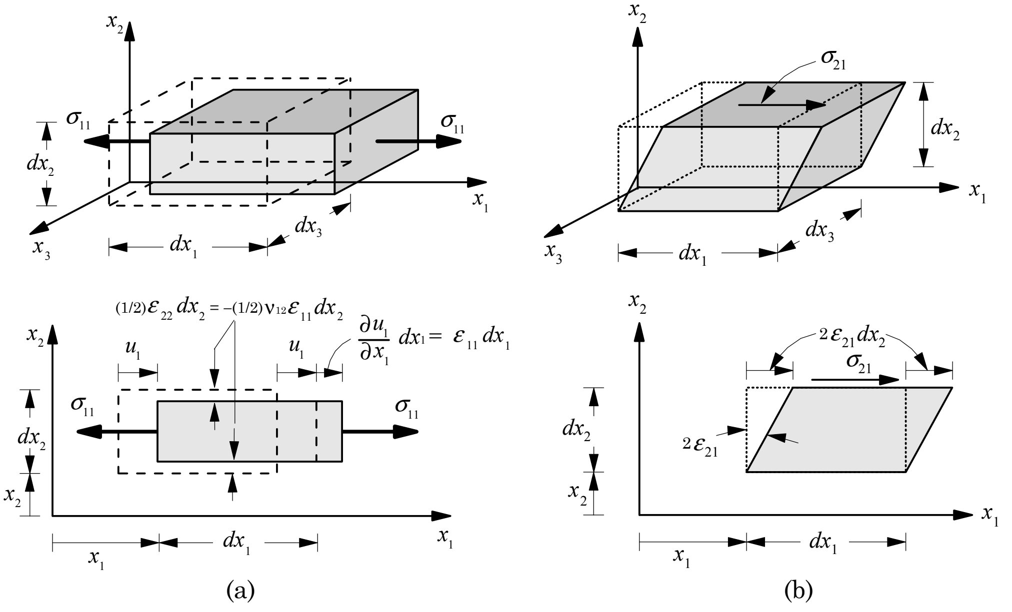

Fig. 6.3.2: Strains produced by applied (a) normal stress σ_{11} and (b) shear stress $\sigma_{21} = \sigma_{12}$ in a cube of material.

where the direction of loading is denoted by the superscript. Similarly, we can write

$$
\begin{aligned}
\varepsilon_{22} &= -\nu_{12}\frac{\sigma_{11}}{E_1} + \frac{\sigma_{22}}{E_2} - \nu_{32}\frac{\sigma_{33}}{E_3} = S_{21}\sigma_{11} + S_{22}\sigma_{22} + S_{23}\sigma_{33},\\
\varepsilon_{33} &= -\nu_{13}\frac{\sigma_{11}}{E_1} - \nu_{23}\frac{\sigma_{22}}{E_2} + \frac{\sigma_{33}}{E_3} = S_{31}\sigma_{11} + S_{32}\sigma_{22} + S_{33}\sigma_{33}.
\end{aligned}
\tag{6.3.21}
$$

The simple shear tests with an orthotropic material give the results

$$
2\varepsilon_{12} = \frac{\sigma_{12}}{G_{12}} = S_{66}\sigma_{12}, \quad 2\varepsilon_{13} = \frac{\sigma_{13}}{G_{13}} = S_{55}\sigma_{13}, \quad 2\varepsilon_{23} = \frac{\sigma_{23}}{G_{23}} = S_{44}\sigma_{23}. \tag{6.3.22}
$$

Recall from Section 3.5.2 that $2\varepsilon_{ij}\ \ (i \neq j)$ is the reduction in the right angle between two material lines parallel to the x_1 and x_2 directions at a point, σ_{ij} $(i \neq j)$ denotes the corresponding shear stress in the x_i–x_j plane, and $G_{ij}\ \ (i \neq j)$ is the shear moduli in the x_i–x_j plane.

Writing Eqs. (6.3.20)–(6.3.22) in matrix form, we obtain

$$
\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix}
=
\begin{bmatrix}
\frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{31}}{E_3} & 0 & 0 & 0 \\
-\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{32}}{E_3} & 0 & 0 & 0 \\
-\frac{\nu_{13}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{G_{13}} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}}
\end{bmatrix}
\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix}
\quad (\{\varepsilon\} = [S]\{\sigma\}), \tag{6.3.23}
$$

where E_1, E_2, and E_3 are Young's moduli in 1, 2, and 3 material directions, respectively; ν_{ij} is Poisson's ratio, defined as the ratio of transverse strain in the jth direction to the axial strain in the ith direction when stressed in the i-direction; and G_{23}, G_{13}, G_{12} are the shear moduli in the 2–3, 1–3, and 1–2 planes, respectively. Because $[S]$ is the inverse of $[C]$ and $[C]$ is symmetric, then $[S]$ is also a symmetric matrix. This in turn implies that the following reciprocal relations hold [i.e., compare the off-diagonal terms in Eq. (6.3.10)]:

$$\frac{\nu_{21}}{E_2} = \frac{\nu_{12}}{E_1}; \quad \frac{\nu_{31}}{E_3} = \frac{\nu_{13}}{E_1}; \quad \frac{\nu_{32}}{E_3} = \frac{\nu_{23}}{E_2},$$

or

$$\frac{\nu_{ij}}{E_i} = \frac{\nu_{ji}}{E_j}, \tag{6.3.24}$$

for $i, j = 1, 2, 3$. The nine independent material coefficients for an orthotropic material are

$$E_1,\ E_2,\ E_3,\ G_{23},\ G_{13},\ G_{12},\ \nu_{12},\ \nu_{13},\ \nu_{23}. \tag{6.3.25}$$

Inversion of the strain-stress relations (6.3.23) give the stress–strain relations in Eq. (6.3.18) with

$$\begin{aligned}
&C_{11} = \frac{E_1}{C_0}\left(1 - \nu_{23}\nu_{32}\right), \quad C_{12} = \frac{E_1}{C_0}\left(\nu_{21} + \nu_{23}\nu_{31}\right) = \frac{E_2}{C_0}\left(\nu_{12} + \nu_{13}\nu_{32}\right),\\
&C_{13} = \frac{E_1}{C_0}\left(\nu_{31} + \nu_{21}\nu_{32}\right) = \frac{E_3}{C_0}\left(\nu_{13} + \nu_{12}\nu_{23}\right), \quad C_{22} = \frac{E_2}{C_0}\left(1 - \nu_{13}\nu_{31}\right),\\
&C_{23} = \frac{E_2}{C_0}\left(\nu_{32} + \nu_{31}\nu_{12}\right) = \frac{E_3}{C_0}\left(\nu_{23} + \nu_{21}\nu_{13}\right), \quad C_{33} = \frac{E_3}{C_0}\left(1 - \nu_{12}\nu_{21}\right),\\
&C_{44} = G_{23} \quad C_{55} = G_{31}, \quad C_{66} = G_{12},\\
&C_0 = 1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{31}\nu_{13} - 2\nu_{21}\nu_{32}\nu_{13}.
\end{aligned} \tag{6.3.26}$$

The difference between ν_{12} and ν_{21} for an orthotropic material is illustrated in Fig. 6.3.3 with two cases of uniaxial stress for a square element of length a. First a stress σ_{11} is applied in the x_1-direction as shown in Fig. 6.3.3(a). The resulting strains are

$$\varepsilon_{11}^{(1)} = \frac{\sigma_{11}}{E_1}, \quad \varepsilon_{22}^{(1)} = -\frac{\nu_{12}}{E_1}\sigma_{11}, \tag{6.3.27}$$

where the direction of loading is denoted by the superscript and the negative sign indicates compression. Next, a stress σ_{22} is applied in the x_2-direction as shown in Fig. 6.3.3(b). The strains are

$$\varepsilon_{11}^{(2)} = -\frac{\nu_{21}}{E_2}\sigma_{22}, \quad \varepsilon_{22}^{(2)} = \frac{\sigma_{22}}{E_2}. \tag{6.3.28}$$

The displacements associated with each of the loads are

$$\begin{aligned}
\Delta_1^{(1)} &= a\frac{\sigma_{11}}{E_1}, & \Delta_2^{(1)} &= -a\frac{\nu_{12}}{E_1}\sigma_{11},\\
\Delta_1^{(2)} &= -a\frac{\nu_{21}}{E_2}\sigma_{22}, & \Delta_2^{(2)} &= a\frac{\sigma_{22}}{E_2},
\end{aligned} \tag{6.3.29}$$

and the reciprocal relation (6.3.24) gives, when $\sigma_{11} = \sigma_{22}$, the equality $\Delta_2^{(1)} = \Delta_1^{(2)}$, which is the statement of *Maxwell's reciprocity relation*, which is discussed in Section 7.4.3.

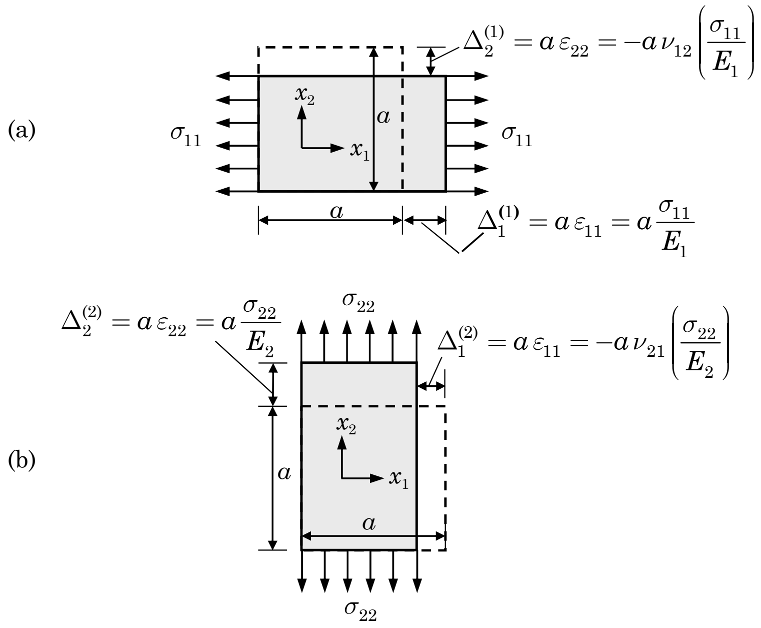

Fig. 6.3.3: Distinction between ν_{12} and ν_{21}. (a) Application of σ_{11}. (b) Application of σ_{22}.

6.3.5 Isotropic Materials

Isotropic materials are those for which the material properties are independent of the direction; that is, there exists an infinite number of material symmetry planes. An isotropic fourth-order tensor can be expressed as [see Eq. (2.5.24)]

$$C_{ijk\ell} = \lambda\delta_{ij}\delta_{k\ell} + \mu\big(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}\big) + \kappa\big(\delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk}\big), \tag{6.3.30}$$

where μ, λ, and κ are called the *Lamé constants*, and summation on repeated indices is implied. In view of the symmetry of $C_{ijk\ell}$ with respect to the first two and the last two indices, the coefficient of κ is zero, giving

$$C_{ijk\ell} = \lambda\delta_{ij}\delta_{k\ell} + 2\mu\delta_{ik}\delta_{j\ell}. \tag{6.3.31}$$

Therefore, Eq. (6.3.4) takes the simple form

$$\boldsymbol{\sigma} = 2\mu\,\boldsymbol{\varepsilon} + \lambda\,\mathrm{tr}(\boldsymbol{\varepsilon})\mathbf{I}, \quad \sigma_{ij} = 2\mu\,\varepsilon_{ij} + \lambda\,\varepsilon_{kk}\,\delta_{ij}, \tag{6.3.32}$$

where tr$(\cdot)$ denotes the *trace* (sum of the diagonal elements) of the enclosed tensor. Thus, only two material parameters, μ and λ, are needed to characterize the mechanical response of an isotropic material. The Lamé constants μ and λ are related to E and ν by

$$\mu = \frac{E}{2(1+\nu)}, \quad \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \quad 2\mu + \lambda = \frac{(1-\nu)E}{(1+\nu)(1-2\nu)}. \tag{6.3.33}$$

The stress–strain relations (6.3.32) can be expressed in terms of E and ν as

$$\boldsymbol{\sigma} = \frac{E}{1+\nu}\boldsymbol{\varepsilon} + \frac{\nu E}{(1+\nu)(1-2\nu)}\,\mathrm{tr}(\boldsymbol{\varepsilon})\,\mathbf{I}, \quad \sigma_{ij} = \frac{E}{1+\nu}\varepsilon_{ij} + \frac{\nu E}{(1+\nu)(1-2\nu)}\,\varepsilon_{kk}\,\delta_{ij}, \tag{6.3.34}$$

and the inverse relations are

$$\boldsymbol{\varepsilon} = \left(\tfrac{1+\nu}{E}\right)\boldsymbol{\sigma} - \tfrac{\nu}{E}\,\mathrm{tr}(\boldsymbol{\sigma})\,\mathbf{I}, \quad \varepsilon_{ij} = \left(\tfrac{1+\nu}{E}\right)\sigma_{ij} - \tfrac{\nu}{E}\,\sigma_{kk}\,\delta_{ij}, \tag{6.3.35}$$

The strain energy density $\rho_0 U_0$ in Eq. (6.3.1) for an isotropic material takes the form

$$\rho_0 U_0(\boldsymbol{\varepsilon}) = \frac{\lambda}{2}(\mathrm{tr}\,\boldsymbol{\varepsilon})^2 + \mu\,\mathrm{tr}(\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}), \quad \rho_0 U_0(\varepsilon_{ij}) = \mu\varepsilon_{ij}\varepsilon_{ij} + \tfrac{1}{2}\lambda(\varepsilon_{kk})^2. \tag{6.3.36}$$

Note that the strain energy density U_0 is positive-definite, that is,

$$U_0(\boldsymbol{\varepsilon}) > 0 \text{ whenever } \boldsymbol{\varepsilon} \neq \mathbf{0}, \text{ and } U_0(\boldsymbol{\varepsilon}) = 0 \text{ only when } \boldsymbol{\varepsilon} = \mathbf{0}. \tag{6.3.37}$$

The coefficients C_{ij} [see Eqs. (6.3.26) and (6.3.9) for the correspondence between the two-subscripted and four-subscripted C's] of Eq. (6.3.26) simplify to [with $E_1 = E_2 = E_3 = E$, $G_{12} = G_{13} = G_{23} = G = E/2(1+\nu)$, $\nu_{12} = \nu_{23} = \nu_{13} = \nu$, and $C_0 = 1 - 3\nu^2 - 2\nu^3 = (1+\nu)^2(1-2\nu)$]

$$\begin{aligned} C_{11} &= C_{22} = C_{33} = 2\mu + \lambda = \frac{(1-\nu)E}{(1+\nu)(1-2\nu)}, \\ C_{12} &= C_{13} = C_{23} = \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \\ C_{44} &= C_{55} = C_{66} = \mu = G = \frac{E}{2(1+\nu)}. \end{aligned} \tag{6.3.38}$$

The stress–strain relations (6.3.34) for an isotropic material can be expressed in matrix form as

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix}. \tag{6.3.39}$$

If the only nonzero normal stress component is $\sigma_{11} = \sigma$ and the only nonzero shear component is $\sigma_{12} = \tau$, and if we denote $\varepsilon_{11} = \varepsilon$ and $2\varepsilon_{12} = \gamma$, then Eq. (6.3.35) gives the uniaxial strain–stress relations,

$$\varepsilon = \frac{1}{E}\sigma \quad \rightarrow \quad \sigma = E\,\varepsilon; \quad \gamma = \frac{2(1+\nu)}{E}\tau \quad \rightarrow \quad \tau = G\,\gamma. \tag{6.3.40}$$

In summary, application of a normal stress to a rectangular block of isotropic or orthotropic material results in only extension in the direction of the applied stress and contraction perpendicular to it, whereas a monoclinic (or anisotropic) material experiences extension in the direction of the applied normal stress, contraction perpendicular to it, and shearing strain, as shown in Fig. 6.3.4. Conversely, the application of a shearing stress to a monoclinic material causes shearing strain as well as normal strains. Also, a normal stress applied to an orthotropic material at an angle to its principal material directions causes it to behave like a monoclinic material.

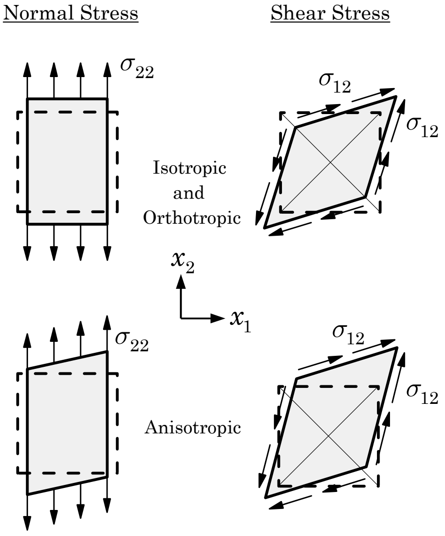

Fig. 6.3.4: Deformation of orthotropic and anisotropic rectangular blocks under uniaxial tension and pure shear.

Example 6.3.1

Consider the thin, filament-wound, closed circular cylindrical pressure vessel in Example 4.3.2, as shown in Fig. 6.3.5. The vessel has an internal diameter of $D_i = 63.5$ cm (25 in.), thickness $h = 2$ cm (0.7874 in.), and pressurized to $p = 1.379$ MPa (200 psi). Assuming a two-dimensional state of stress, determine

(a) stresses σ_{xx}, σ_{yy}, and σ_{xy} in the problem coordinates (x, y, z);

(b) stresses σ_{11}, σ_{22}, and σ_{12} in the material coordinates (x_1, x_2, x_3), with x_1 being tangent to the filament direction;

(c) strains ε_{11}, ε_{22}, and $2\varepsilon_{12}$ in the material coordinates; and

(d) strains ε_{xx}, ε_{yy}, and γ_{xy} in the problem coordinates.

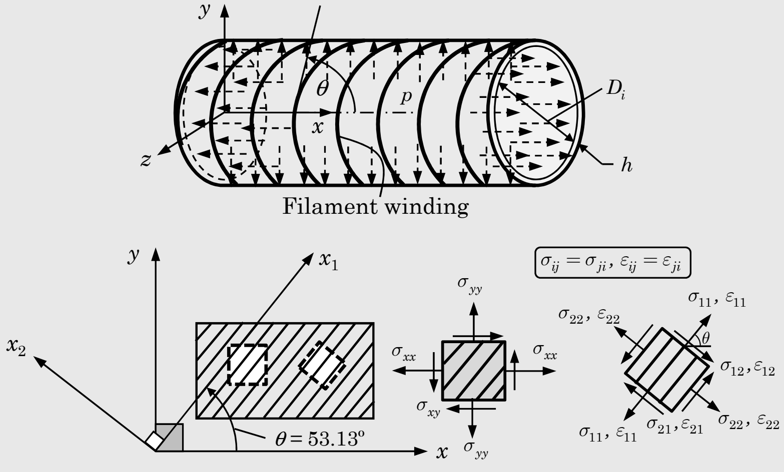

Fig. 6.3.5: A filament-wound cylindrical pressure vessel.

Use a filament winding angle of $\theta = 53.125^\circ$ from the longitudinal axis (x) of the pressure vessel and the following material properties with respect to the material coordinates (typical of graphite-epoxy material): $E_1 = 140$ GPa (20.3 Msi), $E_2 = 10$ GPa (1.45 Msi), $G_{12} = 7$ GPa (1.02 Msi), and $\nu_{12} = 0.3$.

Solution: (a) In Example 4.3.2, the longitudinal stress (σ_{xx}) and circumferential stress (σ_{yy}) in the thin-walled cylindrical pressure vessel were calculated using the formulas

$$\sigma_{xx} = \frac{pD_i}{4h}, \quad \sigma_{yy} = \frac{pD_i}{2h}, \quad \sigma_{xy} = 0. \tag{1}$$

to be $\sigma_{xx} = 10.946$ MPa and $\sigma_{yy} = 21.892$ MPa.

(b) Next, we determine the stresses in the material coordinates (so that we have the shear stress σ_{12} at the fiber–matrix interface, tensile stress σ_{11} in the fiber, and the stress σ_{22} normal to the fiber) using the stress transformation equations (4.3.7)

$$\begin{aligned}
\sigma_{11} &= \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + \sigma_{xy}\,\sin 2\theta,\\
\sigma_{22} &= \sigma_{xx}\sin^2\theta + \sigma_{yy}\cos^2\theta - \sigma_{xy}\,\sin 2\theta,\\
\sigma_{12} &= \tfrac{1}{2}\left(\sigma_{yy} - \sigma_{xx}\right)\sin 2\theta + \sigma_{xy}\,\cos 2\theta.
\end{aligned} \tag{2}$$

We obtain ($\sin\theta = 0.8$, $\cos\theta = 0.6$, $\sin 2\theta = 0.96$, and $\cos 2\theta = -0.28$ for $\theta = 53.13^\circ$)

$$\begin{aligned}
\sigma_{11} &= 10.946 \times (0.6)^2 + 21.892 \times (0.8)^2 = 17.951 \text{ MPa},\\
\sigma_{22} &= 10.946 \times (0.8)^2 + 21.892 \times (0.6)^2 = 14.886 \text{ MPa},\\
\sigma_{12} &= \tfrac{1}{2}\,(21.892 - 10.946) \times 0.96 = 5.254 \text{ MPa}.
\end{aligned}$$

(c) The strains in the material coordinates can be calculated using the strain–stress relations (6.3.23). We have ($\nu_{21}/E_2 = \nu_{12}/E_1$)

$$\varepsilon_{11} = \frac{\sigma_{11}}{E_1} - \nu_{12}\frac{\sigma_{22}}{E_1} = \frac{17.95 \times 10^6}{140 \times 10^9} - 0.3\,\frac{14.885 \times 10^6}{140 \times 10^9} = 0.0963 \times 10^{-3} \text{ m/m},$$

$$\varepsilon_{22} = -\nu_{12}\frac{\sigma_{11}}{E_1} + \frac{\sigma_{22}}{E_2} = -0.3\,\frac{17.95 \times 10^6}{140 \times 10^9} + \frac{14.885 \times 10^6}{10 \times 10^9} = 1.4502 \times 10^{-3} \text{ m/m},$$

$$\varepsilon_{12} = \frac{\sigma_{12}}{2G_{12}} = \frac{5.254 \times 10^6}{2 \times 7 \times 10^9} = 0.3753 \times 10^{-3}.$$

(d) The strains in the (x, y) coordinates can be computed using the transformation equations [see Eq. (3.4.32)]

$$\begin{aligned}
\varepsilon_{xx} &= \varepsilon_{11}\cos^2\theta + \varepsilon_{22}\sin^2\theta - \varepsilon_{12}\,\sin 2\theta,\\
\varepsilon_{yy} &= \varepsilon_{11}\sin^2\theta + \varepsilon_{22}\cos^2\theta + \varepsilon_{12}\,\sin 2\theta,\\
\varepsilon_{xy} &= \tfrac{1}{2}\left(\varepsilon_{11} - \varepsilon_{22}\right)\sin 2\theta + \varepsilon_{12}\,\cos 2\theta.
\end{aligned} \tag{3}$$

We obtain

$$\begin{aligned}
\varepsilon_{xx} &= 10^{-3}\left[0.0963 \times (0.6)^2 + 1.4502 \times (0.8)^2 - 0.3753 \times 0.96\right] = 0.6023 \times 10^{-3} \text{ m/m},\\
\varepsilon_{yy} &= 10^{-3}\left[0.0963 \times (0.8)^2 + 1.4502 \times (0.6)^2 + 0.3753 \times 0.96\right] = 0.9440 \times 10^{-3} \text{ m/m},\\
\varepsilon_{xy} &= 10^{-3}\left[(0.0963 - 1.4502) \times 0.48 + 0.3753 \times (-0.28)\right] = -0.7549 \times 10^{-3}.
\end{aligned}$$

The strains $(\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{xy})$ can also be determined directly from the stresses $(\sigma_{xx}, \sigma_{yy}, \sigma_{xy})$ using the strain–stress relations

$$\begin{aligned}
\varepsilon_{xx} &= \bar{S}_{11}\,\sigma_{xx} + \bar{S}_{12}\,\sigma_{yy} + \bar{S}_{16}\,\sigma_{xy},\\
\varepsilon_{yy} &= \bar{S}_{12}\,\sigma_{xx} + \bar{S}_{22}\,\sigma_{yy} + \bar{S}_{26}\,\sigma_{xy},\\
\varepsilon_{xy} &= \bar{S}_{16}\,\sigma_{xx} + \bar{S}_{26}\,\sigma_{yy} + \bar{S}_{66}\,\sigma_{xy},
\end{aligned} \tag{4}$$

where $\bar{S}_{ij}$ are the transformed elastic compliances referred to the problem coordinates (x, y, z). A transformation law consistent with the tensor transformation equations in Eq. (6.3.13) must be used to write $\bar{S}_{ij}$ in terms of S_{ij} and the angle θ. See the answer to Problem **6.2** for the transformation relations between $\bar{S}_{ij}$ and S_{ij} and between $\bar{C}_{ij}$ and C_{ij}.

6.4 Nonlinear Elastic Constitutive Relations

Most materials exhibit nonlinear elastic behavior for certain strain thresholds, that is, the stress–strain relation is no longer linear, but recovers all its deformation upon the removal of the loads, and Hooke's law is no longer valid. Beyond certain nonlinear elastic range, permanent deformation ensues and the material is said to be inelastic or plastic, as shown in Fig. 6.4.1. Here we briefly review constitutive relations for two well-known nonlinear elastic materials, namely the Mooney–Rivlin and neo-Hookean materials. Further discussion can be found in Truesdell and Noll (1965).

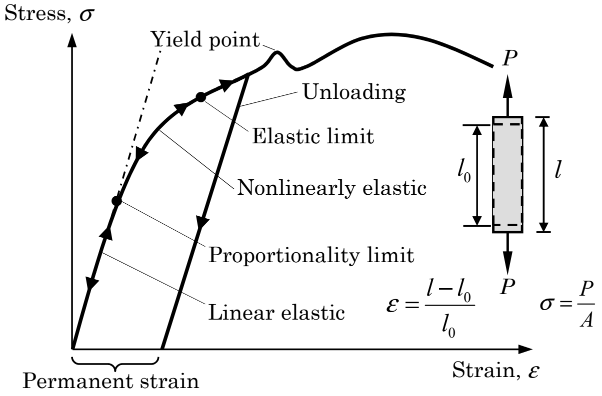

Fig. 6.4.1: A typical stress–strain curve.

Recall from Eq. (6.2.8) that for a hyperelastic material under isothermal conditions there exists a strain energy potential $\Psi = \Psi(\mathbf{F})$ such that

$$\boldsymbol{\sigma}(\mathbf{F}) = \rho \frac{\partial \Psi}{\partial \mathbf{F}} \cdot \mathbf{F}^{\mathrm{T}}, \tag{6.4.1}$$

where ρ is the material density. Some materials (e.g., rubber-like materials) undergo large deformations without appreciable change in volume (i.e., $J \approx 1$). Such materials are called *incompressible* materials. For incompressible elastic materials, the stress tensor is not completely determined by deformation. The hydrostatic pressure p affects the stress. For incompressible elastic materials, we have [see Eq. (6.2.12)]

$$\boldsymbol{\sigma}(\mathbf{F}) = -p\mathbf{I} + \rho \frac{\partial \Psi}{\partial \mathbf{F}} \cdot \mathbf{F}^{\mathrm{T}}, \tag{6.4.2}$$

where p is the hydrostatic pressure.

For a hyperelastic elastic material, Eq. (6.4.1) can also be expressed as

$$\boldsymbol{\sigma}(\mathbf{B}) = 2\rho \frac{\partial \Psi}{\partial \mathbf{B}} \cdot \mathbf{B}, \tag{6.4.3}$$

where the free energy potential Ψ is written as $\Psi = \Psi(\mathbf{B})$ and $\mathbf{B}$ is the left Cauchy–Green (or Finger) tensor $\mathbf{B} = \mathbf{F} \cdot \mathbf{F}^{\mathrm{T}}$ [see Eq. (3.4.4)]. Equations

(6.4.1)–(6.4.3), in general, are nonlinear. The free energy potential Ψ takes different forms for different materials. It is often expressed as a linear combination of unknown parameters and principal invariants of Green strain tensor $\mathbf{E}$, deformation gradient $\mathbf{F}$, or left Cauchy–Green deformation tensor $\mathbf{B}$. The parameters characterize the material and they are determined through suitable experiments.

For incompressible materials, the free energy potential Ψ is taken as a linear function of the principal invariants of $\mathbf{B}$:

$$\Psi = C_1(I_B - 3) + C_2(II_B - 3), \tag{6.4.4}$$

where C_1 and C_2 are constants and I_B and II_B are the two principal invariants of $\mathbf{B}$ (the third invariant III_B is equal to unity for incompressible materials). Materials for which the free energy potential is given by Eq. (6.4.4) are known as the *Mooney–Rivlin material.* The stress tensor in this case has the form

$$\boldsymbol{\sigma} = -p\,\mathbf{I} + \alpha\,\mathbf{B} + \beta\,\mathbf{B}^{-1}, \tag{6.4.5}$$

where α and β are given by

$$\alpha = 2\rho\,\frac{\partial \Psi}{\partial I_B} = 2\rho\,C_1, \quad \beta = -2\rho\,\frac{\partial \Psi}{\partial II_B} = -2\rho\,C_2. \tag{6.4.6}$$

The Mooney–Rivlin incompressible material model is most commonly used to represent the stress–strain behavior of rubber-like solid materials.

If the free energy potential is of the form $\Psi = C_1(I_B - 3)$, that is, $C_2 = 0$, the constitutive equation in Eq. (6.4.5) takes the form

$$\boldsymbol{\sigma} = -p\,\mathbf{I} + 2\rho\,C_1\,\mathbf{B}. \tag{6.4.7}$$

Materials whose constitutive behavior is described by Eq. (6.4.7) are called the *neo-Hookean materials.* The neo-Hookean model provides a reasonable prediction of the constitutive behavior of natural rubber for moderate strains.

6.5 Newtonian Fluids

6.5.1 Introduction

All bulk matter in nature exists in one of two forms (even before they are subjected to forces): solid or fluid. A solid body is characterized by relative immobility of its molecules, whereas a fluid state is characterized by their relative mobility. Fluids can exist either as gases or liquids. In this section we present the constitutive relations for fluids that exhibit the property that stress is proportional to velocity gradients, that is, strain rates. The proportionality parameter is known as the *viscosity* of the fluid, and the relationship is known *Newton's law of viscosity.* Fluids that behave according to Newton's law of viscosity are called *Newtonian fluids.* For such fluids, the constitutive equations for $\boldsymbol{\sigma}$ cannot be derived using the condition (6.1.6) resulting from the entropy inequality. The physics of such fluids requires the symmetric part of the velocity gradient, $\mathbf{D}$, to be an argument, in place of $\mathbf{F}$ used for solids, in the list of dependent variables.

It is assumed that the Eulerian description is used to derive all equations of mechanics.

For viscous fluids the total stress $\boldsymbol{\sigma}$ is decomposed into equilibrium and deviatoric parts. Then the conditions resulting from the entropy inequality permit derivation of constitutive relations for compressible fluids with the equilibrium stress as thermodynamic pressure $p(\rho, \theta)$, which for an incompressible fluid becomes the mechanical pressure $p(\theta)$; the entropy inequality only places the restriction that work done by the deviatoric stress be positive but provides no mechanism for its constitutive relation. Thus, $\boldsymbol{\sigma}$ is assumed to be of the form[3]

$$\begin{aligned} \boldsymbol{\sigma} &= -p(\rho,\theta)\,\mathbf{I} + \boldsymbol{\mathcal{F}}(\mathbf{D}), \text{ for compressible fluids,} \\ \boldsymbol{\sigma} &= -p(\theta)\,\mathbf{I} + \boldsymbol{\mathcal{F}}(\mathbf{D}), \text{ for incompressible fluids,} \end{aligned} \tag{6.5.1}$$

where ρ is the spatial density and θ is the absolute temperature. A fluid is said to be *incompressible* if the volume change is zero:

$$\nabla \cdot \mathbf{v} = 0, \tag{6.5.2}$$

where $\mathbf{v}$ is the velocity vector. A fluid is termed *inviscid* if the viscosity is zero.

6.5.2 Ideal Fluids

An *ideal fluid* is one that is incompressible and has zero viscosity. The most general constitutive equation for an ideal fluid is of the form

$$\boldsymbol{\sigma} = -p(\rho, \theta)\mathbf{I}. \tag{6.5.3}$$

The dependence of p on ρ and θ has been experimentally verified many times over several centuries. The thermomechanical properties of an ideal fluid are the same in all directions, that is, the fluid is isotropic. It can be verified that Eq. (6.5.3) satisfies the frame indifference requirement because

$$\boldsymbol{\sigma}^* = \mathbf{Q} \cdot \boldsymbol{\sigma} \cdot \mathbf{Q}^{\mathrm{T}} = -p\,\mathbf{Q} \cdot \mathbf{I} \cdot \mathbf{Q}^{\mathrm{T}} = -p\,\mathbf{I}.$$

An explicit functional form of $p(\rho, \theta)$, valid for gases over a wide range of temperature and density, is

$$p = R\rho\theta/m, \tag{6.5.4}$$

where R is the universal gas constant, m is the mean molecular mass of the gas, and θ is the absolute temperature. Equation (6.5.4) is known to define a *perfect gas*. When p is only a function of density, the fluid is said to be "barotropic," and the barotropic constitutive model is applicable under isothermal conditions. If p is independent of both ρ and θ (ρ = constant), p is determined from the equations of motion [see Eq. (5.3.14)].

---

[3] The dependence of $\boldsymbol{\mathcal{F}}$ on the vorticity tensor $\mathbf{W}$ is eliminated to satisfy the frame indifference requirement.

6.5.3 Viscous Incompressible Fluids

The general constitutive equation for stress tensor in a fluid motion is assumed to be of the general form in Eq. (6.5.1). Analogous to isotropic materials, isotropic fluids are those for which the shear stress–strain rate relations are of the form [compare with Eq. (6.3.4)]

$$\boldsymbol{\mathcal{F}}(\mathbf{D}) \equiv \boldsymbol{\tau} = \mathbf{C} : \mathbf{D} \quad (\tau_{ij} = C_{ijk\ell}\, D_{ij}). \tag{6.5.5}$$

Here $\boldsymbol{\tau}$ is the viscous stress tensor, $\mathbf{C}$ denotes the fourth-order tensor of viscosities, and $\mathbf{D}$ is the strain rate tensor [symmetric part of the velocity gradient tensor $\mathbf{L}$; see Eqs. (3.6.1) and (3.6.2)]:

$$\mathbf{D} = \tfrac{1}{2}\left[\boldsymbol{\nabla}\mathbf{v} + (\boldsymbol{\nabla}\mathbf{v})^{\mathrm{T}}\right] \quad \left[D_{ij} = \tfrac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\right], \tag{6.5.6}$$

where $\mathbf{v}$ is the velocity vector.

In a majority of cases a viscous fluid is characterized as an isotropic fluid. For an isotropic fluid, we have the constitutive relation [compare with Eq. (6.3.35)]

$$\boldsymbol{\sigma} = -p(\rho,\theta)\,\mathbf{I} + \boldsymbol{\tau}, \quad \boldsymbol{\tau} = 2\mu(\rho,\theta)\,\mathbf{D} + \lambda(\rho,\theta)\,\mathrm{tr}(\mathbf{D})\mathbf{I}, \tag{6.5.7}$$

or in rectangular Cartesian component form

$$\sigma_{ij} = -p\,\delta_{ij} + \tau_{ij}, \quad \tau_{ij} = 2\mu\, D_{ij} + \lambda\, D_{kk}\,\delta_{ij}, \tag{6.5.8}$$

where ρ is the spatial density, θ is the absolute temperature, and λ and μ are the Lamé parameters that have the meaning *bulk viscosity* and *shear viscosity*, respectively.

Equations (6.5.7) and (6.5.8) can be expressed in terms of the deviatoric components of stress and rate of deformation tensors,

$$\boldsymbol{\sigma}' = \boldsymbol{\sigma} - \tilde{\sigma}\mathbf{I}, \quad \mathbf{D}' = \mathbf{D} - \tfrac{1}{3}\,\mathrm{tr}(\mathbf{D})\,\mathbf{I}, \quad \tilde{\sigma} = \tfrac{1}{3}\,\mathrm{tr}(\boldsymbol{\sigma}). \tag{6.5.9}$$

We note that $\sigma'_{ii} = 0$ and $D'_{ii} = 0$. Then the Newtonian constitutive equation (6.5.8) takes the form

$$\begin{aligned} \boldsymbol{\sigma}' &= 2\mu\mathbf{D}' + \left(\tfrac{2}{3}\mu + \lambda\right)\mathrm{tr}(\mathbf{D})\,\mathbf{I} - (\tilde{\sigma} + p)\,\mathbf{I}, \\ \sigma'_{ij} &= 2\mu D'_{ij} + \left(\tfrac{2}{3}\mu + \lambda\right) D_{kk}\delta_{ij} - (\tilde{\sigma} + p)\,\delta_{ij}, \end{aligned} \tag{6.5.10}$$

from which it follows (because $\sigma'_{ii} = 0$ and $D'_{ii} = 0$) that

$$(2\mu + 3\lambda)\, D_{kk} - 3\,(\tilde{\sigma} + p) = 0. \tag{6.5.11}$$

Hence, the last two terms in Eq. (6.5.10) vanish together and we obtain

$$\boldsymbol{\sigma}' = 2\mu\mathbf{D}', \quad \sigma'_{ij} = 2\mu D'_{ij}. \tag{6.5.12}$$

Note that the mean stress $\tilde{\sigma}$ is equal to the thermodynamic pressure $-p$ if and only if one of the following two conditions is satisfied ($D_{kk} = \boldsymbol{\nabla}\cdot\mathbf{v}$):

$$\text{Fluid is incompressible:} \qquad \boldsymbol{\nabla}\cdot\mathbf{v} = 0, \tag{6.5.13}$$

$$\text{Stokes condition:} \qquad K = 2\mu + 3\lambda = 0. \tag{6.5.14}$$

In general, the Stokes condition does not hold. Thus, the constitutive equation for viscous, isotropic, incompressible fluids reduces to

$$\boldsymbol{\sigma} = -p\,\mathbf{I} + \boldsymbol{\tau}, \quad \boldsymbol{\tau} = 2\mu\,\mathbf{D}, \quad (\sigma_{ij} = -p\,\delta_{ij} + \tau_{ij}, \quad \tau_{ij} = 2\mu\,D_{ij}), \tag{6.5.15}$$

and p represents the mean normal stress or *hydrostatic pressure.* For inviscid fluids, the constitutive equation for the stress tensor has the form

$$\boldsymbol{\sigma} = -p\,\mathbf{I} \quad (\sigma_{ij} = -p\,\delta_{ij}). \tag{6.5.16}$$

We note that Eq. (6.5.15) does not hold for compressible fluids, unless the Stokes condition (6.5.14) is satisfied. Equation (6.5.7) is valid for compressible fluids, with p being the thermodynamic pressure.

6.6 Generalized Newtonian Fluids

6.6.1 Introduction

Fluids for which the viscosity of the fluid may be a function of the strain rate tensor (or its invariants) but the form of the constitutive equations is similar to those of the Newtonian fluid are called *generalized Newtonian fluids.* Generalized Newtonian fluids include motor oils and high molecular weight liquids such as polymers, slurries, pastes, and other complex mixtures. The processing and transporting of such fluids are central problems in the chemical, food, plastics, petroleum, and polymer industries. We note that the generalized Newtonian constitutive models presented in this section for viscous fluids are only a few of the many available in literature [see Reddy and Gartling (2001)].

Most generalized Newtonian fluids exhibit a shear rate dependent viscosity, with “shear thinning” characteristic (i.e., decreasing viscosity with increasing shear rate). Other characteristics associated with generalized Newtonian fluids are elasticity, memory effects, the Weissenberg effect, and the curvature of the free surface in an open channel flow. A discussion of these and other non-Newtonian effects is presented in the book by Bird, Armstrong, and Hassager (1971).

Generalized Newtonian fluids can be classified into two groups: (1) inelastic fluids or fluids without memory and (2) viscoelastic fluids, in which memory effects are significant. For inelastic fluids the viscosity depends on the rate of deformation of the fluid, much like nonlinear elastic solids. Viscoelastic fluids exhibit time-dependent “memory”; that is, the motion of a material point depends not only on the present stress state, but also on the deformation history of the material element. This history dependence leads to very complex constitutive equations.

The constitutive equation for the stress tensor for a generalized Newtonian fluid can be expressed as

$$\boldsymbol{\sigma} = -p\,\mathbf{I} + \boldsymbol{\tau} \quad (\sigma_{ij} = -p\,\delta_{ij} + \tau_{ij}), \tag{6.6.1}$$

where $\boldsymbol{\tau}$ is known as the viscous or extra stress tensor.

6.6.2 Inelastic fluids

The viscosity for inelastic fluids is found to depend on the rate of deformation tensor **D**. Often the viscosity is expressed as a function of the principal invariants of the rate of deformation tensor **D**

$$\mu = \mu\left(J_1, J_2, J_3\right), \tag{6.6.2}$$

where the J_1, J_2, and J_3 are the principal invariants of **D**,

$$\begin{aligned} J_1 &= \operatorname{tr}\left(\mathbf{D}\right) = D_{ii}, \\ J_2 &= \frac{1}{2}\operatorname{tr}\left(\mathbf{D}^2\right) = \frac{1}{2}D_{ij}D_{ij}, \\ J_3 &= \frac{1}{3}\operatorname{tr}\left(\mathbf{D}^3\right) = \frac{1}{3}D_{ij}D_{jk}D_{ki}, \end{aligned} \tag{6.6.3}$$

where $\operatorname{tr}(\cdot)$ denotes the trace of the enclosed tensor. Note that J_2 and J_3 defined above are different from $J_2 = \frac{1}{2}\left(J_1^2 - \mathbf{D}:\mathbf{D}\right)$ and $J_3 = |\mathbf{D}|$ defined in Eq. (3.4.36).

For an incompressible fluid, $J_1 = \boldsymbol{\nabla}\cdot\mathbf{v} = 0$. Also, there is no theoretical or experimental evidence to suggest that the viscosity depends on J_3; thus, the dependence on the third invariant is eliminated. Equation (6.6.2) reduces to

$$\mu = \mu(J_2). \tag{6.6.4}$$

The viscosity can also depend on the thermodynamic state of the fluid, which for incompressible fluids usually implies a dependence only on the temperature. Equation (6.6.4) gives the general functional form for the viscosity function, and experimental observations and a limited theoretical base are used to provide specific forms of Eq. (6.6.4) for non-Newtonian viscosities. A variety of inelastic models have been proposed and correlated with experimental data, as discussed by Bird et al. (1971). Several of the most useful and popular models are presented next; see Reddy and Gartling (2001).

6.6.2.1 Power-law model

The simplest and most familiar non-Newtonian viscosity model is the power-law model, which has the form

$$\mu = K J_2^{(n-1)/2}, \tag{6.6.5}$$

where n and K are parameters, which are, in general, functions of temperature; n is termed the *power-law index* and K is called *consistency.* Fluids with an index $n < 1$ are termed *shear thinning* or *pseudoplastic.* A few materials are *shear thickening* or *dilatant* and have an index $n > 1$. The Newtonian viscosity is obtained with $n = 1$. The admissible range of the index n is bounded below by zero because of stability considerations.

When considering nonisothermal flows, the following empirical relations for n and K are used:

$$n = n_0 + B\left(\frac{\theta - \theta_0}{\theta_0}\right), \tag{6.6.6}$$

$$K = K_0 \exp\left(-A[\theta - \theta_0]/\theta_0\right), \tag{6.6.7}$$

where θ denotes the temperature and the subscript 0 indicates a reference value; A and B are material constants.

6.6.2.2 Carreau model

A major deficiency in the power-law model is that it fails to predict upper and lower limiting viscosities for extreme values of the deformation rate. This problem is alleviated in the Carreau model:

$$\mu = \mu_\infty + (\mu_0 - \mu_\infty)\left(1 + [\lambda\, J_2]^2\right)^{(n-1)/2}, \tag{6.6.8}$$

wherein μ_0 and μ_∞ are the initial and infinite shear rate viscosities, respectively, and λ is a time constant.

6.6.2.3 Bingham model

The *Bingham fluid* differs from most other fluids in that it can sustain an applied stress without fluid motion occurring. The fluid possesses a yield stress, τ_0, such that when the applied stresses are below τ_0 no motion occurs; when the applied stresses exceed τ_0 the material flows, with the viscous stresses being proportional to the excess of the stress over the yield condition. Typically, the constitutive equation after yield is taken to be Newtonian (Bingham model), though other forms such as a power-law equation are possible. In a general form, the Bingham model can be expressed as

$$\boldsymbol{\tau} = \left(\frac{\tau_0}{\sqrt{J_2}} + 2\mu\right)\mathbf{D} \quad \text{when} \quad \frac{1}{2}\mathrm{tr}(\boldsymbol{\tau}^2) \geq \tau_0^2, \tag{6.6.9}$$

$$\boldsymbol{\tau} = 0 \quad \text{when} \quad \frac{1}{2}\mathrm{tr}(\boldsymbol{\tau}^2) < \tau_0^2. \tag{6.6.10}$$

From Eq. (6.6.9) the apparent viscosity of the material beyond the yield point is $\left(\tau_0/\sqrt{J_2} + 2\mu\right)$. For a Herschel–Buckley fluid the μ in Eq. (6.6.9) is given by Eq. (6.6.5). The inequalities in Eqs. (6.6.9) and (6.6.10) describe a von Mises yield criterion.

6.6.3 Viscoelastic Constitutive Models

For a viscoelastic fluid, the constitutive equation for the extra-stress $\boldsymbol{\tau}$ in Eq. (6.6.1) is time dependent. Such a relationship is often expressed in abstract form where the current extra-stress is related to the history of deformation in the fluid as

$$\boldsymbol{\tau} = \mathcal{F}\left[\mathbf{G}(s),\ 0 < s < \infty\right], \tag{6.6.11}$$

where $\mathcal{F}$ is a tensor-valued functional, $\mathbf{G}$ is a finite deformation tensor (related to the Cauchy–Green tensor), and $s = t - t'$ is the time lapse from time t' to the present time, t. Fluids that obey constitutive equations of the form in Eq. (6.6.11) are called *simple fluids.* The functional form in Eq. (6.6.11) is not useful for general flow problems, and therefore numerous approximations of

Eq. (6.6.11) have been proposed in several different forms. Several of them are reviewed here.

The two major categories of approximate constitutive relations include the differential and integral models. For a differential model the extra-stress is determined from a differential equation that relates the stress and stress rate to the flow kinematics. The integral model represents the extra-stress in terms of an integral over past time of the fluid deformation history. In general, the specific choice is dictated by the ability of a given model to predict the non-Newtonian effects expected in a particular application.

6.6.3.1 Differential models

Constitutive models for viscoelastic fluids in differential equation form are preferable due to the ease with which they can be incorporated into the conservation and balance equations, and the resulting equations are simple to handle in a computational framework. Due to rheology of the fluid (fading memory) the deviatoric part of the stress tensor, called *extra-stress tensor*, is time dependent, and thus the constitutive models are differential equations in time between deviatoric stress tensor and the strain rate tensor.

The constitutive models for viscoelastic fluids in differential form can also be constructed using a purely phenomenological approach based on our understanding of the physics. Such models, for example, one-dimensional spring and dash-pot models discussed for viscoelastic solids in Chapter 9, serve to describe the observed physical response. However, such models do not have a thermodynamic basis, and their extension to two and three dimensions is based on an analogy with elastic constitutive relations.

In this section, we consider differential constitutive theories using the deviatoric stress tensor (derived using the theory of generators and invariants) for fluids. The well-known differential constitutive equations are generally associated with Oldroyd, Maxwell, and Jeffrey. First we define various types of material time derivatives used in these models. In the spatial description the material time derivative of a symmetric second-order tensor can be defined in several ways, all of which are frame invariant. Let $\mathbf{S}$ denote a second-order tensor. Then the *upper-convected* (or co-deformational or contravariant) derivative is defined by

$$\overset{\nabla}{\mathbf{S}} = \frac{\partial \mathbf{S}}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} \mathbf{S} - \mathbf{L} \cdot \mathbf{S} - (\mathbf{L} \cdot \mathbf{S})^{\mathrm{T}}, \tag{6.6.12}$$

and the *lower-convected* derivative is defined as

$$\overset{\Delta}{\mathbf{S}} = \frac{\partial \mathbf{S}}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} \mathbf{S} + \mathbf{L}^{\mathrm{T}} \cdot \mathbf{S} + \mathbf{S}^{\mathrm{T}} \cdot \mathbf{L}, \tag{6.6.13}$$

where $\mathbf{v}$ is the velocity vector and $\mathbf{L}$ is the velocity gradient tensor

$$\mathbf{L} = (\boldsymbol{\nabla} \mathbf{v})^{\mathrm{T}} \quad \left(L_{ij} = \frac{\partial v_i}{\partial x_j} \right). \tag{6.6.14}$$

Since both Eqs. (6.6.12) and (6.6.13) are objective (not shown here) convected derivatives, their linear combination is also objective:

$$\overset{\circ}{\mathbf{S}} = (1-\alpha)\overset{\nabla}{\mathbf{S}} + \alpha\overset{\Delta}{\mathbf{S}}, \quad 0 \leq \alpha \leq 1. \tag{6.6.15}$$

Equation (6.6.15) can be viewed as the definition of a general convected derivative, which reduces to Eq. (6.6.12) for $\alpha = 0$ and to Eq. (6.6.13) for $\alpha = 1$. When $\alpha = 0.5$ [average of Eq. (6.6.12) and Eq. (6.6.13)] the convected derivative in Eq. (6.6.15) is termed a *corotational* or the *Jaumann derivative.* The selection of one type of derivative over other is usually based on the physical plausibility of the constitutive equation, that is, matching experimental data.

The simplest differential constitutive models are the upper- and lower-convected Maxwell fluids, which are defined by the following equations:

$$\text{Upper-convected Maxwell fluid:} \quad \boldsymbol{\tau} + \lambda\overset{\nabla}{\boldsymbol{\tau}} = 2\mu^p\,\mathbf{D}, \tag{6.6.16}$$

$$\text{Lower-convected Maxwell fluid:} \quad \boldsymbol{\tau} + \lambda\overset{\Delta}{\boldsymbol{\tau}} = 2\mu^p\,\mathbf{D} \tag{6.6.17}$$

where λ is the relaxation time for the fluid, μ^p is its viscosity, and $\mathbf{D}$ is the rate of deformation tensor. The upper-convected Maxwell model in Eq. (6.6.16) has been used extensively in testing numerical algorithms; the lower-convected and corotational forms of the Maxwell fluid predict physically unrealistic behavior and are not generally used.

Johnson–Segalman model. By employing the general convected derivative (6.6.15) in a Maxwell-like model, the Johnson–Segalman model is produced:

$$\boldsymbol{\tau} + \lambda\overset{\circ}{\boldsymbol{\tau}} = 2\mu^p\,\mathbf{D}. \tag{6.6.18}$$

Phan Thien–Tanner model. By slightly modifying Eq. (6.6.18) to include a variable coefficient for τ, the Phan Thien–Tanner model is obtained:

$$Y(\boldsymbol{\tau})\boldsymbol{\tau} + \lambda\overset{\circ}{\boldsymbol{\tau}} = 2\mu^p\,\mathbf{D}, \tag{6.6.19}$$

where

$$Y(\boldsymbol{\tau}) = 1 + (\epsilon\lambda/\mu^p)\;\mathrm{tr}(\boldsymbol{\tau}) \tag{6.6.20}$$

and ϵ is a constant. This equation is somewhat better than (6.6.18) in representing actual material behavior.

Oldroyd model. The Johnson–Segalman and Phan Thien–Tanner models suffer from a common defect. For a monotonically increasing shear rate, there is a region where the shear stress decreases, which is a physically unrealistic behavior. To correct this anomaly, the constitutive equations are altered using the following procedure. First, the extra-stress is decomposed into two partial stresses, $\boldsymbol{\tau}^s$ and $\boldsymbol{\tau}^p$, such that

$$\boldsymbol{\tau} = \boldsymbol{\tau}^s + \boldsymbol{\tau}^p, \tag{6.6.21}$$

where $\boldsymbol{\tau}^s$ is a purely viscous and $\boldsymbol{\tau}^p$ is a viscoelastic stress component. Then $\boldsymbol{\tau}^s$ and $\boldsymbol{\tau}^p$ are expressed in terms of the rate of deformation tensor $\mathbf{D}$, using the Johnson–Segalman fluid as an example, as

$$\boldsymbol{\tau}^s = 2\mu^s\,\mathbf{D}, \quad \boldsymbol{\tau}^p + \lambda\overset{\circ}{\boldsymbol{\tau}}{}^p = 2\mu^p\,\mathbf{D}. \tag{6.6.22}$$

Finally, the partial stresses in Eqs. (6.6.21) and (6.6.22) are eliminated to produce a new constitutive relation

$$\boldsymbol{\tau} + \lambda\overset{\circ}{\boldsymbol{\tau}} = 2\bar{\mu}\Big(\mathbf{D} + \lambda'\overset{\circ}{\mathbf{D}}\Big), \tag{6.6.23}$$

where $\bar{\mu} = (\mu^s + \mu^p)$ and $\lambda' = \lambda\mu^s/\bar{\mu}$; and λ' is a retardation time. The constitutive equation in Eq. (6.6.23) is known as a type of Oldroyd fluid. For particular choices of the convected derivative in Eq. (6.6.23), specific models can be generated. When $\alpha = 0$ ($\overset{\circ}{\boldsymbol{\tau}} \rightarrow \overset{\nabla}{\boldsymbol{\tau}}$), then Eq. (6.6.23) becomes the Oldroyd B fluid, and $\alpha = 1$ ($\overset{\circ}{\boldsymbol{\tau}} \rightarrow \overset{\Delta}{\boldsymbol{\tau}}$) produces the Oldroyd A fluid. In order to ensure a monotonically increasing shear stress, the inequality $\mu^s \geq \mu^p/8$ must be satisfied. The stress decomposition in Eq. (6.6.21) can also be used with the Phan Thien–Tanner model to produce a correct shear stress behavior.

White–Metzner model. In all of the above constitutive equations the material parameters, λ and μ^p, were assumed to be constants. For some constitutive equations the constancy of these parameters leads to material (or viscometric) functions that do not accurately represent the behavior of real elastic fluids. For example, the shear viscosity predicted by a Maxwell fluid is a constant, when in fact viscoelastic fluids normally exhibit a shear thinning behavior. This situation can be remedied to some degree by allowing the parameters λ and μ^p to be functions of the invariants of the rate of deformation tensor $\mathbf{D}$. Using the upper-convected Maxwell fluid as an example, then

$$\boldsymbol{\tau} + \lambda(J_2)\,\overset{\nabla}{\boldsymbol{\tau}} = 2\mu^p\,(J_2)\mathbf{D}, \tag{6.6.24}$$

where J_2 is the second invariant of the rate of deformation tensor $\mathbf{D}$ [see Eq. (6.6.3)]. The constitutive equation in Eq. (6.6.24) is termed a White–Metzner model. White–Metzner forms of other differential models, such as the Oldroyd fluids, have also been developed and used in various situations.

6.6.3.2 Integral models

An approximate integral model for a viscoelastic fluid represents the extra-stress in terms of an integral over the past history of the fluid deformation. A general form for a single integral model can be expressed as

$$\boldsymbol{\tau} = \int_{-\infty}^{t} 2m(t-t')\mathbf{H}(t,t')\,dt', \tag{6.6.25}$$

where t is the current time, m is a scalar memory function (or relaxation kernel), and $\mathbf{H}$ is a nonlinear deformation tensor between the past time t' and current time t.

There are many possible forms for both the memory function m and the deformation tensor $\mathbf{H}$. Normally the memory function is a decreasing function of the time lapse $s = t - t'$. Typical of such a function is the exponential given by

$$m(t - t') = m(s) = \frac{\mu_0}{\lambda^2}\, e^{-s/\lambda}, \tag{6.6.26}$$

where the parameters μ_0, λ, and s were defined previously. Like the choice of a convected derivative in a differential model, the selection of a deformation measure for use in Eq. (6.6.25) is somewhat arbitrary. One particular form that has received some attention is given by

$$\mathbf{H} = \phi_1(J_B, \tilde{J}_B)\mathbf{B} + \phi_2(J_B, \tilde{J}_B)\tilde{\mathbf{B}}. \tag{6.6.27}$$

In Eq. (6.6.27) $\tilde{\mathbf{B}}$ is the Cauchy strain tensor, $\mathbf{B}$ is its inverse, called the Finger tensor [see Eq. (3.4.22)], and ϕ_1 and ϕ_2 are scalar functions of the invariants of the deformation tensors, $J_B = \text{tr}(\mathbf{B})$ and $\tilde{J}_B = \text{tr}(\tilde{\mathbf{B}})$. The form of the deformation measure in Eq. (6.6.27) is still quite general, though specific choices for the functions ϕ_i and the memory function m lead to several well-known constitutive models. Among these are the Kaye–BKZ fluid and the Lodge rubber-like liquid.

As a specific example of an integral model, we consider the Maxwell fluid. Setting $\phi_1 = 1$ and $\phi_2 = 0$ in Eq. (6.6.27) and using the memory function of Eq. (6.6.26), we obtain a constitutive equation of the form

$$\boldsymbol{\tau} = \frac{\mu_0}{\lambda^2} \int_{-\infty}^{t} \exp\left[-(t - t')/\lambda\right] \left[\mathbf{B}(t') - \mathbf{I}\right] dt'. \tag{6.6.28}$$

The constitutive equation (6.6.28) is an integral equivalent to the upper-convected Maxwell model shown in differential form in Eq. (6.6.16). Note that in this case, the extra-stress is given in an explicit form but its evaluation requires that the strain history be known for each fluid particle. Although the Maxwell fluid has both differential and integral forms, this is generally not true for other constitutive equations. A discussion of additional integral models can be found in the book by Bird, Armstrong, and Hassager (1971).

6.7 Heat Transfer

6.7.1 Introduction

Heat transfer is a branch of engineering that deals with the transfer of thermal energy within a medium or from one medium to another due to a temperature difference. Heat transfer may take place in one or more of the three basic forms: *conduction*, *convection*, and *radiation*. The transfer of heat within a medium due to diffusion process is called conduction heat transfer. *Fourier's law* states that the heat flow is proportional to the temperature gradient. The proportionality parameter is known as the *thermal conductivity*. Note that for heat conduction to occur there must be temperature differences between neighboring points.

Convection heat transfer is the energy transport effected by the motion of a fluid. The convection heat transfer between two dissimilar media is governed by *Newton's law of cooling.* It states that the heat flow is proportional to the difference of the temperatures between the two media. The proportionality parameter is called the *convection heat transfer coefficient* or *film conductance.* For heat convection to occur there must be a fluid or another medium that can transport energy to and from the primary medium.

Radiation is a mechanism that is different from three transport processes we discussed so far, namely, (1) *momentum transport* in Newtonian fluids that is proportional to the velocity gradient, (2) *energy transport by conduction* that is proportional to the negative of the temperature gradient, and (3) *energy transport by convection* that is proportional to the difference in temperatures of the body and the moving fluid in contact with the body. Thermal radiation is an electromagnetic mechanism, which allows energy transport with the speed of light through regions of space that are devoid of any matter. Radiant energy exchange between surfaces or between a region and its surroundings is described by the *Stefan–Boltzmann law*, which states that the radiant energy transmitted is proportional to the difference of the fourth power of the temperatures of the surfaces. The proportionality parameter is known as the *Stefan–Boltzmann* parameter.

6.7.2 Fourier's Heat Conduction Law

The Fourier heat conduction law states that the heat flow $\mathbf{q}$ is related to the temperature gradient by the relation

$$\mathbf{q} = -\mathbf{k}\cdot\boldsymbol{\nabla}\theta \quad (q_i = -k_{ij}\frac{\partial\theta}{\partial x_j}), \tag{6.7.1}$$

where $\mathbf{k}$ is the thermal conductivity tensor of order two. The negative sign in (6.7.1) indicates that heat flows downhill on the temperature scale. The balance of energy (5.4.11) requires that $(e = c\,\theta)$

$$\rho c\frac{D\theta}{Dt} = \Phi - \boldsymbol{\nabla}\cdot\mathbf{q} + \rho\, r_h, \quad \Phi = \boldsymbol{\tau} : \mathbf{D}, \tag{6.7.2}$$

which, in view of Eq. (6.7.1), becomes

$$\rho c\frac{D\theta}{Dt} = \Phi + \boldsymbol{\nabla}\cdot(\mathbf{k}\cdot\boldsymbol{\nabla}\theta) + \rho\, r_h, \tag{6.7.3}$$

where $\rho\, r_h$ is the internal heat generation per unit volume, ρ is the density, and c is the specific heat of the material (assumed to be independent of time t).

For heat transfer in a solid medium ($\mathbf{v} = 0$), Eq. (6.7.3) reduces to

$$\rho c\frac{\partial\theta}{\partial t} = \boldsymbol{\nabla}\cdot(\mathbf{k}\cdot\boldsymbol{\nabla}\theta) + \rho\, r_h, \tag{6.7.4}$$

which forms the subject of the field of conduction heat transfer. For a fluid medium, Eq. (6.7.3) becomes

$$\rho c\left(\frac{\partial\theta}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}\theta\right) = \Phi + \boldsymbol{\nabla}\cdot(\mathbf{k}\cdot\boldsymbol{\nabla}\theta) + \rho\, r_h, \tag{6.7.5}$$

where $\mathbf{v}$ is the velocity field, and Φ is the viscous dissipation function.

6.7.3 Newton's Law of Cooling

At a solid–fluid interface the heat flux is related to the difference between the temperature θ at the interface and that in the fluid

$$q_n \equiv \hat{\mathbf{n}} \cdot \mathbf{q} = h\left(\theta - \theta_{\text{fluid}}\right), \tag{6.7.6}$$

where $\hat{\mathbf{n}}$ is the unit normal to the surface of the body and h is known as the *heat transfer coefficient* or *film conductance*. This relation is known as Newton's law of cooling, which also defines h. Clearly, Eq. (6.7.6) defines a boundary condition on the bounding surface of a conducting medium.

6.7.4 Stefan–Boltzmann Law

The heat flow from surface 1 to surface 2 by radiation is governed by the Stefan–Boltzman law:

$$q_n = \sigma\left(\theta_1^4 - \theta_2^4\right), \tag{6.7.7}$$

where θ_1 and θ_2 are the temperatures of surfaces 1 and 2, respectively, and σ is the Stefan–Boltzman constant. Again, Eq. (6.7.7) defines a boundary condition on the surface 1 of a body.

6.8 Constitutive Relations for Coupled Problems

6.8.1 Electromagnetics

Problems involving the coupling of electromagnetic fields with fluid and thermal transport have a broad spectrum of applications ranging from astrophysics to manufacturing and to electromechanical devices and sensors. A good introduction to electromagnetic field theory is available in the textbook by Jackson (1975). Here we present a brief discussion of pertinent equations for the sake of completeness[4].

6.8.1.1 Maxwell's equations

The appropriate mathematical description of electromagnetic phenomena in a conducting material region, Ω_C, is given by the following Maxwell's equations [see Reddy and Gartling (2001) and Jackson (1975), and references therein]:

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{6.8.2}$$

$$\boldsymbol{\nabla} \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}, \tag{6.8.3}$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0, \tag{6.8.4}$$

$$\boldsymbol{\nabla} \cdot \mathbf{D} = \rho, \tag{6.8.5}$$

where $\mathbf{E}$ is the electric field intensity, $\mathbf{H}$ is the magnetic field intensity, $\mathbf{B}$ is the magnetic flux density, $\mathbf{D}$ is the electric flux (displacement) density, $\mathbf{J}$ is the conduction current density, and ρ is the source charge density. Equation (6.8.1)

---

[4] Note that the notation used here for various fields is standard in the literature; unfortunately, some of the symbols used here were already used previously for other variables.

is referred to as *Faraday's law*, Eq. (6.8.2) as *Ampere's law* (as modified by Maxwell), and Eq. (6.8.4) as *Gauss' law.* A continuity condition on the current density is also defined by

$$\nabla \cdot \mathbf{J} = \frac{\partial \rho}{\partial t}\,. \tag{6.8.5}$$

Note that only three of the preceding five equations are independent; either Eqs. (6.8.1), (6.8.2), and (6.8.4) or Eqs. (6.8.1), (6.8.2), and (6.8.5) form valid sets of equations for the field variables.

6.8.1.2 Constitutive relations

To complete the formulation, the constitutive relations for the material are required. The fluxes are functionally related to the field variables by

$$\mathbf{D} = \mathcal{F}_D(\mathbf{E}, \mathbf{B}), \tag{6.8.6}$$

$$\mathbf{H} = \mathcal{F}_H(\mathbf{E}, \mathbf{B}), \tag{6.8.7}$$

$$\mathbf{J} = \mathcal{F}_J(\mathbf{E}, \mathbf{B}), \tag{6.8.8}$$

where the response functions $\mathcal{F}_D$, $\mathcal{F}_H$, and $\mathcal{F}_J$ may also depend on external variables such as temperature θ and mechanical stress $\boldsymbol{\sigma}$. The form of the material response due to applied **E** or **B** fields can vary strongly depending on the microstructure and the strength of the material and on the magnitude and time-dependent nature of the applied field.

Conductive and Dielectric Materials. For conducting materials, the standard response function $\mathcal{F}_J$ gives Ohm's law, which relates the current density **J** to the electric field intensity **E**

$$\mathbf{J} = \mathbf{k}_\sigma \cdot \mathbf{E}, \tag{6.8.9}$$

where $\mathbf{k}_\sigma$ is the conductivity tensor. For isotropic materials, we have $\mathbf{k}_\sigma = k_\sigma \mathbf{I}$, where k_σ is a scalar and **I** is the unit tensor. In general, the conductivity may be a function of **E** or an external variable such as temperature. This form of Ohm's law applies to stationary conductors. If the conductive material is moving in a magnetic field, then Eq. (6.8.9) is modified to read

$$\mathbf{J} = \mathbf{k}_\sigma \cdot \mathbf{E} + \mathbf{k}_\sigma \cdot (\mathbf{v} \times \mathbf{B}), \tag{6.8.10}$$

where **v** is the velocity vector describing the motion of the conductor and **B** is the magnetic flux vector.

For dielectric materials, the standard response function $\mathcal{F}_D$ relates the electric flux density **D** to the electric field **E** and polarization vector **P**:

$$\mathbf{D} = \epsilon_0 \cdot \mathbf{E} + \mathbf{P}, \tag{6.8.11}$$

where ϵ_0 is the permittivity of free space. The polarization is generally related to the electric field through

$$\mathbf{P} = \epsilon_0 \mathbf{S}_e \cdot \mathbf{E} + \mathbf{P}_0, \tag{6.8.12}$$

where $\mathbf{S}_e$ is the electric susceptibility tensor that accounts for the different types of polarization, and $\mathbf{P}_0$ is the remnant polarization that may be present in some materials.

Magnetic Materials. For magnetic materials, the standard response function $\boldsymbol{\mathcal{F}}_H$ relates the magnetic field intensity $\mathbf{H}$ to the magnetic flux $\mathbf{B}$

$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}, \tag{6.8.13}$$

where μ_0 is the permeability of free space and $\mathbf{M}$ is the magnetization vector. The magnetization vector $\mathbf{M}$ can be related to either the magnetic flux $\mathbf{B}$ or magnetic field intensity $\mathbf{H}$ by

$$\mathbf{M} = \frac{1}{\mu_0}\frac{\mathbf{S}_m}{(\mathbf{I} + \mathbf{S}_m)} \cdot \mathbf{B} + \mathbf{M}_0, \tag{6.8.14}$$

$$\mathbf{M} = \mathbf{S}_m \cdot \mathbf{H} + (\mathbf{I} + \mathbf{S}_m) \cdot \mathbf{M}_0, \tag{6.8.15}$$

where $\mathbf{S}_m$ is the magnetic susceptibility for the material, $\mathbf{M}_0$ is the remnant magnetization, and $\mathbf{I}$ is the unit tensor. If the susceptibility is negative, the material is diamagnetic, whereas a positive susceptibility defines a paramagnetic material. Generally, these susceptibilities are quite small and are often neglected. Ferromagnetic materials have large positive susceptibilities and produce a nonlinear (hysteretic) relationship between $\mathbf{B}$ and $\mathbf{H}$. These materials may also exhibit spontaneous and remnant magnetization.

Electromagnetic Forces and Volume Heating. The coupling of electromagnetic fields with a fluid or thermal problem occurs through the dependence of material properties on electromagnetic field quantities and the production of electromagnetic-induced body forces and volumetric energy production. The Lorentz body force per unit volume in a conductor due to the presence of electric currents and magnetic fields is given by

$$\mathbf{F}_B = \rho\mathbf{E} + \mathbf{J} \times \mathbf{B}, \tag{6.8.16}$$

where, in the general case, the current is defined by Eq. (6.8.10). The first term on the right-hand side of Eq. (6.8.16) is the electric field contribution to the Lorentz force; the magnetic term $\mathbf{J} \times \mathbf{B}$ is usually of more interest in applied mechanics problems. The energy generation or Joule heating in a conductor is described by

$$Q_J = \mathbf{J} \cdot \mathbf{E}, \tag{6.8.17}$$

which takes on a more familiar form if the simplified ($\mathbf{v} = \mathbf{0}$) form of Eq. (6.8.10) is used to produce

$$Q_J = \sigma^{-1}(\mathbf{J} \cdot \mathbf{J}), \tag{6.8.18}$$

where σ is the conductivity. The aforementioned forces and heat source occur in the fluid momentum and energy equations, respectively.

6.8.2 Thermoelasticity

The use of the entropy density η as an independent variable is not convenient. A more convenient thermal variable is the temperature θ, as it is fairly easy to measure and control. The constitutive equations of thermoelasticity are derived by assuming the existence of the *Helmholtz free-energy potential* $\Psi = U_0(\theta, \boldsymbol{\varepsilon}) - \eta(\varepsilon_{ij})\,\theta = \Psi(\theta, \boldsymbol{\varepsilon})$:

$$\rho_0\,\Psi(\varepsilon_{ij}, \theta) = \rho_0\,U_0 - \rho_0\,(\theta - \theta_0)\eta = \frac{1}{2} C_{ijk\ell}\,\varepsilon_{ij}\,\varepsilon_{k\ell} - \beta_{ij}\,\varepsilon_{ij}\,(\theta - \theta_0) - \frac{\rho_0\,c_v}{2\theta_0}(\theta - \theta_0)^2 \tag{6.8.19}$$

such that

$$\sigma_{ij} = \rho_0 \frac{\partial \Psi}{\partial \varepsilon_{ij}} = C_{ijk\ell}\,\varepsilon_{k\ell} - \beta_{ij}\,(\theta - \theta_0), \quad \rho_0 \eta = -\rho_0 \frac{\partial \Psi}{\partial \theta} = \beta_{ij}\,\varepsilon_{ij} + \frac{\rho_0\,c_v}{\theta_0}(\theta - \theta_0), \tag{6.8.20}$$

where θ is the temperature measured from a reference value θ_0, η is the entropy density, and β_{ij} are material coefficients. In arriving at Eq. (6.8.20), it is assumed that η and σ_{ij} are initially zero (see the answer to Problem **6.35**), and c_v β_{ij}, and $C_{ijk\ell}$ are values at the reference state. Inverting the stress–strain relations in Eq. (6.8.20), we obtain

$$\varepsilon_{ij} = S_{ijk\ell}\,\sigma_{k\ell} + \alpha_{ij}(\theta - \theta_0), \tag{6.8.21}$$

where $S_{ijk\ell}$ are the elastic compliances, and α_{ij} are the thermal coefficients of expansion, and they are related to β_{ij} by

$$\beta_{ij} = C_{ijk\ell}\,\alpha_{k\ell}. \tag{6.8.22}$$

6.8.3 Hygrothermal Elasticity

The moisture adsorption problem is mathematically similar to the heat conduction problem. The moisture concentration c in a solid is described by Fick's law (analogous to Fourier's heat conduction law):

$$\mathbf{q}_f = -\mathbf{D} \cdot \boldsymbol{\nabla} c \tag{6.8.23}$$

and the diffusion process is governed by

$$\frac{\partial c}{\partial t} = -\boldsymbol{\nabla} \cdot \mathbf{q}_f + \phi_f, \tag{6.8.24}$$

where $\mathbf{D}$ denotes the *mass diffusivity tensor* of order two, $\mathbf{q}_f$ is the moisture flux vector, and ϕ_f is the moisture source in the domain. The negative sign in Eq. (6.8.24) indicates that moisture seeps from a higher concentration to a lower concentration. The boundary conditions involve specifying the moisture concentration or the flux normal to the boundary:

$$c = \hat{c}(s, t) \quad \text{on} \quad \Gamma_1, \tag{6.8.25}$$

$$\mathbf{n} \cdot \mathbf{q}_f = \hat{q}_f(s, t) \quad \text{on} \quad \Gamma_2, \tag{6.8.26}$$

where $\Gamma = \Gamma_1 \cup \Gamma_2$, and $\Gamma_1 \cap \Gamma_2 = \emptyset$, and quantities with a hat are specified functions on the respective boundaries.

The moisture-induced strains $\{\varepsilon\}^M$ are given by

$$\{\varepsilon\}^M = \{\alpha_M\}c, \tag{6.8.27}$$

where $\{\alpha_M\}$ is the vector of *coefficients of hygroscopic expansion*. Thus, the hygrothermal strains have the same form as the thermal strains. The total strains are given by

$$\{\varepsilon\} = [S]\{\sigma\} + \{\alpha_T\}(\theta - \theta_0) + \{\alpha_M\}(c - c_0), \tag{6.8.28}$$

where $\{\alpha_T\}$ is the vector of *coefficients of thermal expansion*, and θ_0 and c_0 are reference values of temperature and concentration, respectively, from which the strains and stresses are measured. In view of the similarity between the thermal and moisture strains, thermoelasticity and hygroelasticity problems share the same solution approach.

6.8.4 Electroelasticity

Electroelasticity deals with the phenomena caused by interactions between electric and mechanical fields. The *piezoelectric effect* is one such phenomenon, and it is concerned with the effect of the electric charge on the deformation. A structure with piezoelectric layers receives actuation through an applied electric field, and the piezoelectric layers send electric signals that are used to measure the motion or deformation of the laminate. In these problems, the electric charge that is applied to actuate a structure provides an additional body force to the stress analysis problem, much the same way a temperature field induces a body force through thermal strains.

The piezoelectric effect is described by the *polarization vector* **P**, which represents the electric moment per unit volume or polarization charge per unit area. It is related to the stress tensor by the relation

$$\mathbf{P} = \mathbf{d} \cdot \boldsymbol{\sigma} \quad \text{or} \quad P_i = d_{ijk}\sigma_{jk}, \tag{6.8.29}$$

where **d** is the third-order tensor of piezoelectric moduli. The inverse effect relates the electric field vector **E** to the linear strain tensor ε by

$$\boldsymbol{\varepsilon} = \mathbf{E} \cdot \mathbf{d} \quad \text{or} \quad \varepsilon_{ij} = d_{kij}E_k. \tag{6.8.30}$$

Note that d_{kij} is symmetric with respect to indices i and j because of the symmetry of ε_{ij} (note that $i, j, k = 1, 2, 3$).

The *pyroelectic effect* is another phenomenon that relates temperature changes to polarization of a material. For a temperature change from a reference temperature θ_0, the change in polarization vector $\Delta\mathbf{P}$ is given by

$$\Delta\mathbf{P} = \mathbf{p}(\theta - \theta_0), \tag{6.8.31}$$

where **p** is the vector of pyroelectric coefficients.

The coupling between the mechanical, thermal, and electrical fields can be established using thermodynamical principles and Maxwell's relations. Analogous to the strain energy potential U_0 for elasticity and the Helmholtz free-energy potential Ψ for thermoelasticity, we assume the existence of a function Φ, called the electric *Gibb's free-energy potential* or *enthalpy function*,

$$\begin{aligned}\rho_0\Phi(\varepsilon_{ij}, E_i, \theta) &= \rho_0\, U_0 - \rho_0\,\mathbf{E}\cdot\mathbf{D} - \rho_0\,\eta(\theta-\theta_0)\\ &= \frac{1}{2}C_{ijk\ell}\,\varepsilon_{ij}\,\varepsilon_{k\ell} - e_{ijk}\,\varepsilon_{ij}\,E_k - \beta_{ij}\,\varepsilon_{ij}\,(\theta-\theta_0)\\ &\quad -\frac{1}{2}\epsilon_{k\ell}\,E_k\,E_\ell - p_k\,E_k(\theta-\theta_0) - \frac{\rho_0 c_v}{2\theta_0}(\theta-\theta_0)^2, \qquad (6.8.32)\end{aligned}$$

such that

$$\sigma_{ij} = \rho_0\frac{\partial\Phi}{\partial\varepsilon_{ij}}, \quad \rho_0 D_i = -\rho_0\frac{\partial\Phi}{\partial E_i}, \quad \rho_0\,\eta = -\rho_0\frac{\partial\Phi}{\partial\theta}, \qquad (6.8.33)$$

where σ_{ij} are the components of the stress tensor $\boldsymbol{\sigma}$, D_i are the components of the electric displacement vector $\mathbf{D}$, and η is the entropy. Use of Eq. (6.8.32) in Eq. (6.8.33) gives the constitutive equations of a deformable piezoelectric medium:

$$\sigma_{ij} = C_{ijk\ell}\,\varepsilon_{k\ell} - e_{ijk}\,E_k - \beta_{ij}(\theta-\theta_0), \qquad (6.8.34)$$

$$\rho_0\,D_k = e_{ijk}\,\varepsilon_{ij} + \epsilon_{k\ell}\,E_\ell + p_k(\theta-\theta_0), \qquad (6.8.35)$$

$$\rho_0\,\eta = \beta_{ij}\,\varepsilon_{ij} + p_k\,E_k + \frac{\rho_0 c_v}{\theta_0}(\theta-\theta_0), \qquad (6.8.36)$$

where $C_{ijk\ell}$ are the elastic moduli, e_{ijk} are the piezoelectric moduli, ϵ_{ij} are the dielectric constants, p_k are the pyroelectric constants, β_{ij} are the stress-temperature expansion coefficients, c_v is the specific heat (at constant strain or volume) per unit mass, and θ_0 is the reference temperature. In single-subscript notation for stresses and strains, Eqs. (6.8.34)–(6.8.36) can be expressed as

$$\sigma_i = C_{ij}\,\varepsilon_j - e_{ik}\,E_k - \beta_i(\theta-\theta_0), \qquad (6.8.37)$$

$$\rho_0\,D_k = e_{ik}\,\varepsilon_i + \epsilon_{k\ell}\,E_\ell + p_k(\theta-\theta_0), \qquad (6.8.38)$$

$$\rho_0\,\eta = \beta_i\,\varepsilon_i + p_k\,E_k + \frac{\rho_0 c_v}{\theta_0}\,(\theta-\theta_0). \qquad (6.8.39)$$

Note that the range of summation in Eqs. (6.8.37)–(6.8.39) is different for different terms: $i, j = 1, 2, \cdots, 6; k, \ell = 1, 2, 3$. For the general anisotropic material, there are 21 independent elastic constants, 18 piezoelectric constants, 6 dielectric constants, 3 pyroelectric constants, and 6 thermal expansion coefficients.

Maxwell's equation governing the electric displacement vector $\mathbf{D}$ is given by

$$\boldsymbol{\nabla}\cdot\mathbf{D} = 0. \qquad (6.8.40)$$

It is often assumed that the electric field $\mathbf{E}$ is derivable from an electric scalar potential function ϕ:

$$\mathbf{E} = -\boldsymbol{\nabla}\phi. \qquad (6.8.41)$$

This assumption allows us to write Eq. (6.8.40), in view of Eq. (6.8.38), as

$$\frac{\partial}{\partial x_1}\left(\epsilon_{11}\frac{\partial\phi}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\epsilon_{22}\frac{\partial\phi}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(\epsilon_{33}\frac{\partial\phi}{\partial x_3}\right) + f_e = 0, \qquad (6.8.42)$$

where

$$f_e = -\frac{\partial}{\partial x_k}\left[e_{k\ell}\varepsilon_\ell + p_k(\theta-\theta_0)\right]. \qquad (6.8.43)$$

6.9 Summary

This chapter was dedicated to a discussion of the constitutive equations for Hookean solids, Newtonian fluids, and heat transfer in solids. Constitutive models of solids and fluids are derived using the entropy inequality or conditions resulting from the entropy inequality. Beginning with a discussion of the constitutive rules or axioms, frame indifference, and restrictions placed by the entropy inequality, general constitutive relations for the stress tensor, entropy, and heat flux were derived. Then the generalized Hooke's law governing linear elastic solids, Newtonian relations for viscous fluids, and the Fourier heat conduction equation for heat transfer in solids are presented. The generalized Hooke's law is specialized to monoclinic materials, orthotropic materials, and isotropic materials using material symmetries. Constitutive relations for nonlinear elastic solids, generalized Newtonian fluids, and coupled problems (e.g., electromagnetics, thermoelasticity, hygrothermal elasticity, and electroelasticity) are also presented for the sake of completeness.

The constitutive relations presented in this chapter along with the field equations developed in Chapter 5 will be used in Chapters 7 and 8 to analyze some typical boundary-value problems of solid mechanics, fluid mechanics, and heat transfer. The main results of this chapter that are of importance in the coming chapters are summarized here.

Hookean deformable solids (infinitesimal strains)

$$\boldsymbol{\sigma} = 2\mu\,\boldsymbol{\varepsilon} + \lambda\,\mathrm{tr}(\boldsymbol{\varepsilon})\mathbf{I}, \quad \sigma_{ij} = 2\mu\,\varepsilon_{ij} + \lambda\,\varepsilon_{kk}\,\delta_{ij} \tag{6.9.1}$$

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left[\boldsymbol{\nabla}\mathbf{u} + (\boldsymbol{\nabla}\mathbf{u})^{\mathrm{T}}\right], \quad \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{6.9.2}$$

Newtonian fluids (compressible)

$$\boldsymbol{\sigma} = -p(\rho,\theta)\,\mathbf{I} + 2\mu(\rho,\theta)\,\mathbf{D} + \lambda(\rho,\theta)\,\mathrm{tr}(\mathbf{D})\mathbf{I}$$

$$\sigma_{ij} = -p\,\delta_{ij} + 2\mu\,\varepsilon_{ij} + \lambda\,\varepsilon_{kk}\,\delta_{ij} \tag{6.9.3}$$

Newtonian Fluids(incompressible)

$$\boldsymbol{\sigma} = -p(\theta)\,\mathbf{I} + 2\mu\,\mathbf{D}, \quad \sigma_{ij} = -p(\theta)\,\delta_{ij} + 2\mu\,D_{ij} \tag{6.9.4}$$

$$\mathbf{D} = \frac{1}{2}\left[\boldsymbol{\nabla}\mathbf{v} + (\boldsymbol{\nabla}\mathbf{v})^{\mathrm{T}}\right], \quad D_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \tag{6.9.5}$$

Heat transfer

$$\mathbf{q} = -\mathbf{k}\cdot\boldsymbol{\nabla}\theta, \quad q_i = -k_{ij}\,\frac{\partial \theta}{\partial x_j} \tag{6.9.6}$$

In general, the derivation of constitutive equations of a fluid or solid matter is quite involved. The presentation here is made simple keeping in mind the introductory nature of the present course. For a detailed and advanced study of the subject, the reader may consult the books by Truesdell and Toupin (1965) and Truesdell and Noll (1965).

Problems

HOOKEAN SOLIDS

6.1 Recall from Examples 3.4.3 and 4.3.1 that under the coordinate transformation

$$\begin{aligned}\hat{\mathbf{e}}_1 &= \cos\theta\,\hat{\mathbf{e}}_x + \sin\theta\,\hat{\mathbf{e}}_y,\\ \hat{\mathbf{e}}_2 &= -\sin\theta\,\hat{\mathbf{e}}_x + \cos\theta\,\hat{\mathbf{e}}_y,\\ \hat{\mathbf{e}}_3 &= \hat{\mathbf{e}}_z,\end{aligned} \tag{1}$$

the stress components and strain components ε_i and σ_i are given in terms of the components $\sigma_{xx}, \sigma_{yy}, \cdots$ and $\varepsilon_{xx}, \varepsilon_{yy}, \cdots$ by [see Eqs. (3.4.33) and (4.3.7)]

$$\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\\ \varepsilon_4\\ \varepsilon_5\\ \varepsilon_6\end{Bmatrix} = \begin{bmatrix}\cos^2\theta & \sin^2\theta & 0 & 0 & 0 & \frac{1}{2}\sin 2\theta\\ \sin^2\theta & \cos^2\theta & 0 & 0 & 0 & -\frac{1}{2}\sin 2\theta\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & \cos\theta & -\sin\theta & 0\\ 0 & 0 & 0 & \sin\theta & \cos\theta & 0\\ -\sin 2\theta & \sin 2\theta & 0 & 0 & 0 & \cos 2\theta\end{bmatrix} \begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \varepsilon_{yz}\\ \varepsilon_{xz}\\ \varepsilon_{xy}\end{Bmatrix}, \quad \{\bar{\varepsilon}\} = [T^\theta]\{\varepsilon\}, \tag{2}$$

$$\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \sigma_4\\ \sigma_5\\ \sigma_6\end{Bmatrix} = \begin{bmatrix}\cos^2\theta & \sin^2\theta & 0 & 0 & 0 & \sin 2\theta\\ \sin^2\theta & \cos^2\theta & 0 & 0 & 0 & -\sin 2\theta\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & \cos\theta & -\sin\theta & 0\\ 0 & 0 & 0 & \sin\theta & \cos\theta & 0\\ -\frac{1}{2}\sin 2\theta & \frac{1}{2}\sin 2\theta & 0 & 0 & 0 & \cos 2\theta\end{bmatrix} \begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \sigma_{yz}\\ \sigma_{xz}\\ \sigma_{xy}\end{Bmatrix}, \quad \{\bar{\sigma}\} = [R^\theta]\{\sigma\}. \tag{3}$$

Show that

$$[\bar{S}] = [T^\theta][S][T^\theta]^{\mathrm{T}}, \qquad [\bar{C}] = [R^\theta][C][R^\theta]^{\mathrm{T}}, \tag{4}$$

where $[\bar{S}]$ is the matrix of compliance coefficients and $[\bar{C}]$ is the matrix of stiffness coefficients with respect to the (x_1, x_2, x_3) coordinates and $[S]$ is the matrix of compliance coefficients and $[C]$ is the matrix of stiffness coefficients with respect to the (x, y, z) coordinates.

6.2 Under the coordinate transformation

$$\begin{aligned}\hat{\bar{\mathbf{e}}}_1 &= \cos\theta\,\hat{\mathbf{e}}_1 + \sin\theta\,\hat{\mathbf{e}}_2,\\ \hat{\bar{\mathbf{e}}}_2 &= -\sin\theta\,\hat{\mathbf{e}}_1 + \cos\theta\,\hat{\mathbf{e}}_2,\\ \hat{\bar{\mathbf{e}}}_3 &= \hat{\mathbf{e}}_3,\end{aligned}$$

determine $\bar{S}_{ij}$ in terms of S_{ij} and $\bar{C}_{ij}$ in terms of C_{ij}.

6.3 Given the transformation

$$\hat{\bar{\mathbf{e}}}_1 = \hat{\mathbf{e}}_1, \quad \hat{\bar{\mathbf{e}}}_2 = \hat{\mathbf{e}}_2, \quad \hat{\bar{\mathbf{e}}}_3 = -\hat{\mathbf{e}}_3, \tag{1}$$

determine the stress components $\bar{\sigma}_{ij}$ in terms of σ_{ij}, strain components $\bar{\varepsilon}_{ij}$ in terms of ε_{ij}, and the elasticity coefficients $\bar{C}_{ij}$ in terms of C_{ij}.

6.4 Establish the following relations between the Lame' constants μ and λ and engineering constants E, ν, and K:

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \quad \mu = G = \frac{E}{2(1+\nu)}, \quad K = \frac{E}{3(1-2\nu)}.$$

6.5 Determine the longitudinal stress σ_{xx} and the hoop stress σ_{yy} in a thin-walled circular cylindrical pressure vessel with closed ends; that is, establish Eq. (1) of Example 6.3.1. Assume an internal pressure of p, internal diameter D_i, and thickness h.

6.6 Determine the stress tensor components at a point in 7075-T6 aluminum alloy body ($E = 72$ GPa, and $G = 27$ GPa) if the strain tensor at the point has the following components with respect to the Cartesian basis vectors $\hat{\mathbf{e}}_i$:

$$[\varepsilon] = \begin{bmatrix}200 & 100 & 0\\ 100 & 300 & 400\\ 0 & 400 & 0\end{bmatrix} \times 10^{-6} \text{ m/m}\,.$$

6.7 For the state of stress and strain given in Problem **6.6**, determine the principal invariants of the stress and strain tensors.

6.8 The components of strain tensor at a point in a body made of structural steel are

$$[\varepsilon] = \begin{bmatrix} 36 & 12 & 30 \\ 12 & 40 & 0 \\ 30 & 0 & 25 \end{bmatrix} \times 10^{-6} \text{ m/m} .$$

Assuming that the Lamé constants for the structural steel are $\lambda = 207$ GPa (30×10^6 psi) and $\mu = 79.6$ GPa (11.54×10^6 psi), determine the principal invariants of the stress and strain tensors.

6.9 The components of a stress tensor at a point in a body made of structural steel are

$$[\sigma] = \begin{bmatrix} 42 & 12 & 30 \\ 12 & 15 & 0 \\ 30 & 0 & -5 \end{bmatrix} \text{ MPa.}$$

Assuming that the Lamé constants for structural steel are $\lambda = 207$ GPa (30×10^6 psi) and $\mu = 79.6$ GPa (11.54×10^6 psi), determine the principal invariants of the strain tensor.

6.10 *Plane stress-reduced constitutive relations.* Beginning with the strain-stress relations in Eq. (6.3.23) for an orthotropic material in a two-dimensional case (i.e., $\sigma_{33} = \sigma_{13} = \sigma_{23} = 0$), determine the two-dimensional stress–strain relations.

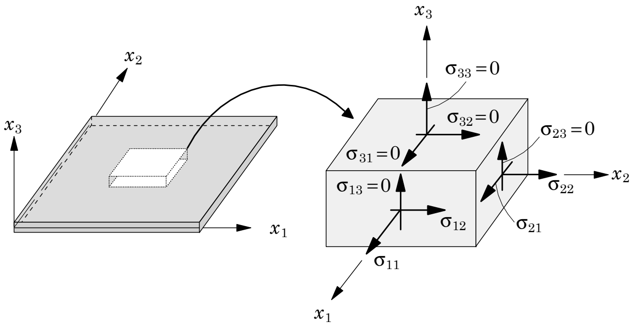

Fig. P6.10

6.11 Given the strain energy potential

$$\Psi(\mathbf{E}) = \frac{\lambda}{2}(\mathrm{tr}\,\mathbf{E})^2 + \mu\,\mathrm{tr}(\mathbf{E}\cdot\mathbf{E}),$$

determine the second Piola–Kirchhoff stress tensor $\mathbf{S}$ in terms of the Green strain tensor $\mathbf{E}$.

6.12 Given the strain energy potential for the case of infinitesimal deformations

$$\Psi(\varepsilon) = \frac{\lambda}{2}(\mathrm{tr}\,\varepsilon)^2 + \mu\,\mathrm{tr}(\varepsilon\cdot\varepsilon),$$

determine the strain energy function $\Psi(\boldsymbol{\sigma})$ in terms of the stress tensor $\boldsymbol{\sigma}$.

6.13 Assuming that the strain energy density $\Psi = U_0(\boldsymbol{\sigma})$ is positive-definite, that is, $U_0 \geq 0$, with $U_0 = 0$ if and only if $\boldsymbol{\sigma} = \mathbf{0}$, determine the restrictions placed on the elastic parameters E, K, and ν by considering the following stress states: (a) uniaxial stress state with $\sigma_{11} = \sigma$; (b) pure shear stress state, $\sigma_{12} = \tau$; and (c) hydrostatic stress state, $\sigma_{11} = \sigma_{22} = \sigma_{33} = p$.

6.14 A material is *transversely isotropic* at a point if it is symmetric with respect to an arbitrary rotation about a given axis. Aligned fiber-reinforced composites provide examples of transversely isotropic materials (see Fig. P6.14). Take the x_3-axis as the axis of symmetry with the transformation matrix

$$[L] = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where θ is arbitrary. Show that the stress–strain relations of a transversely isotropic material are of the form

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(C_{11} - C_{12}) \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix}.$$

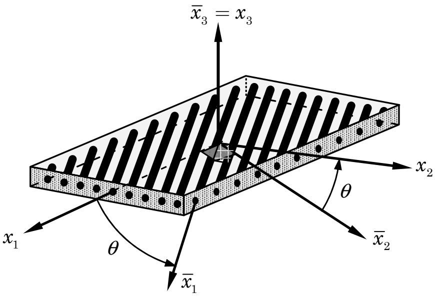

Fig. P6.14

6.15 The stress–strain relations of an isotropic material in the cylindrical coordinate system are

$$\begin{aligned} \sigma_{rr} &= 2\mu\,\varepsilon_{rr} + \lambda\,(\varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{zz}), \\ \sigma_{\theta\theta} &= 2\mu\,\varepsilon_{\theta\theta} + \lambda\,(\varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{zz}), \\ \sigma_{zz} &= 2\mu\,\varepsilon_{zz} + \lambda\,(\varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{zz}), \\ \sigma_{r\theta} &= 2\mu\,\varepsilon_{r\theta}, \quad \sigma_{rz} = 2\mu\,\varepsilon_{rz}, \quad \sigma_{\theta z} = 2\mu\,\varepsilon_{\theta z}. \end{aligned}$$

Express the relations in terms of the displacements (u_r, u_θ, u_z).

6.16 Express the stress–strain relations of an isotropic material in the spherical coordinate system and express the result in terms of the displacements (u_R, u_ϕ, u_θ).

6.17 Given the displacement field in an isotropic body

$$u_r = U(r), \quad u_\theta = 0, \quad u_z = 0, \tag{1}$$

where $U(r)$ is a function of only r, determine the stress components in the cylindrical coordinate system.

6.18 Given the displacement field in an isotropic body

$$u_R = U(R), \quad u_\phi = 0, \quad u_\theta = 0, \tag{1}$$

where $U(R)$ is a function of only R, determine the stress components in the spherical coordinate system.

6.19 *The Navier equations.* Show that for an isotropic, incompressible solid with infinitesimal deformations (i.e., $\boldsymbol{\sigma} \approx \mathbf{S}$ and $\mathbf{F} \cdot \mathbf{S} \approx \mathbf{S}$), the equation of motion (5.3.11), $\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \rho_0 \mathbf{f} = \rho_0 \ddot{\mathbf{u}}$, can be expressed as

$$\rho_0 \frac{\partial^2 \mathbf{u}}{\partial t^2} = \rho_0 \mathbf{f} - \boldsymbol{\nabla} p + (\lambda + \mu)\, \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{u}) + \mu \nabla^2 \mathbf{u}.$$

NEWTONIAN FLUIDS

6.20 Given the following motion of an isotropic continuum,

$$\boldsymbol{\chi}(\mathbf{X}) = (X_1 + kt^2X_2^2)\,\hat{\mathbf{e}}_1 + (X_2 + ktX_2)\,\hat{\mathbf{e}}_2 + X_3\,\hat{\mathbf{e}}_3,$$

determine the components of the viscous stress tensor as a function of position and time.

6.21 Express the upper and lower convective derivatives of Eqs. (6.6.12) and (6.6.13) in Cartesian component form.

6.22 Interpret the Lamé constant μ by considering the flow field

$$v_1 = f(x_2), \quad v_2 = 0, \quad v_3 = 0,$$

where f is a known function of x_1.

6.23 For viscous compressible flows (in spatial description), show that

$$\tilde{\sigma} - p = \left(\lambda + \tfrac{2}{3}\mu\right)\frac{1}{\rho}\frac{D\rho}{Dt},$$

where $\tilde{\sigma} = -\sigma_{ii}/3$ is the mean stress and p is the thermodynamic pressure.

6.24 *The Navier–Stokes equations.* Show that for a compressible fluid, the Cauchy equations of motion (5.3.10) can be expressed as

$$\rho\frac{D\mathbf{v}}{Dt} = \rho\mathbf{f} - \boldsymbol{\nabla}p + (\lambda + \mu)\,\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{v}) + \mu\nabla^2\mathbf{v}.$$

Simplify the equation for (a) an incompressible fluid and (b) hydrostatic state of stress.

6.25 Show that for an incompressible fluid the equation of motion simplifies to

$$\frac{D}{Dt}(\rho\mathbf{v}) = \rho\,\mathbf{f} - \boldsymbol{\nabla}p + \mu\nabla^2\mathbf{v}.$$

6.26 Show that for the two-dimensional flow of an incompressible Newtonian fluid with $\boldsymbol{\nabla}\times\mathbf{f} = 0$, where $\mathbf{f}$ is the body force vector (measured per unit volume), the vorticity $\mathbf{w}$ [see Eq. (3.6.5)] satisfies the diffusion equation

$$\rho\frac{D\mathbf{w}}{Dt} = \mu\nabla^2\mathbf{w}.$$

6.27 *Stokesian fluid.* A Stokesian fluid is one in which (a) the stress tensor $\boldsymbol{\sigma}$ is a continuous function of the rate of deformation tensor $\mathbf{D}$ and the local thermodynamic state (i.e., may depend on temperature), but independent of other kinematic variables; (b) $\boldsymbol{\sigma}$ is not an explicit function of position $\mathbf{x}$; (c) constitutive behavior is isotropic; and (d) the stress is hydrostatic when the rate of deformation is zero, $\mathbf{D} = \mathbf{0}$. Consider the following constitutive equation for a Stokesian fluid:

$$\boldsymbol{\sigma} = -p\,\mathbf{I} + \mu\mathbf{D} + \beta\,\mathbf{D}\cdot\mathbf{D} \quad (\sigma_{ij} = -p\,\delta_{ij} + \mu D_{ij} + \beta\,D_{ik}D_{kj}).$$

Write the equations of motion (5.3.10) in terms of p and $\mathbf{D}$ for a Stokesian fluid. Note that a linear Stokesian fluid is a Newtonian fluid.

6.28 *Irrotational motion.* The velocity field $\mathbf{v}$ is said to be irrotational when the vorticity is zero, $\mathbf{w} = \mathbf{0}$. Then there exists a velocity potential $\phi(\mathbf{x}, t)$ such that $\mathbf{v} = \boldsymbol{\nabla}\phi$. Show that the Navier–Stokes equations of Problem **6.24** can be expressed in the form

$$\rho\boldsymbol{\nabla}\left[\frac{\partial\phi}{\partial t} + \tfrac{1}{2}(\boldsymbol{\nabla}\phi)^2\right] = \rho\,\mathbf{f} - \boldsymbol{\nabla}p + (\lambda + 2\mu)\boldsymbol{\nabla}(\nabla^2\phi).$$

6.29 Show that in the case of irrotational body force $\mathbf{f} = -\boldsymbol{\nabla}V$ and when p is a function only of ρ

$$\frac{\partial\phi}{\partial t} + \tfrac{1}{2}(\boldsymbol{\nabla}\phi)^2 + V + P(\rho) - \frac{1}{\rho}(\lambda + 2\mu)\nabla^2\phi = g(t),$$

where $P(\rho) = \int_{p_0}^{p} dp/\rho$, p_0 is a constant, and $g(t)$ is a function of time only.

HEAT TRANSFER

6.30 Show that for an isotropic Newtonian fluid the energy equation can be expressed in the form

$$\rho\frac{De}{Dt} = \boldsymbol{\nabla}\cdot(k\boldsymbol{\nabla}\theta) - p\,J_1 + (\lambda+2\mu)J_1^2 - 4\mu J_2 + \rho\,r,$$

where J_1 and J_2 are the principal invariants of $\mathbf{D}$ [see Eq. (3.4.36)], and k is the conductivity.

6.31 Show that for an isotropic Newtonian fluid the energy equation can be expressed in the form

$$\rho\theta\frac{D\eta}{Dt} = \boldsymbol{\nabla}\cdot(k\boldsymbol{\nabla}\theta) + (\lambda+2\mu)J_1^2 - 4\mu J_2 + \rho\,r,$$

where θ is the absolute temperature, η is the entropy, J_1 and J_2 are the principal invariants of $\mathbf{D}$, and k is the conductivity. *Hint:* $\theta\,d\eta = de + p\,d(1/\rho)$ and $d/dt = D/Dt$.

6.32 The thermal stress coefficients, β_{ij}, measure the increases in the stress components per unit decrease in temperature with no change in the strain, that is,

$$\beta_{ij} = -\frac{\partial\sigma_{ij}}{\partial\theta}\bigg|_{\boldsymbol{\varepsilon}=\text{const}}.$$

Deduce from the above equation the result

$$\beta_{ij} = \rho_0\frac{\partial\eta}{\partial\varepsilon_{ij}}.$$

6.33 The specific heat at constant strain is defined by

$$c_v = \frac{\partial e}{\partial\theta}\bigg|_{\boldsymbol{\varepsilon}=\text{const}}.$$

Deduce from the above equation the result

$$c_v = -\theta\,\frac{\partial^2\Psi}{\partial\theta^2}.$$

6.34 Consider a reference state at zero strain and temperature θ_0, and expand $\Psi(\theta,\boldsymbol{\varepsilon})$ in Taylor's series about this state up to quadratic terms in θ and ε_{ij} to derive the constitutive equations, Eq. (6.8.20), for linear thermoelasticity. Specialize the relations to the isotropic case.