7
LINEARIZED ELASTICITY

You cannot depend on your eyes when your imagination is out of focus.

— Mark Twain (1835–1910)

Research is to see what everybody else has seen, and to think what nobody else has thought.

— Albert Szent-Gyoergi (1893-1986)

7.1 Introduction

This chapter is dedicated to the study of deformation and stress in solid bodies under a prescribed set of forces and kinematic constraints. In a majority of problems, we assume that stresses and strains are small so that linear strain-displacement relations and Hooke's law are valid, and we use appropriate governing equations derived using the Lagrangian description in the previous chapters to solve them for stresses and displacements. In the linearized elasticity we assume that the geometric changes are so small that we neglect squares of the displacement gradients, that is, $|\boldsymbol{\nabla}\mathbf{u}|^2 \approx 0$, and do not make a distinction between the deformed and undeformed geometries, between the second Piola–Kirchhoff stress tensor $\mathbf{S}$ and the Cauchy stress tensor $\boldsymbol{\sigma}$, and between the current coordinates $\mathbf{x}$ and the material coordinates $\mathbf{X}$ (and use $\boldsymbol{\sigma}$ and $\mathbf{x}$). Mathematically, we seek solutions to coupled partial differential equations over an elastic domain occupied by the reference (or undeformed) configuration of the body, subject to specified boundary conditions on displacements or forces. Such problems are called boundary value problems of elasticity.

Most practical problems of even linearized elasticity involve geometries that are complicated, and analytical solutions to such problems cannot be obtained. Therefore, the objective here is to familiarize the reader with certain solution methods as applied to simple boundary value problems. Boundary value problems discussed in most elasticity books are about the same, and they illustrate the methodologies used in the analytical solution of problems of elasticity. Although is a book on a first course in continuum mechanics, typical solid mechanics problems discussed in most elasticity books, for example, Timoshenko and Goodier (1970), Slaughter (2002), and Sadd (2004) are covered. The methods discussed here may not be directly useful in solving practical engineering problems, but the discussion provides certain insights into the formulation and solution of boundary value problems. These insights are useful irrespective of the specific problems or methods of solution presented here.

7.2 Governing Equations

7.2.1 Preliminary Comments

It is useful to summarize the equations of linearized elasticity for use in the later sections of this chapter. The governing equations of a three-dimensional elastic body involve (1) 6 strain-displacement relations among 9 variables, namely 6 components of strain tensor $\boldsymbol{\varepsilon}$ and 3 components of displacement vector $\mathbf{u}$; (2) 3 equations of motion among 6 components of stress tensor $\boldsymbol{\sigma}$, assuming symmetry of the stress tensor; and (3) 6 stress–strain equations among 6 stress and 6 strain components that are already counted. Thus, there are a total of 15 coupled equations among 15 scalar field variables. These equations are listed here in vector form and Cartesian, cylindrical, and spherical component forms for an isotropic body occupying a domain Ω with closed boundary Γ in the reference configuration. Figures 7.2.1(a)–(c) show the normal stress components in the three coordinate systems; shear stress components should be obvious (as well as all of the strain components).

7.2.2 Summary of Equations

All of the equations derived in Chapters 3, 4, 5, and 6 in material description are presented here. Throughout this chapter, we use the following notations: $\mathbf{x} = \mathbf{X}$, $\boldsymbol{\varepsilon} = \mathbf{E}$, and $\boldsymbol{\sigma} = \mathbf{S}$.

7.2.2.1 Strain-displacement equations

The linearized strain-displacement relations are summarized here:

Vector form:

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left[\boldsymbol{\nabla}\mathbf{u} + (\boldsymbol{\nabla}\mathbf{u})^{\mathrm{T}}\right] \tag{7.2.1}$$

Rectangular Cartesian component form: (u_x, u_y, u_z)

$$\begin{aligned}
\varepsilon_{xx} &= \frac{\partial u_x}{\partial x}, & \varepsilon_{xy} &= \tfrac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)\\
\varepsilon_{xz} &= \tfrac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right), & \varepsilon_{yy} &= \frac{\partial u_y}{\partial y}\\
\varepsilon_{yz} &= \tfrac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right), & \varepsilon_{zz} &= \frac{\partial u_z}{\partial z}
\end{aligned} \tag{7.2.2}$$

Component form in cylindrical coordinates: (u_r, u_θ, u_z)

$$\begin{aligned}
\varepsilon_{rr} &= \frac{\partial u_r}{\partial r}, & \varepsilon_{r\theta} &= \tfrac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)\\
\varepsilon_{rz} &= \tfrac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right), & \varepsilon_{\theta\theta} &= \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}\\
\varepsilon_{z\theta} &= \tfrac{1}{2}\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right), & \varepsilon_{zz} &= \frac{\partial u_z}{\partial z}
\end{aligned} \tag{7.2.3}$$

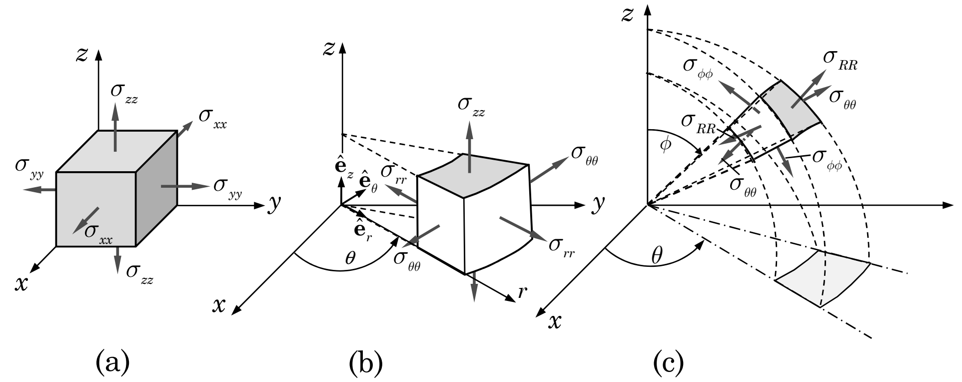

Fig. 7.2.1: Components of a second-order tensor (stress) on a typical volume element in (a) Cartesian, (b) cylindrical, and (c) spherical coordinate systems.

Component form in spherical coordinates: (u_R, u_ϕ, u_θ)

$$
\begin{aligned}
\varepsilon_{RR} &= \frac{\partial u_R}{\partial R}, \quad \varepsilon_{\phi\phi} = \frac{1}{R}\left(\frac{\partial u_\phi}{\partial \phi} + u_R\right) \\
\varepsilon_{R\phi} &= \tfrac{1}{2}\left(\frac{1}{R}\frac{\partial u_R}{\partial \phi} + \frac{\partial u_\phi}{\partial R} - \frac{u_\phi}{R}\right) \\
\varepsilon_{R\theta} &= \tfrac{1}{2}\left(\frac{1}{R\sin\phi}\frac{\partial u_R}{\partial \theta} + \frac{\partial u_\theta}{\partial R} - \frac{u_\theta}{R}\right) \\
\varepsilon_{\phi\theta} &= \tfrac{1}{2}\frac{1}{R}\left(\frac{1}{\sin\phi}\frac{\partial u_\phi}{\partial \theta} + \frac{\partial u_\theta}{\partial \phi} - u_\theta \cot\phi\right) \\
\varepsilon_{\theta\theta} &= \frac{1}{R\sin\phi}\left(\frac{\partial u_\theta}{\partial \theta} + u_R \sin\phi + u_\phi \cos\phi\right)
\end{aligned}
\tag{7.2.4}
$$

7.2.2.2 Equations of motion

The linearized equations of motion, under the assumption that $\boldsymbol{\sigma}^{\mathrm{T}} = \boldsymbol{\sigma}$, are summarized. The equilibrium equations are obtained by setting the acceleration terms to zero. Here $\mathbf{f}$ is the body force vector measured per unit mass.

Vector form

$$
\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \rho_0 \mathbf{f} = \rho_0 \frac{\partial^2 \mathbf{u}}{\partial t^2} \tag{7.2.5}
$$

Rectangular Cartesian component form: $(\sigma_{xx}, \sigma_{yy}, \sigma_{xy}, \cdots)$

$$
\begin{aligned}
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + \rho_0 f_x &= \rho_0 \frac{\partial^2 u_x}{\partial t^2} \\
\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + \rho_0 f_y &= \rho_0 \frac{\partial^2 u_y}{\partial t^2} \\
\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + \rho_0 f_z &= \rho_0 \frac{\partial^2 u_z}{\partial t^2}
\end{aligned}
\tag{7.2.6}
$$

Component form in cylindrical coordinates: $(\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{r\theta}, \cdots)$

$$\begin{aligned}\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) + \rho_0 f_r = \rho_0 \frac{\partial^2 u_r}{\partial t^2}\\ \frac{\partial \sigma_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{\sigma_{\theta r} + \sigma_{r\theta}}{r} + \rho_0 f_\theta = \rho_0 \frac{\partial^2 u_\theta}{\partial t^2}\\ \frac{\partial \sigma_{zr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{z\theta}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{zr}}{r} + \rho_0 f_z = \rho_0 \frac{\partial^2 u_z}{\partial t^2}\end{aligned} \tag{7.2.7}$$

Component form in spherical coordinates: $(\sigma_{RR}, \sigma_{\phi\phi}, \sigma_{R\phi}, \cdots)$

$$\begin{aligned}&\frac{\partial \sigma_{RR}}{\partial R} + \frac{1}{R}\frac{\partial \sigma_{R\phi}}{\partial \phi} + \frac{1}{R\sin\phi}\frac{\partial \sigma_{R\theta}}{\partial \theta} + \frac{1}{R}(2\sigma_{RR} - \sigma_{\phi\phi} - \sigma_{\theta\theta} + \sigma_{R\phi}\cot\phi)\\ &\qquad + \rho_0 f_R = \rho_0 \frac{\partial^2 u_R}{\partial t^2}\\ &\frac{\partial \sigma_{\phi R}}{\partial R} + \frac{1}{R}\frac{\partial \sigma_{\phi\phi}}{\partial \phi} + \frac{1}{R\sin\phi}\frac{\partial \sigma_{\phi\theta}}{\partial \theta} + \frac{1}{R}[(\sigma_{\phi\phi} - \sigma_{\theta\theta})\cot\phi + \sigma_{R\phi} + 2\sigma_{\phi R}]\\ &\qquad + \rho_0 f_\phi = \rho_0 \frac{\partial^2 u_\phi}{\partial t^2}\\ &\frac{\partial \sigma_{\theta R}}{\partial R} + \frac{1}{R}\frac{\partial \sigma_{\theta\phi}}{\partial \phi} + \frac{1}{R\sin\phi}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{1}{R}[(\sigma_{\phi\theta} + \sigma_{\theta\phi})\cot\phi + \sigma_{R\theta}]\\ &\qquad + \rho_0 f_\theta = \rho_0 \frac{\partial^2 u_\theta}{\partial t^2}\end{aligned} \tag{7.2.8}$$

7.2.2.3 Constitutive equations

The stress–strain equations of a linear, isotropic, elastic body are presented here.

Vector form

$$\boldsymbol{\sigma} = 2\mu\boldsymbol{\varepsilon} + \lambda\,(\mathrm{tr}\,\boldsymbol{\varepsilon})\,\mathbf{I} \tag{7.2.9}$$

Rectangular Cartesian, cylindrical, and spherical component forms

$$\begin{Bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & \nu & 0 & 0 & 0\\ \nu & 1-\nu & \nu & 0 & 0 & 0\\ \nu & \nu & 1-\nu & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2}\end{bmatrix}\begin{Bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ 2\varepsilon_{23}\\ 2\varepsilon_{13}\\ 2\varepsilon_{12}\end{Bmatrix} \tag{7.2.10}$$

Here the subscripts 1, 2, and 3 take x, y, and z for rectangular Cartesian coordinates; r, θ, and z for cylindrical coordinates; and R, ϕ, and θ for spherical coordinates. The Lamé constants μ and λ are related to Young's modulus E and Poisson's ratio ν by

$$\mu = G = \frac{E}{2(1+\nu)}\,, \quad \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}\,. \tag{7.2.11}$$

Equations (7.2.1)–(7.2.10) are valid for all problems of linearized elasticity; various problems differ from each other only in (a) geometry of the domain, (b) boundary conditions, and (c) values of the material parameters E and ν. The general form of the boundary condition is presented next.

7.2.2.4 Boundary conditions

Vector form

$$\mathbf{u} = \hat{\mathbf{u}} \ \text{ on } \ \Gamma_u, \quad \hat{\mathbf{n}} \cdot \boldsymbol{\sigma} = \hat{\mathbf{t}} \ \text{ on } \ \Gamma_\sigma. \tag{7.2.12}$$

Component form

$$u_i = \hat{u}_i \ \text{ on } \ \Gamma_u, \quad \sigma_{ij}\, n_j = \hat{t}_i \ \text{ on } \ \Gamma_\sigma, \tag{7.2.13}$$

where Γ_σ and Γ_u are disjoint portions (except for a point) of the boundary whose union is equal to the total boundary Γ, and quantities with a hat are specified values. Note that only one element of the pair (t_i, u_i), for any $i = 1, 2, 3$, may be specified at a point on the boundary. The indices (1, 2, 3) may take the values of (x, y, z), (r, θ, z), and (R, ϕ, θ).

7.2.2.5 Compatibility conditions

In addition to the 15 equations listed in (7.2.1), (7.2.5), and (7.2.9), there are 6 *compatibility conditions* among 6 components of strain:

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{\varepsilon})^{\mathrm{T}} = \mathbf{0}, \quad e_{ikr} e_{j\ell s} \varepsilon_{ij,k\ell} = 0. \tag{7.2.14}$$

Recall that the compatibility equations are necessary and sufficient conditions on the strain field to ensure the existence of a corresponding displacement field. Associated with each displacement field, there is a unique strain field as given by Eq. (7.2.1) and there is no need to use the compatibility conditions. The compatibility conditions are required only when the strain or stress field is given and the displacement field is to be determined.

In most formulations of boundary value problems of elasticity, one does not use the 15 equations in 15 unknowns. Most often, the 15 equations are reduced to either 3 equations in terms of displacement field or 6 equations in terms of stress field. The two sets of equations are presented next.

7.2.3 The Navier Equations

The 15 equations can be combined into 3 equations by substituting strain-displacement equations into the stress–strain relations and the result into the equations of equilibrium. We shall carry out this process using the Cartesian component form (in index notation) and then express the final result in vector as well as Cartesian component forms.

The Cartesian component form of Eq. (7.2.9) is

$$\sigma_{ij} = \mu\,(u_{i,j} + u_{j,i}) + \lambda u_{k,k} \delta_{ij}. \tag{7.2.15}$$

Substituting into Eq. (7.2.5), we arrive at the equations

$$\begin{aligned}\rho_0 \frac{\partial^2 u_i}{\partial t^2} &= \sigma_{ji,j} + \rho_0 f_i \\ &= \mu \left(u_{i,jj} + u_{j,ij} \right) + \lambda u_{k,ki} + \rho_0 f_i \\ &= \mu u_{i,jj} + (\mu + \lambda) u_{j,ji} + \rho_0 f_i. \end{aligned} \tag{7.2.16}$$

Thus, we have

$$\begin{aligned}\mu \nabla^2 \mathbf{u} + (\mu + \lambda) \boldsymbol{\nabla} \left(\boldsymbol{\nabla} \cdot \mathbf{u} \right) + \rho_0 \mathbf{f} &= \rho_0 \frac{\partial^2 \mathbf{u}}{\partial t^2}, \\ \mu u_{i,jj} + (\mu + \lambda) u_{j,ji} + \rho_0 f_i &= \rho_0 \frac{\partial^2 u_i}{\partial t^2}. \end{aligned} \tag{7.2.17}$$

These are called *Lamé–Navier equations* of elasticity, and they represent the equilibrium equations expressed in terms of the displacement field. The boundary conditions (7.2.13) can be expressed in terms of the displacement field as

$$\left[n_j \mu \left(u_{i,j} + u_{j,i} \right) + n_i \lambda u_{k,k} \right] = \hat{t}_i \ \text{ on } \ \Gamma_\sigma, \quad u_i = \hat{u}_i \ \text{ on } \ \Gamma_u. \tag{7.2.18}$$

Equations (7.2.17) and (7.2.18) together describe the boundary value problem of linearized elasticity.

7.2.4 The Beltrami–Michell Equations

Alternative to the formulation of Section 7.2.3, the 12 equations from (7.2.5) and (7.2.9) and 6 equations from (7.2.14) can be combined into 6 equations in terms of the stress field. Substitution of the constitutive (strain-stress) equations

$$\varepsilon_{ij} = \frac{1}{E} \left[(1 + \nu) \sigma_{ij} - \nu \sigma_{kk} \delta_{ij} \right] \tag{7.2.19}$$

into the compatibility equations (7.2.14) yields

$$\begin{aligned} 0 &= e_{ikr} e_{j\ell s} \, \varepsilon_{ij,k\ell} \\ &= e_{ikr} e_{j\ell s} \left[(1 + \nu) \sigma_{ij,k\ell} - \nu \sigma_{mm,k\ell} \delta_{ij} \right] \\ &= (1 + \nu) e_{ikr} e_{j\ell s} \sigma_{ij,k\ell} - \nu e_{ikr} e_{i\ell s} \sigma_{mm,k\ell} \\ &= (1 + \nu) e_{ikr} e_{j\ell s} \sigma_{ij,k\ell} - \nu \left(\delta_{k\ell} \delta_{rs} - \delta_{ks} \delta_{\ell r} \right) \sigma_{mm,k\ell} \\ &= (1 + \nu) e_{ikr} e_{j\ell s} \sigma_{ij,k\ell} - \nu \left(\delta_{rs} \sigma_{mm,kk} - \sigma_{mm,rs} \right). \end{aligned} \tag{7.2.20}$$

In view of the identity

$$\begin{aligned} e_{ikr} e_{j\ell s} = \begin{vmatrix} \delta_{ij} & \delta_{i\ell} & \delta_{is} \\ \delta_{kj} & \delta_{k\ell} & \delta_{ks} \\ \delta_{rj} & \delta_{r\ell} & \delta_{rs} \end{vmatrix} &= \delta_{ij} \delta_{k\ell} \delta_{rs} - \delta_{ij} \delta_{ks} \delta_{r\ell} - \delta_{kj} \delta_{i\ell} \delta_{rs} + \delta_{kj} \delta_{r\ell} \delta_{is} \\ &\quad + \delta_{rj} \delta_{i\ell} \delta_{ks} - \delta_{rj} \delta_{k\ell} \delta_{is}, \end{aligned} \tag{7.2.21}$$

Eq. (7.2.20) simplifies to

$$\delta_{rs} \sigma_{ii,jj} - \sigma_{ii,rs} - (1 + \nu) \left(\delta_{rs} \sigma_{ij,ij} + \sigma_{rs,ii} - \sigma_{is,ir} - \sigma_{ir,is} \right) = 0. \tag{7.2.22}$$

Contracting the indices r and s ($s \to r$) gives

$$2\sigma_{ii,jj} - (1+\nu)\left(\sigma_{ij,ij} + \sigma_{jj,ii}\right) = 0.$$

Simplifying the above result, we obtain

$$\sigma_{ii,jj} = \frac{(1+\nu)}{(1-\nu)}\sigma_{ij,ij}. \tag{7.2.23}$$

Substituting this result back into Eq. (7.2.22) leads to

$$\sigma_{ij,kk} + \frac{1}{1+\nu}\sigma_{kk,ij} = \frac{\nu}{1-\nu}\sigma_{rs,rs}\delta_{ij} + \sigma_{kj,ki} + \sigma_{ki,kj}. \tag{7.2.24}$$

Next, we use the equilibrium equations to compute the second derivative of the stress components, $\sigma_{rs,rk} = -\rho_0\, f_{s,k}$. We have

$$\sigma_{ij,kk} + \frac{1}{1+\nu}\sigma_{kk,ij} = -\frac{\nu\rho_0}{1-\nu} f_{k,k}\delta_{ij} - \rho_0\left(f_{j,i} + f_{i,j}\right), \tag{7.2.25}$$

or in vector form

$$\nabla^2\boldsymbol{\sigma} + \frac{1}{1+\nu}\boldsymbol{\nabla}[\boldsymbol{\nabla}\,(\mathrm{tr}\,\boldsymbol{\sigma})] = -\frac{\nu\rho_0}{1-\nu}(\boldsymbol{\nabla}\cdot\mathbf{f})\,\mathbf{I} - \rho_0\left[\boldsymbol{\nabla}\mathbf{f} + (\boldsymbol{\nabla}\mathbf{f})^{\mathrm{T}}\right]. \tag{7.2.26}$$

The 6 equations in (7.2.25) or (7.2.26), called *Michell's equations*, provide the necessary and sufficient conditions for an equilibrated stress field to be compatible with the displacement field in the body. The traction boundary conditions in Eq. (7.2.13) are valid for this formulation.

When the body force is uniform, we have $\boldsymbol{\nabla}\cdot\mathbf{f} = 0$ and $\boldsymbol{\nabla}\mathbf{f} = \mathbf{0}$, and Michell's equations (7.2.26) reduce to *Beltrami's equations*

$$\nabla^2\boldsymbol{\sigma} + \frac{1}{1+\nu}\boldsymbol{\nabla}[\boldsymbol{\nabla}\,(\mathrm{tr}\,\boldsymbol{\sigma})] = \mathbf{0}, \quad \sigma_{ij,kk} + \frac{1}{1+\nu}\sigma_{kk,ij} = 0. \tag{7.2.27}$$

7.3 Solution Methods

7.3.1 Types of Problems

The equilibrium problems, also called boundary value problems, of elasticity can be classified into three types on the basis of the nature of specified boundary conditions. They are outlined next.

Type I. Boundary value problems in which all specified boundary conditions are of the displacement type

$$\mathbf{u} = \hat{\mathbf{u}} \quad \text{on} \quad \Gamma, \tag{7.3.1}$$

are called boundary value problems of type I or *displacement boundary value problems.*

Type II. Boundary value problems in which all specified boundary conditions are of the traction type,

$$\mathbf{t} = \hat{\mathbf{t}} \quad \text{on} \quad \Gamma, \tag{7.3.2}$$

are called boundary value problems of type II or *stress boundary value problems.* Such boundary value problems are rare because most practical problems involve specifying displacements that eliminate rigid-body motion.

Type III. Boundary value problems in which all specified boundary conditions are of the mixed type,

$$\mathbf{u} = \hat{\mathbf{u}} \ \text{ on } \ \Gamma_u \quad \text{and} \quad \mathbf{t} = \hat{\mathbf{t}} \ \text{ on } \ \Gamma_\sigma, \tag{7.3.3}$$

are called boundary value problems of type III or *mixed boundary value problems.* Most practical problems, including contact problems, fall into this category.

7.3.2 Types of Solution Methods

An *exact solution* of a problem is one that satisfies the governing differential equation(s) at every point of the domain as well as the boundary conditions exactly. In general, finding exact solutions of elasticity problems is not simple owing to complicated geometries and boundary conditions. An *approximate solution* is one that satisfies governing differential equations as well as the boundary conditions approximately. *Numerical solutions* are approximate solutions that are developed using a numerical method, such as the finite difference method, the finite element method, the boundary element method, and other methods. Often one seeks approximate solutions of practical problems using numerical methods. The phrase *analytical solution* is used to indicate that the solution, exact or approximate, is obtained using analytical means rather than by numerical methods. Also, one may obtain exact solution to an idealized (or approximate) mathematical model of the actual problem. Most of the exact solutions found in textbooks fall into this category.

There are several types of solution methods for finding analytical solutions [see Slaughter (2002)]. The most common methods are described here.

1. The *inverse method* is one in which one finds the solution for displacement, strain, and stress fields by solving the governing equations of elasticity, and then tries to find a problem with geometry and boundary conditions to which the fields correspond. This approach is more common with mathematicians than with engineers.

2. The *semi-inverse method* is one in which the solution form in terms of unknown functions is arrived at with the help of a qualitative understanding of the problem characteristics. The unknown functions are determined to satisfy the governing equations. In identifying a solution form, often assumptions are made about the displacement or stress field (in addition to the constitutive behavior) to reduce a three-dimensional problem to a two-dimensional or even one-dimensional problem. Very few problems of elasticity have exact solutions, and the assumed fields in most cases are approximate. The semi-inverse method is the most commonly used approach in solid mechanics.

3. The *method of potentials* is one in which potential functions (with unknowns) are introduced to trivially satisfy some or all of the governing equations, and the functions are then determined using the remaining governing equations as well as boundary conditions of the problem. The potential functions are then used to determine stresses, strains, and displacements.

4. *Variational methods* are those that make use of extremum (i.e., minimum or maximum) and stationary principles, which are equivalent to the governing equations and some of the boundary conditions of the problem. The principles are cast in terms of strain energy, work done by loads, and kinetic energy of the system. The variational methods have the added advantage of being approximate methods. Variational methods form the basis of certain numerical methods such as the finite element method.

Other analytical methods include complex variable methods, integral transform methods, perturbation methods, method of multiple scales, and so on. In the remainder of this chapter, we consider mostly the semi-inverse method and the method of potentials to formulate and solve certain problems of linearized elasticity.

7.3.3 Examples of the Semi-inverse Method

In the first problem (spherical pressure vessel) considered in this section, the displacement field is assumed in terms of an unknown function, and then the equations of elasticity or their equivalents are used to determine differential equations governing the unknown function. In the second problem (deformation of a prismatic bar under its own weight), the state of stress is assumed in terms of an unknown function and the equations of elasticity are used to determine the unknown function, strains, and displacements. In the first problem, even though the semi-inverse method is used, the assumed form of the solution happens to be exact. This is not the case in most problems of elasticity. These two examples illustrate the general methodology of solving elasticity problems by the semi-inverse method. The key element of the approach is to gain sufficient qualitative understanding of solution (displacements and stresses) before identifying the solution form.

Example 7.3.1

Consider an isotropic, hollow spherical pressure vessel of internal radius a and outside radius b. The vessel is pressurized at $r = a$ as well as at $r = b$ with pressures p_a and p_b, respectively, as shown in Fig. 7.3.1(a). Determine the displacements, strains, and stresses in the pressure vessel.

Solution: We use the spherical coordinate system to formulate the problem. Based on the spherical symmetry of the geometry, boundary conditions, and material properties, we note that the solution also exhibits spherical symmetry, that is, the solution does not depend on ϕ and θ coordinates [see Fig. 7.3.1(b)]. In fact, the only nonzero displacement is u_R, and it is only a function of the radial distance R. Thus, this three-dimensional elasticity problem can be formulated as a two-dimensional one without any approximation.

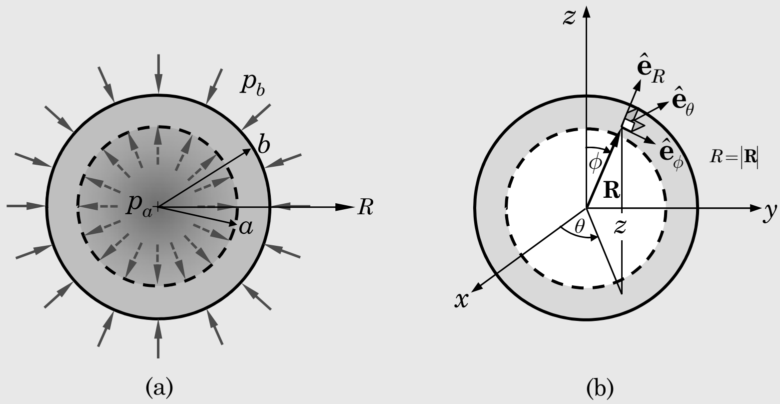

Fig. 7.3.1: A spherical pressure vessel.

For this problem, only the following stress boundary conditions (BVP type II) are known:

$$\begin{aligned} &\text{At } R = a: \quad \hat{\mathbf{n}} = -\hat{\mathbf{e}}_R, \quad \mathbf{t} = p_a\,\hat{\mathbf{e}}_R \quad \text{or} \quad \sigma_{RR} = -p_a, \quad \sigma_{R\phi} = \sigma_{R\theta} = 0, \\ &\text{At } R = b: \quad \hat{\mathbf{n}} = \hat{\mathbf{e}}_R, \quad \mathbf{t} = -p_b\,\hat{\mathbf{e}}_R \quad \text{or} \quad \sigma_{RR} = -p_b, \quad \sigma_{R\phi} = \sigma_{R\theta} = 0. \end{aligned} \tag{7.3.4}$$

Based on our qualitative understanding of the solution to the problem, we begin with the assumed displacement field

$$u_R = U(R), \quad u_\phi = u_\theta = 0, \tag{7.3.5}$$

where $U(R)$ is an unknown function to be determined such that the equations of elasticity and boundary conditions of the problem are satisfied. If we cannot find $U(R)$ that satisfies the governing equations, then we must abandon the assumption in Eq. (7.3.5).

The only nonzero strains associated with the displacement field (7.3.5) are [see Eq. (7.2.4)]

$$\varepsilon_{RR} = \frac{dU}{dR}, \quad \varepsilon_{\phi\phi} = \varepsilon_{\theta\theta} = \frac{1}{R}U(R). \tag{7.3.6}$$

The nonzero stresses are

$$\begin{aligned} \sigma_{RR} &= 2\mu\varepsilon_{RR} + \lambda\left(\varepsilon_{RR} + \varepsilon_{\phi\phi} + \varepsilon_{\theta\theta}\right) = (2\mu + \lambda)\frac{dU}{dR} + 2\lambda\frac{U}{R}\,, \\ \sigma_{\phi\phi} &= 2\mu\varepsilon_{\phi\phi} + \lambda\left(\varepsilon_{RR} + \varepsilon_{\phi\phi} + \varepsilon_{\theta\theta}\right) = 2(\mu + \lambda)\frac{U}{R} + \lambda\frac{dU}{dR}\,, \\ \sigma_{\theta\theta} &= \sigma_{\phi\phi}. \end{aligned} \tag{7.3.7}$$

The last two equations of equilibrium, Eq. (7.2.8) without the body force and acceleration terms, are trivially satisfied, and the first equation reduces to

$$\frac{d\sigma_{RR}}{dR} + \frac{1}{R}\left(2\sigma_{RR} - \sigma_{\phi\phi} - \sigma_{\theta\theta}\right) = 0, \tag{7.3.8}$$

which can be expressed in terms of the displacement function $U(R)$ using Eq. (7.3.7)

$$\begin{aligned} &(2\mu + \lambda)\frac{d^2U}{dR^2} + \frac{2\lambda}{R}\frac{dU}{dR} - 2\lambda\frac{U}{R^2} \\ &+\frac{1}{R}\left[2(2\mu + \lambda)\frac{dU}{dR} + 4\lambda\frac{U}{R} - 2\lambda\frac{dU}{dR} - 4(\mu + \lambda)\frac{U}{R}\right] = 0. \end{aligned} \tag{7.3.9}$$

Simplifying the expression, we obtain

$$R^2\frac{d^2U}{dR^2} + 2R\frac{dU}{dR} - 2U = 0. \tag{7.3.10}$$

The linear differential equation (7.3.10) can be transformed to one with constant coefficients by a change of independent variable, $R = e^{\xi}$ (or $\xi = \ln R$). Using the chain rule of differentiation, we obtain

$$\frac{dU}{dR} = \frac{dU}{d\xi}\frac{d\xi}{dR} = \frac{1}{R}\frac{dU}{d\xi}, \quad \frac{d^2U}{dR^2} = \frac{d}{dR}\left(\frac{1}{R}\frac{dU}{d\xi}\right) = \frac{1}{R^2}\left(-\frac{dU}{d\xi} + \frac{d^2U}{d\xi^2}\right).$$

Substituting the above expressions into (7.3.10), we obtain

$$\frac{d^2U}{d\xi^2} + \frac{dU}{d\xi} - 2U = 0. \tag{7.3.11}$$

Seeking solution in the form $U(\xi) = e^{m\xi}$ and substituting it into Eq. (7.3.11), we obtain $(m-1)(m+2) = 0$. Hence, the general solution to the problem is

$$U(\xi) = c_1 e^{\xi} + c_2 e^{-2\xi}. \tag{7.3.12}$$

Changing back to the original independent variable R, the radial displacement is

$$u_R(R) = U(R) = c_1 R + \frac{c_2}{R^2}, \tag{7.3.13}$$

where the constants c_1 and c_2 are to be determined using the boundary conditions in Eq. (7.3.4). Hence, we must compute σ_{RR},

$$\begin{aligned}\sigma_{RR} &= (2\mu+\lambda)\left(c_1 - c_2\frac{2}{R^3}\right) + 2\lambda\left(c_1 + c_2\frac{1}{R^3}\right)\\ &= (2\mu+3\lambda)c_1 - 4\mu c_2\frac{1}{R^3}.\end{aligned} \tag{1}$$

Applying the stress boundary conditions in (7.3.5) and (7.3.6), we obtain

$$\begin{aligned}(2\mu+3\lambda)c_1 - \frac{4\mu c_2}{a^3} &= -p_a,\\ (2\mu+3\lambda)c_1 - \frac{4\mu c_2}{b^3} &= -p_b.\end{aligned} \tag{2}$$

Solving for the constants c_1 and c_2, we obtain

$$c_1 = \frac{1}{(2\mu+3\lambda)}\left(\frac{p_a a^3 - p_b b^3}{b^3 - a^3}\right), \quad c_2 = \frac{a^3 b^3}{4\mu}\left(\frac{p_a - p_b}{b^3 - a^3}\right). \tag{3}$$

Finally, the displacement u_R and stresses σ_{RR}, $\sigma_{\phi\phi}$, and $\sigma_{\theta\theta}$ in the sphere are given by

$$u_R(R) = \frac{R}{(2\mu+3\lambda)}\left(\frac{p_a a^3 - p_b b^3}{b^3 - a^3}\right) + \frac{a^3 b^3}{4\mu R^2}\left(\frac{p_a - p_b}{b^3 - a^3}\right), \tag{7.3.14}$$

$$\begin{aligned}\sigma_{RR} &= \left(\frac{p_a a^3 - p_b b^3}{b^3 - a^3}\right) - \frac{a^3 b^3}{R^3}\left(\frac{p_a - p_b}{b^3 - a^3}\right),\\ \sigma_{\phi\phi} = \sigma_{\theta\theta} &= \left(\frac{p_a a^3 - p_b b^3}{b^3 - a^3}\right) + \frac{a^3 b^3}{2R^3}\left(\frac{p_a - p_b}{b^3 - a^3}\right).\end{aligned} \tag{7.3.15}$$

Since the off-diagonal elements of the stress tensor are zero, that is, $\sigma_{R\phi} = \sigma_{R\theta} = \sigma_{\phi\theta} = 0$, σ_{RR}, $\sigma_{\phi\phi}$, and $\sigma_{\theta\theta}$ are the principal stresses, with $\hat{\mathbf{e}}_R$, $\hat{\mathbf{e}}_\phi$, and $\hat{\mathbf{e}}_\theta$ being the principal directions, respectively.

Example 7.3.2

Consider a prismatic bar with dimensions $2a \times 2b \times L$ and mass density ρ in a gravitational field $\mathbf{g} = -g\,\hat{\mathbf{e}}_3$. The top surface of the bar is attached to a rigid support in such a way that $u = v = w = 0$ at $x = y = 0, z = L$, as shown in Fig. 7.3.2. Use the semi-inverse method to determine the displacements i the body.

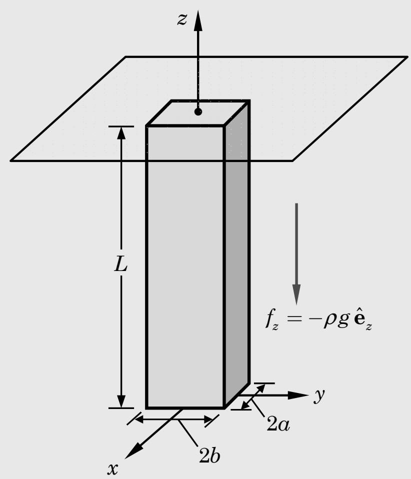

Fig. 7.3.2: Deformation of a prismatic bar under its own weight.

Solution: First we summarize the boundary conditions. We have

$$\mathbf{u}(0,0,L) = \mathbf{0}, \quad \mathbf{t}(x,y,0) = \mathbf{0}, \quad \mathbf{t}(x,\pm b,z) = \mathbf{0}, \quad \mathbf{t}(\pm a,y,z) = \mathbf{0}. \tag{7.3.16}$$

Thus, we find that

$$\sigma_{xx} = \sigma_{yy} = \sigma_{xy} = \sigma_{xz} = \sigma_{yz} = 0,$$

on the boundary, except at the point $x = y = 0$ and $z = L$. Since there are no other geometric constraints (that is, the body is free to change its geometry), it does not develop the stresses σ_{xx}, σ_{yy}, σ_{xy}, σ_{xz}, and σ_{yz}. Thus, we use the semi-inverse method, where we assume that

$$\sigma_{zz} = S(z), \quad \sigma_{xx} = \sigma_{yy} = \sigma_{xy} = \sigma_{xz} = \sigma_{yz} = 0. \tag{7.3.17}$$

The boundary conditions require that $S(0) = 0$. The first two equations of equilibrium are satisfied trivially and the third equation reduces to

$$\frac{dS}{dz} = \rho g \quad \rightarrow \quad S(z) = \gamma\,z + c \quad (\gamma = \rho g). \tag{1}$$

The constant of integration, c, is zero in order to satisfy the boundary condition $S(0) = 0$. Thus, the stress field is

$$\sigma_{zz} = \gamma\,z, \quad \sigma_{xx} = \sigma_{yy} = \sigma_{xy} = \sigma_{xz} = \sigma_{yz} = 0. \tag{2}$$

The stress compatibility conditions in Eq. (7.2.26) are trivially satisfied.

The strains are given by

$$\varepsilon_{xx} = \varepsilon_{yy} = -\frac{\nu}{E}\,\gamma\,z, \quad \varepsilon_{zz} = \frac{1}{E}\gamma\,z, \quad \varepsilon_{xy} = \varepsilon_{xz} = \varepsilon_{yz} = 0. \tag{3}$$

The corresponding displacement field is determined from the strain-displacement boundary conditions:

$$\begin{aligned} \varepsilon_{zz} &= \frac{1}{E}\gamma\,z, \quad \rightarrow \quad u_z = \frac{1}{2E}\,\gamma\,z^2 + h(x,y), \\ 2\varepsilon_{xz} &= \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} = 0, \quad \rightarrow \quad \frac{\partial u_x}{\partial z} = -\frac{\partial h}{\partial x}, \end{aligned} \tag{4}$$

where h is a function to be determined. Integrating Eq. (4), we obtain

$$u_x = -\frac{\partial h}{\partial x} z + g(x,y), \tag{5}$$

where g is a function to be determined. Similarly,

$$2\varepsilon_{yz} = \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} = 0, \quad \frac{\partial u_y}{\partial z} = -\frac{\partial h}{\partial y}, \quad u_y = -\frac{\partial h}{\partial y} z + f(x,y), \tag{6}$$

where f is a function to be determined. Now comparing ε_{xx} from Eq. (3) with that computed from Eq. (5), we obtain

$$-\frac{\nu}{E}\,\gamma\, z = -\frac{\partial^2 h}{\partial x^2} z + \frac{\partial g}{\partial x}\,.$$

We see that, because it must hold for any z,

$$\frac{\partial^2 h}{\partial x^2} = \frac{\nu}{E}\gamma, \quad \frac{\partial g}{\partial x} = 0 \quad \to \quad g = G(y). \tag{7}$$

Similarly, comparing ε_{yy} from Eq. (3) with that computed from Eq. (6), we obtain

$$-\frac{\nu}{E}\,\gamma\, z = -\frac{\partial^2 h}{\partial y^2} z + \frac{\partial f}{\partial y}\,,$$

we see that, since it must hold for any z,

$$\frac{\partial^2 h}{\partial y^2} = \frac{\nu}{E}\gamma, \quad \frac{\partial f}{\partial y} = 0 \quad \to \quad f = F(x). \tag{8}$$

From $\varepsilon_{xy} = 0$, we see that

$$-2\frac{\partial^2 h}{\partial x \partial y} z + \frac{\partial g}{\partial y} + \frac{\partial f}{\partial x} = 0. \tag{9}$$

This gives the result

$$\frac{\partial^2 h}{\partial x \partial y} = 0, \quad \frac{dG}{dy} + \frac{dF}{dx} = 0 \quad \to G(y) = c_1 y + c_2, \quad F(x) = -c_1 x + c_3. \tag{10}$$

Conditions in Eqs. (6)–(8) imply that h is of the form

$$h(x,y) = \frac{\nu}{2E}\,\gamma(x^2 + y^2) + c_4 x + c_5 y + c_6, \tag{7.3.18}$$

where c_i are constants. In summary, we have

$$\begin{aligned} u_x &= -\frac{\partial h}{\partial x} z + g(x,y) = -\frac{\nu}{E}\,\gamma x z - c_4 z + c_1 y + c_2, \\ u_y &= -\frac{\partial h}{\partial y} z + f(x,y) = -\frac{\nu}{E}\,\gamma y z - c_5 z - c_1 x + c_3, \\ u_z &= \frac{\gamma}{2E}\left[z^2 + \nu(x^2 + y^2)\right] + c_4 x + c_5 y + c_6. \end{aligned} \tag{7.3.19}$$

The displacement boundary conditions in Eq. (7.3.16) give $c_2 = c_3 = 0$, and $c_6 = -\gamma L^2/2E$, which correspond to the translational rigid-body motions. To remove the six rigid-body rotations, we may require

$$\frac{\partial u_x}{\partial y} = \frac{\partial u_y}{\partial x} = \frac{\partial u_y}{\partial z} = \frac{\partial u_z}{\partial y} = \frac{\partial u_y}{\partial z} = \frac{\partial u_x}{\partial z} = \frac{\partial u_z}{\partial x} = 0 \quad \text{at } x = y = 0, \text{ and } z = L,$$

which yield all other constants to be zero, giving the final displacement field

$$\begin{aligned} u_x &= -\frac{\partial h}{\partial x} z + g(x,y) = -\frac{\nu}{E}\,\gamma x z, \\ u_y &= -\frac{\partial h}{\partial y} z + f(x,y) = -\frac{\nu}{E}\,\gamma y z, \\ u_z &= \frac{\gamma}{2E}\left[z^2 + \nu(x^2 + y^2)\right]. \end{aligned} \tag{7.3.20}$$

7.3.4 Stretching and Bending of Beams

In this section, we use the semi-inverse method to formulate equations governing stretching and bending of prismatic members. Using a set of assumptions concerning the kinematics of deformation of the members, the form of the displacement field is identified. We consider the prismatic bar shown in Fig. 7.3.3. The bar has a length L and has rectangular cross section of dimensions $b \times h$, b being the width and h being the height, such that $b < h << L$. We set up a coordinate system such that the x-axis is along the length of the beam through its geometric centroid, y-axis is transverse to the length of the beam, and the z-axis is out of the plane of the page, as shown in Fig. 7.3.3. A distributed load $q(x)$ (measured per unit length) acts along the length of the beam in the xy-plane in the positive y-direction, a distributed load $f(x)$ (measured per unit length) acts along the center line of the beam in the x-direction, and a point load F_0 acts at a distance $x = a$ from the left end. The bar is geometrically constrained at the right end in such a way that all three displacements are zero there. Thus, the boundary conditions are

$$\begin{aligned}
&\mathbf{u}(L,y,z) = \mathbf{0}, \quad \sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0 \text{ on faces } z = \pm b/2 \text{ for all } x, y,\\
&\sigma_{yy}(x,h/2,0) = q(x), \quad \sigma_{yy}(x,-h/2,0) = 0, \quad \sigma_{xy}(x,\pm h/2,0) = 0, \quad (7.3.21)\\
&\sigma_{yz}(x,\pm h/2,z) = 0, \quad \sigma_{xx}(0,y,z) = 0, \quad \sigma_{xy}(0,y,z) = 0, \quad \sigma_{xz}(0,y,z) = 0.
\end{aligned}$$

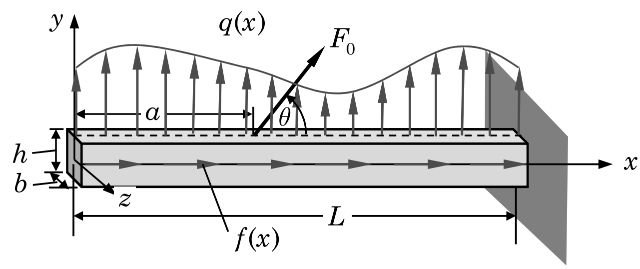

Fig. 7.3.3: A prismatic bar under various loads.

Solving the problem for exact displacements, strains, and stresses that satisfy the boundary conditions in Eq. (7.3.21) and the equilibrium equations of elasticity is an impossible task. We can formulate it as an equivalent problem of finding the solution that satisfies statically equivalent[1] stress boundary conditions and through-thickness-integrated equations of elasticity. Such formulation reduces the three-dimensional elasticity problem to a one-dimensional elasticity problem, known as the *beam bending* problem. Once again, we use the semi-inverse method, and assume a form the displacement field.

We seek a solution $(u_x, u_y, 0)$ based on the following assumptions: the transverse normal lines, such as AB shown in Fig. 7.3.4(a), (1) remain straight, (2)

[1]The phrase "statically equivalent" means that the two distributions of forces have the same resultant force and resultant moment.

are inextensible, and (3) rotate such that they remain normal to the middle surface after deformation. These assumptions are known as the *Euler–Bernoulli hypothesis* of beam bending. The first two assumptions together amount to neglecting Poisson's effect and the transverse normal strain (i.e., $\varepsilon_{yy} = 0$). The third assumption is to neglect the transverse shear strain $\varepsilon_{xy} = 0$. We assume that the deformation is only two-dimensional (in the plane of the page). This requires that the applied loads be in the xy plane so that stretching and bending are in the xy plane, and there is no rotation about the x axis.

The Euler–Bernoulli hypothesis is satisfied by the following form of the displacement field:

$$\begin{aligned} \mathbf{u} &= \left[u(x) - y\frac{\partial u_y}{\partial x}\right]\hat{\mathbf{e}}_x + u_y\,\hat{\mathbf{e}}_y, \\ u_x &= u(x) - y\frac{dv}{dx}, \quad u_y = v(x), \quad u_z = 0, \end{aligned} \tag{7.3.22}$$

where $u(x)$ and $v(x)$ are functions to be determined by requiring that the equilibrium equations of elasticity are satisfied in an integral sense, as explained shortly. From the assumed form of the displacement field, we see that the displacement component u_x consists of two parts: stretching displacement $u(x)$ of all lines parallel to the x-axis and the displacement $-y(dv/dx)$ due to bending action, which is proportional to the distance y measured from the middle plane. The transverse displacement $u_y = v(x)$ is independent of the y-coordinate, a consequence of the inextensibility assumption.

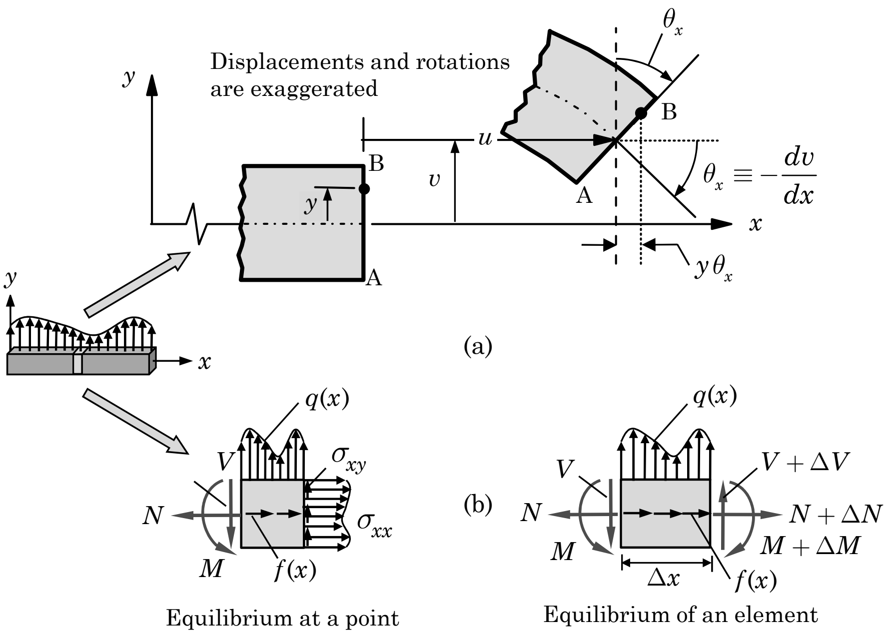

Fig. 7.3.4: Bending of a beam. (a) Kinematics of deformation. (b) Equilibrium of an element of the beam.

The only nonzero strain and corresponding stress components corresponding to the assumed displacement field are [$\nu = 0$ in writing the stress–strain relation $\sigma_{xx} = (2\mu + \lambda)\varepsilon_{xx} = E\varepsilon_{xx}$ but not in the relation $2\mu = 2G = E/(1+\nu)$]

$$\varepsilon_{xx} = \frac{du}{dx} - y\frac{d^2v}{dx^2}, \tag{7.3.23}$$

$$\sigma_{xx} = E\left(\frac{du}{dx} - y\frac{d^2v}{dx^2}\right), \tag{7.3.24}$$

where E is Young's modulus of the material.

Since we cannot satisfy the equations of equilibrium, Eq. (7.2.6), without the inertia terms, at every point of the beam, we derive equations of equilibrium by considering a typical element of the beam, as shown in Fig. 7.3.4(b). Summing the forces and moments on the element, we obtain

$$\text{sum of the forces in the } x\text{-direction:} \quad \frac{dN}{dx} + f(x) = 0, \tag{7.3.25}$$

$$\text{sum of the forces in the } z\text{-direction:} \quad \frac{dV}{dx} + q(x) = 0, \tag{7.3.26}$$

$$\text{sum of the moments about the } y\text{-axis:} \quad V - \frac{dM}{dx} = 0, \tag{7.3.27}$$

where $N(x)$ is the axial force, $M(x)$ is the bending moment, and $V(x)$ is the shear force. These quantities are known as the *stress resultants*, and they can be defined in terms of the stresses σ_{xx} and σ_{xy} as

$$N(x) = \int_A \sigma_{xx}\, dA, \quad M(x) = \int_A y\,\sigma_{xx}\, dA, \quad V(x) = \int_A \sigma_{xy}\, dA, \tag{7.3.28}$$

where $A = bh$ is the cross-sectional area. One can show that the equilibrium equations (7.3.25)–(7.3.27) are equivalent to the following two stress equilibrium equations ($\sigma_{xz} = \sigma_{zz} = \sigma_{yz} = 0$; hence, the third equation of equilibrium is trivially satisfied):

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0, \qquad \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0.$$

This is left as an exercise for the reader (see Problem **7.13**).

From the constitutive relation $\sigma_{xy} = 2G\varepsilon_{xy}$, we have $\sigma_{xy} = 0$ and, therefore, $V = 0$ from Eq. (7.3.28). Although the transverse shear force V is zero from the kinematic assumptions made here, in reality it cannot be zero as it is responsible for supporting the applied vertical loads on the beam, as can be seen from Eq. (7.3.27). This is the flaw in the Euler–Bernoulli beam theory, which can be overcome by ignoring the definition of V in Eq. (7.3.28) and calculating it using Eq. (7.3.27). That is, substitute for V from Eq. (7.3.27) into Eq. (7.3.26) and obtain only two equations of equilibrium:

$$-\frac{dN}{dx} = f(x), \qquad -\frac{d^2M}{dx^2} = q(x). \tag{7.3.29}$$

The stress resultants (N, M) can be related back to the unknown functions (u, v) as [because the x-axis is taken through the geometric centroid of the cross section, we have $\int_A y dA = 0$]:

$$N(x) = \int_A \sigma_{xx}\, dA = EA\frac{du}{dx}, \quad M(x) = \int_A y\, \sigma_{xx}\, dA = -EI\frac{d^2v}{dx^2}, \tag{7.3.30}$$

where I is the moment of inertia about the axis of bending (i.e., z-axis). From Eqs. (7.3.24) and (7.3.30), we can express σ_{xx} in terms of the stress resultants N and M as

$$\sigma_{xx} = E\varepsilon_{xx} = \frac{N(x)}{A} + \frac{M(x)y}{I}. \tag{7.3.31}$$

Finally, we have two equations of equilibrium governing u and v

$$-\frac{d}{dx}\left(EA\frac{du}{dx}\right) = f(x), \quad \frac{d^2}{dx^2}\left(EI\frac{d^2v}{dx^2}\right) = q(x). \tag{7.3.32}$$

Note that the two equations are not coupled, that is, each equation can be solved independent of the other. Indeed, when no axial loads are applied on the beam, we have $u = 0$ everywhere in the beam. Conversely, when no bending loads are applied on the beam, we have $v = 0$ everywhere. The former case is known as the beam bending problem and the latter as the bar problem. The two equations in (7.3.32) are subjected to boundary conditions of the type

$$u = \hat{u}, \quad N = \hat{N}; \quad v = \hat{v}, \quad -\frac{dv}{dx} = \hat{\theta}, \quad M = \hat{M}, \quad V = \hat{V}, \tag{7.3.33}$$

Only one element of each of the following three pairs should be specified at a boundary point:

$$(u, N), \quad (v, V), \quad (\theta, M). \tag{7.3.34}$$

This completes the formulation of the Euler–Bernoulli beam theory. Next, we consider an example.

Example 7.3.3

Consider the cable-supported beam shown in Fig. 7.3.5(a). The beam as well as the cable are made of homogeneous, linear elastic, isotropic materials, with constant geometric properties. Determine the displacements (u, v) and the force F_c in the cable.

Solution: Figure 7.3.5(b) contains the effect of the cable force on the beam. We begin with the first equation in (7.3.32) and integrate it twice with respect to x and obtain

$$E_bA_b\frac{du}{dx} = c_1, \quad E_bA_bu(x) = c_1x + c_2, \tag{1}$$

where the constants of integration, c_1 and c_2, are determined using the boundary conditions

$$u(L_b) = 0, \quad \left[E_bA_b\frac{du}{dx}\right]_{x=0} = -F_c\cos\alpha. \tag{2}$$

We obtain $c_1 = -F_c\cos\alpha$ and $c_2 = F_cL_b\cos\alpha$, and the solution becomes

$$u(x) = \frac{F_cL_b}{E_bA_b}\left(1 - \frac{x}{L_b}\right)\cos\alpha. \tag{3}$$

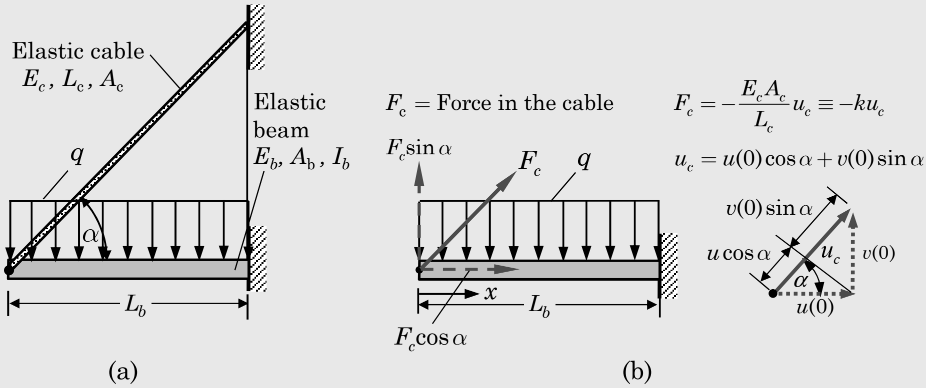

Fig. 7.3.5: A cable-supported beam.

Next, we consider the second equation in (7.3.32) and integrate it four times with respect to x and obtain

$$
\begin{aligned}
\frac{d}{dx}\left(E_b I_b \frac{d^2 v}{dx^2}\right) &= -qx + c_3, \\
E_b I_b \frac{d^2 v}{dx^2} &= -q\frac{x^2}{2} + c_3 x + c_4, \\
E_b I_b \frac{dv}{dx} &= -q\frac{x^3}{6} + c_3\frac{x^2}{2} + c_4 x + c_5, \\
E_b I_b\, v(x) &= -q\frac{x^4}{24} + c_3\frac{x^3}{6} + c_4\frac{x^2}{2} + c_5 x + c_6,
\end{aligned}
\tag{4}
$$

where the constants of integration, c_3, c_4, c_5, and c_6 are obtained with the help of the boundary conditions

$$
V(0) = -F_c \sin\alpha, \quad M(0) = 0, \quad \left[\frac{dv}{dx}\right]_{x=L_b} = 0, \quad v(L_b) = 0. \tag{5}
$$

We obtain

$$
c_3 = F_c \sin\alpha, \quad c_4 = 0, \quad c_5 = \frac{qL_b^3}{6} - \frac{F_c L_b^2}{2}\sin\alpha\,, \quad c_6 = -\frac{qL_b^4}{8} + \frac{F_c L_b^3}{3}\sin\alpha.
$$

The solution is given by

$$
v(x) = -\frac{qL_b^4}{24E_b I_b}\left[3 - 4\frac{x}{L_b} + \left(\frac{x}{L_b}\right)^4\right] + \frac{F_c L_b^3}{6E_b I_b}\left[2 - 3\frac{x}{L_b} + \left(\frac{x}{L_b}\right)^3\right]\sin\alpha. \tag{6}
$$

The displacements at $x = 0$ are

$$
u(0) = \frac{F_c L_b}{E_b A_b}\cos\alpha, \quad v(0) = -\frac{qL_b^4}{8E_b I_b} + \frac{F_c L_b^3}{3E_b I_b}\sin\alpha. \tag{7}
$$

To determine the cable force, F_c, first we note that

$$
u_c = u(0)\,\cos\alpha + v(0)\,\sin\alpha, \tag{8}
$$

and calculate F_c from (u_c is in the opposite direction to F_c)

$$
F_c = -\frac{E_c A_c}{L_c}u_c = -\frac{E_c A_c}{L_c}\left(\frac{F_c L_b}{E_b A_b}\cos^2\alpha + \frac{F_c L_b^3}{3E_b I_b}\sin^2\alpha - \frac{qL_b^4}{8E_b I_b}\sin\alpha\right), \tag{9}
$$

or

$$
F_c = \frac{qL_b^4}{8E_b I_b}\sin\alpha\left[\frac{L_c}{E_c A_c} + \frac{L_b}{E_b A_b}\cos^2\alpha + \frac{L_b^3}{3E_b I_b}\sin^2\alpha\right]^{-1}. \tag{10}
$$

7.3.5 Superposition Principle

An advantage of linear boundary value problems is that the principle of superposition holds. The principle of superposition is said to hold for a solid body if the displacements obtained under two sets of boundary conditions and forces are equal to the sum of the displacements that would be obtained by applying each set of boundary conditions and forces separately.

To be more specific, consider the following two sets of boundary conditions and forces

$$\textbf{Set 1:}\quad \mathbf{u} = \mathbf{u}^{(1)} \text{ on } \Gamma_u; \quad \mathbf{t} = \mathbf{t}^{(1)} \text{ on } \Gamma_\sigma; \quad \mathbf{f} = \mathbf{f}^{(1)} \text{ in } \Omega, \tag{7.3.35}$$

$$\textbf{Set 2:}\quad \mathbf{u} = \mathbf{u}^{(2)} \text{ on } \Gamma_u; \quad \mathbf{t} = \mathbf{t}^{(2)} \text{ on } \Gamma_\sigma; \quad \mathbf{f} = \mathbf{f}^{(2)} \text{ in } \Omega, \tag{7.3.36}$$

where the specified data ($\mathbf{u}^{(1)}$, $\mathbf{t}^{(1)}$, $\mathbf{f}^{(1)}$) and ($\mathbf{u}^{(2)}$, $\mathbf{t}^{(2)}$, $\mathbf{f}^{(2)}$) are independent of the deformation. Suppose that the solution to the two problems be $\mathbf{u}^{(1)}(\mathbf{x})$ and $\mathbf{u}^{(2)}(\mathbf{x})$, respectively. The superposition of the two sets of boundary conditions is

$$\mathbf{u} = \mathbf{u}^{(1)} + \mathbf{u}^{(2)} \text{ on } \Gamma_u; \quad \mathbf{t} = \mathbf{t}^{(1)} + \mathbf{t}^{(2)} \text{ on } \Gamma_\sigma; \quad \mathbf{f} = \mathbf{f}^{(1)} + \mathbf{f}^{(2)} \text{ in } \Omega. \tag{7.3.37}$$

Because of the linearity of the elasticity equations, the solution of the boundary value problem with the superposed data is $\mathbf{u}(\mathbf{x}) = \mathbf{u}^{(1)}(\mathbf{x}) + \mathbf{u}^{(2)}(\mathbf{x})$ in Ω. This is known as the *superposition principle.*

The principle of superposition can be used to represent a linear problem with complicated boundary conditions and/or loads as a combination of linear problems that are equivalent to the original problem. Example 7.3.4 illustrates this point.

Example 7.3.4

Consider the indeterminate beam shown in Fig. 7.3.6(a). Determine the deflection of point A using the principle of superposition.

Solution: The problem can be viewed as one equivalent to the two beam problems shown in Fig. 7.3.6(b). The sum of the deflections from each problem is the solution of the original problem. Within the restrictions of the linear Euler–Bernoulli beam theory, the deflections are linear functions of the loads. Therefore, the principle of superposition is valid. Thus, the transverse displacement of the original beam can be determined as the sum of the displacements of the individual beams shown in Fig. 7.3.6(b):

$$v(x) = \frac{q_0 L^4}{24EI}\left[3 - 4\,\frac{x}{L} + \left(\frac{x}{L}\right)^4\right] - \frac{F_s L^3}{6EI}\left[2 - 3\,\frac{x}{L} + \left(\frac{x}{L}\right)^3\right]. \tag{1}$$

In particular, the deflection v_{A} at point A is equal to the sum of v_{A}^q and v_{A}^s due to the distributed load q_0 and spring force F_s, respectively, at point A:

$$v_{\mathrm{A}} = v_{\mathrm{A}}^q + v_{\mathrm{A}}^s = \frac{q_0 L^4}{8EI} - \frac{F_s L^3}{3EI}. \tag{2}$$

Because the spring force F_s is equal to kv_{A}, we can calculate v_{A} from

$$v_{\mathrm{A}} = \frac{q_0 L^4}{8EI\left(1 + \frac{kL^3}{3EI}\right)}. \tag{3}$$

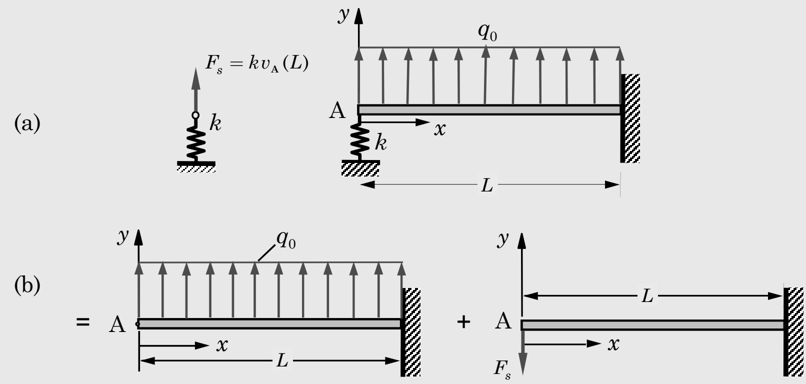

Fig. 7.3.6: Representation of an indeterminate beam as a superposition of two determinate beams.

7.3.6 Uniqueness of Solutions

Although the existence of solutions is a difficult question to answer, the uniqueness of solutions is rather easy to prove for linear boundary value problems of elasticity. Consider the problem of finding the solution to the Navier equations (7.2.17) of linearized elasticity, for a given body force $\mathbf{f}$ and boundary conditions

$$\mathbf{u} = \hat{\mathbf{u}} \quad \text{on} \quad \Gamma_u, \tag{7.3.38}$$

$$\mathbf{t} = \hat{\mathbf{t}} \quad \text{on} \quad \Gamma_\sigma. \tag{7.3.39}$$

Now suppose that for this set of loads and boundary conditions, there exist two distinct solutions, $\mathbf{u}^{(1)}(\mathbf{x}, t)$ and $\mathbf{u}^{(2)}(\mathbf{x}, t)$. Associated with the two displacement fields, we can compute the strains and stress fields ($\boldsymbol{\varepsilon}^{(1)}$, $\boldsymbol{\sigma}^{(1)}$) and ($\boldsymbol{\varepsilon}^{(2)}$, $\boldsymbol{\sigma}^{(2)}$). Then the difference $\mathbf{u}^d(\mathbf{x}, t) \equiv \mathbf{u}^{(1)}(\mathbf{x}, t) - \mathbf{u}^{(2)}(\mathbf{x}, t)$ satisfies the homogeneous form of the Navier equation (with $\mathbf{f}^d = \mathbf{f}^{(1)} - \mathbf{f}^{(2)} = \mathbf{0}$, because the applied forces and boundary values are the same for both solutions)

$$\mu \boldsymbol{\nabla}^2 \mathbf{u}^d + (\mu + \lambda) \boldsymbol{\nabla} \left(\boldsymbol{\nabla} \cdot \mathbf{u}^d \right) = \mathbf{0} \quad \text{in} \quad \Omega, \tag{7.3.40}$$

as well as the homogeneous forms of the boundary conditions

$$\mathbf{u}^d = \mathbf{0} \quad \text{on} \quad \Gamma_u, \tag{7.3.41}$$

$$\mathbf{t}^d = \mathbf{0} \quad \text{on} \quad \Gamma_\sigma. \tag{7.3.42}$$

Because no work is done on the body by external forces (because $\mathbf{f}^d$ and $\mathbf{t}^d$ are zero), the strain energy density U_0 stored in the body is zero. Noting that the strain energy density U_0 (measured per unit volume)

$$U_0(\boldsymbol{\varepsilon}) = \frac{\lambda}{2} (\operatorname{tr} \boldsymbol{\varepsilon})^2 + \mu \operatorname{tr}(\boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon}), \quad U_0(\varepsilon_{ij}) = \mu \varepsilon_{ij} \varepsilon_{ij} + \tfrac{1}{2} \lambda (\varepsilon_{kk})^2, \tag{7.3.43}$$

is a positive-definite function of the strains [see Eqs. (6.3.36) and (6.3.37)],

$$U_0(\boldsymbol{\varepsilon}) > 0 \text{ whenever } \boldsymbol{\varepsilon} \neq \mathbf{0}, \text{ and } U_0(\boldsymbol{\varepsilon}) = 0 \text{ only when } \boldsymbol{\varepsilon} = \mathbf{0}, \tag{7.3.44}$$

we conclude that the strain field $\boldsymbol{\varepsilon}^d$ is zero and hence the stress field $\boldsymbol{\sigma}^d$ is also zero:

$$\boldsymbol{\varepsilon}^d = \boldsymbol{\varepsilon}^{(1)} - \boldsymbol{\varepsilon}^{(2)} = \mathbf{0}, \quad \boldsymbol{\sigma}^d = \boldsymbol{\sigma}^{(1)} - \boldsymbol{\sigma}^{(2)} = \mathbf{0}, \tag{7.3.45}$$

implying that the strain and stress fields associated with the two distinct displacements $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$ are the same, that is, they are unique. Also, $\boldsymbol{\varepsilon}^d = \mathbf{0}$ implies that $\boldsymbol{\nabla}\mathbf{u}^d = \mathbf{0}$, which corresponds to a rigid-body motion. For type I and type III problems, the displacement boundary conditions eliminate the rigid-body motion and, therefore, the displacements are unique for type I and type III problems. For boundary value problems of type II, the displacements are determined within the quantities representing rigid-body motions.

7.4 Clapeyron's, Betti's, and Maxwell's Theorems

7.4.1 Clapeyron's Theorem

The principle of superposition is *not* valid for energies because they are quadratic functions of displacements or forces. In other words, when a linear elastic body $\mathcal{B}$ is subjected to more than one external force, the total work done due to external forces is *not* equal to the sum of the works that are obtained by applying the single forces separately. However, there exist theorems that relate the work done in linear elastic solids by two different forces applied in different orders. We will consider them in this section.

Recall from Chapter 6 that the strain energy density due to linear elastic deformation is given by[2]

$$\begin{aligned} U_0 &= \tfrac{1}{2}\,\boldsymbol{\varepsilon} : \mathbf{C} : \boldsymbol{\varepsilon} = \tfrac{1}{2}\,\boldsymbol{\varepsilon} : \boldsymbol{\sigma} \\ &= \tfrac{1}{2}\,C_{ijkl}\,\varepsilon_{kl}\,\varepsilon_{ij} = \tfrac{1}{2}\,\sigma_{ij}\,\varepsilon_{ij}. \end{aligned} \tag{7.4.1}$$

The total strain energy stored in the body $\mathcal{B}$ occupying the region Ω with surface Γ is equal to

$$U = \int_\Omega U_0 \; d\mathbf{x} = \tfrac{1}{2}\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\varepsilon} \; d\mathbf{x} = \tfrac{1}{2}\int_\Omega \sigma_{ij}\,\varepsilon_{ij} \; d\mathbf{x}. \tag{7.4.2}$$

The total work done by the body force $\mathbf{f}$ (measured per unit volume) and surface traction $\mathbf{t}$ (measured per unit area) in moving through their respective displacements $\mathbf{u}$ is given by

$$W_E = \int_\Omega \mathbf{f}\cdot\mathbf{u}\,d\mathbf{x} + \oint_\Gamma \mathbf{t}\cdot\mathbf{u}\,ds. \tag{7.4.3}$$

When $\mathbf{u} = \mathbf{0}$ on a portion Γ_u of the boundary Γ, the surface integral in Eq. (7.4.3) becomes

$$\int_{\Gamma_\sigma} \mathbf{t}\cdot\mathbf{u}\,ds, \quad \text{where} \quad \Gamma_\sigma = \Gamma - \Gamma_u.$$

[2]In this chapter U_0 is measured per unit volume as opposed to per unit mass.

Owing to the symmetry of the stress tensor, $\sigma_{ij} = \sigma_{ji}$, we can write $\sigma_{ij}\varepsilon_{ij} = \sigma_{ij}\, u_{i,j}$. Consequently, the strain energy U can be expressed as

$$\begin{aligned} U &= \tfrac{1}{2}\int_\Omega \sigma_{ij}\,\varepsilon_{ij}\, d\mathbf{x} = \tfrac{1}{4}\int_\Omega \sigma_{ij}\,(u_{i,j}+u_{j,i})\; d\mathbf{x} = \tfrac{1}{2}\int_\Omega \sigma_{ij}\, u_{i,j}\, d\mathbf{x} \\ &= -\tfrac{1}{2}\int_\Omega \sigma_{ij,j}\, u_i\, d\mathbf{x} + \tfrac{1}{2}\oint_\Gamma n_j\sigma_{ij}u_i\, ds \\ &= \tfrac{1}{2}\int_\Omega f_i\, u_i\, d\mathbf{x} + \tfrac{1}{2}\oint_\Gamma t_i u_i\, ds = \tfrac{1}{2}\int_\Omega \mathbf{f}\cdot\mathbf{u}\, d\mathbf{x} + \tfrac{1}{2}\oint_\Gamma \mathbf{t}\cdot\mathbf{u}\, ds, \end{aligned}$$

where, in arriving at the last line, we have used the stress equilibrium equation $\sigma_{ij,j} + f_i = 0$, Cauchy's formula $t_i = \sigma_{ij}n_j$, and the divergence theorem. Thus, the total strain energy stored in a body undergoing linear elastic deformation is also equal to the one-half of the work done by applied forces

$$\tfrac{1}{2}\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\varepsilon}\; d\mathbf{x} = \tfrac{1}{2}\int_\Omega \mathbf{f}\cdot\mathbf{u}\, d\mathbf{x} + \tfrac{1}{2}\oint_\Gamma \mathbf{t}\cdot\mathbf{u}\, ds. \tag{7.4.4}$$

The first term on the right-hand side represents the work done by body force $\mathbf{f}$ in moving through the displacement $\mathbf{u}$ while the second term represents the work done by surface force $\mathbf{t}$ in moving through the displacements $\mathbf{u}$ during the deformation. Equation (7.4.4) is known as *Clapeyron's theorem*. The next three examples illustrate the usefulness of the theorem.

Example 7.4.1

Consider a linear elastic spring with spring constant k. Let F be the external force applied on the spring to elongate it and u be the resulting elongation of the spring (see Fig. 7.4.1). Verify Clapeyron's theorem.

Solution: The internal force developed in the spring is $F_s = ku$. The work done by F_s in moving through an increment of displacement du is $F_s\, du$. The total strain energy stored in the spring is

$$U = \int_0^u F_s\, du = \int_0^u ku\, du = \frac{1}{2}ku^2. \tag{7.4.5}$$

The work done by external force F is equal to $F\,u$. But by equilibrium, $F = F_s = ku$. Hence,

$$U = \frac{1}{2}ku^2 = \frac{1}{2}F\,u,$$

which proves Clapeyron's theorem.

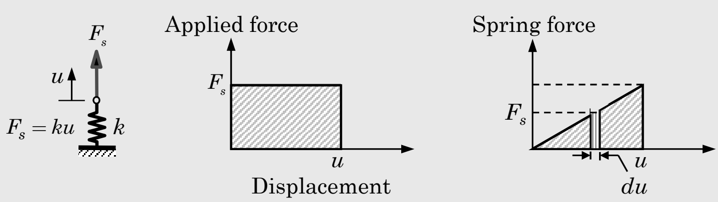

Fig. 7.4.1: Strain energy stored in a linear elastic spring.

Example 7.4.2

Consider a uniform elastic bar of length L, cross-sectional area A, and modulus of elasticity E. The bar is fixed at $x = 0$ and subjected to a tensile force of P at $x = L$, as shown in Fig. 7.4.2. Determine the axial displacement $u(L)$ using Clapeyron's theorem.

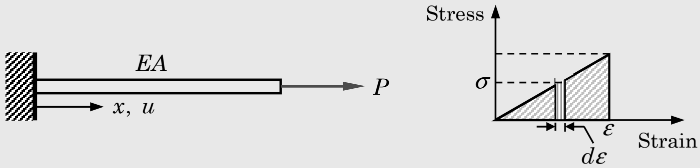

Fig. 7.4.2: A bar subjected to an end load.

Solution: If the axial displacement in the bar is equal to $u(x)$, then the work done by external point force P is equal to $W = Pu(L)$. The strain energy in the bar is given by

$$U = \frac{1}{2}\int_A\int_0^L \sigma_{xx}\varepsilon_{xx}\, dx\, dA = \frac{EA}{2}\int_0^L \varepsilon_{xx}^2\, dx = \frac{EA}{2}\int_0^L \left(\frac{du}{dx}\right)^2 dx = \frac{1}{2}\int_0^L \frac{N^2}{EA}\, dx. \quad (7.4.6)$$

Hence, by Clapeyran's theorem we have

$$\frac{Pu(L)}{2} = \frac{EA}{2}\int_0^L \left(\frac{du}{dx}\right)^2 dx.$$

To make use of the above equation to determine $u(x)$, let us assume that $u(x) = u(L)x/L$, which certainly satisfies the geometric boundary condition, $u(0) = 0$. Then we have

$$u(L) = \frac{EA}{P}\int_0^L \left(\frac{du}{dx}\right)^2 dx = \frac{EA}{PL}[u(L)]^2,$$

or $u(L) = PL/AE$ and the solution is $u(x) = Px/AE$, which happens to coincide with the exact solution to the problem.

Example 7.4.3

Consider a cantilever beam of length L and flexural rigidity EI and bent by a point load F at the free end (see Fig. 7.4.3). Determine $v(0)$ using Clapeyron's theorem.

Solution: By Clapeyron's theorem we have

$$\frac{1}{2}Fv(0) = \frac{1}{2}\int_A\int_0^L \sigma_{xx}\varepsilon_{xx}\, dx\, dA.$$

But according to the Euler–Bernoulli beam theory, the strain and stress in the beam are given by

$$\varepsilon_{xx} = -y\frac{d^2v}{dx^2}, \quad \sigma_{xx} = E\varepsilon_{xx} = -Ey\frac{d^2v}{dx^2}, \quad (7.4.7)$$

where v is the transverse deflection. Then we have

$$\begin{aligned}\frac{1}{2}Fv(0) &= \frac{1}{2}\int_A\int_0^L E\varepsilon_{xx}^2\, dx\, dA = \frac{1}{2}\int_A\int_0^L Ey^2\left(\frac{d^2v}{dx^2}\right)^2 dA\, dx\\ &= \frac{1}{2}\int_0^L EI\left(\frac{d^2v}{dx^2}\right)^2 dx = \frac{1}{2}\int_0^L \frac{M^2}{EI}\, dx, \quad (7.4.8)\end{aligned}$$

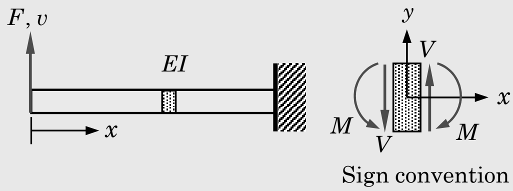

Fig. 7.4.3: A beam subjected to an end load.

where $M(x)$ is the bending moment at x

$$M(x) = \int_A y\sigma_{xx}\, dA = -E \int_A y^2 \frac{d^2v}{dx^2}\, dA = -EI\frac{d^2v}{dx^2}. \tag{7.4.9}$$

Equation (7.4.8) can be used to determine the deflection $v(0)$. The bending moment at any point x is $M(x) = -Fx$. Hence, we have

$$Fv(0) = \frac{1}{EI}\int_0^L F^2x^2\, dx = \frac{F^2L^3}{3EI} \quad \text{or} \quad v(0) = \frac{FL^3}{3EI}. \tag{7.4.10}$$

7.4.2 Betti's Reciprocity Theorem

Consider the equilibrium state of a linear elastic solid under the action of two different external forces, $\mathbf{F}_1$ and $\mathbf{F}_2$, as shown in Fig. 7.4.4. Since the order of application of the forces is arbitrary for linearized elasticity, we suppose that force $\mathbf{F}_1$ is applied first. Let W_1 be the work done by $\mathbf{F}_1$. Then, we apply force $\mathbf{F}_2$ at some other point of the body, which does work W_2. This work is the same as that produced by force $\mathbf{F}_2$, if it alone were acting on the body. However, when force $\mathbf{F}_2$ is applied, force $\mathbf{F}_1$ (which is already acting on the body) does additional work because its point of application is displaced due to the deformation caused by force $\mathbf{F}_2$. Let us denote this work by W_{12}, which is the work done by force F_1 due to the application of force F_2. Thus the total work done by the application of forces $\mathbf{F}_1$ and $\mathbf{F}_2, \mathbf{F}_1$ first and $\mathbf{F}_2$ next, is

$$W = W_1 + W_2 + W_{12}. \tag{7.4.11}$$

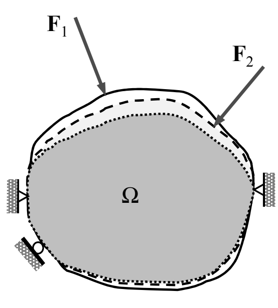

Fig. 7.4.4: Configurations of an elastic body due to the application of loads $\mathbf{F}_1$ and $\mathbf{F}_2$. —- Undeformed configuration. - - - - Deformed configuration after the application of $\mathbf{F}_1$. ...... Deformed configuration after the application of $\mathbf{F}_2$.

Work W_{12}, which can be positive or negative, is zero if and only if the displacement of the point of application of force $\mathbf{F}_1$ produced by force $\mathbf{F}_2$ is zero or perpendicular to the direction of $\mathbf{F}_1$. Now suppose that we change the order of application of the forces, that is, force $\mathbf{F}_2$ is applied first and force $\mathbf{F}_1$ is applied next. Then the total work done is equal to

$$\bar{W} = W_1 + W_2 + W_{21}, \tag{7.4.12}$$

where W_{21} (note the order of the subscripts) is the work done by force F_2 due to the application of force F_1. The work done in both cases should be the same because, at the end, the body is loaded by the same pair of external forces. Thus, we have $W = \bar{W}$, or

$$W_{12} = W_{21}. \tag{7.4.13}$$

Equation (7.4.13) is a mathematical statement of Betti's (1823–1892) reciprocity theorem: *If a linear elastic body is subjected to two different sets of forces, the work done by the first system of forces in moving through the displacements produced by the second system of forces is equal to the work done by the second system of forces in moving through the displacements produced by the first system of forces.* Applied to a three-dimensional elastic body Ω with closed surface s, Eq. (7.4.13) takes the form

$$\int_\Omega \mathbf{f}^{(1)} \cdot \mathbf{u}^{(2)}\, d\mathbf{x} + \oint_s \mathbf{t}^{(1)} \cdot \mathbf{u}^{(2)}\, ds = \int_\Omega \mathbf{f}^{(2)} \cdot \mathbf{u}^{(1)}\, d\mathbf{x} + \oint_s \mathbf{t}^{(2)} \cdot \mathbf{u}^{(1)}\, ds, \tag{7.4.14}$$

where $\mathbf{u}^{(i)}$ are the displacements produced by body forces $\mathbf{f}^{(i)}$ and surface forces $\mathbf{t}^{(i)}$. The usefulness of Betti's (also Maxwell's) reciprocity theorem is that it allows us to compute the the displacements or forces at points other than where the forces are applied; that is, the theorem does not allow us to determine the displacement of a point where the force is applied.

The proof of Betti's reciprocity theorem is straightforward. Let W_{12} denote the work done by forces $(\mathbf{f}^{(1)}, \mathbf{t}^{(1)})$ acting through the displacement $\mathbf{u}^{(2)}$ produced by the forces $(\mathbf{f}^{(2)}, \mathbf{t}^{(2)})$. Then

$$\begin{aligned} W_{12} &= \int_\Omega \mathbf{f}^{(1)} \cdot \mathbf{u}^{(2)}\, d\mathbf{x} + \oint_s \mathbf{t}^{(1)} \cdot \mathbf{u}^{(2)}\, ds \\ &= \int_\Omega f_i^{(1)} u_i^{(2)}\, d\mathbf{x} + \oint_s t_i^{(1)} u_i^{(2)}\, ds \\ &= \int_\Omega f_i^{(1)} u_i^{(2)}\, d\mathbf{x} + \oint_s n_j \sigma_{ji}^{(1)} u_i^{(2)}\, ds \\ &= \int_\Omega f_i^{(1)} u_i^{(2)}\, d\mathbf{x} + \int_\Omega \left(\sigma_{ji}^{(1)} u_i^{(2)} \right)_{,j}\, d\mathbf{x} \\ &= \int_\Omega \left(\sigma_{ij,j}^{(1)} + f_i^{(1)} \right) u_i^{(2)}\, d\mathbf{x} + \int_\Omega \sigma_{ij}^{(1)} u_{i,j}^{(2)}\, d\mathbf{x} \\ &= \int_\Omega \sigma_{ij}^{(1)} u_{i,j}^{(2)}\, d\mathbf{x} = \int_\Omega \sigma_{ij}^{(1)} \varepsilon_{ij}^{(2)}\, d\mathbf{x}. \end{aligned} \tag{7.4.15}$$

Using Hooke's law $\sigma_{ij}^{(1)} = C_{ijk\ell}\,\varepsilon_{k\ell}^{(1)}$, we obtain

$$W_{12} = \int_\Omega C_{ijk\ell}\,\varepsilon_{k\ell}^{(1)}\,\varepsilon_{ij}^{(2)}\;d\mathbf{x}. \tag{7.4.16}$$

Since $C_{ijk\ell} = C_{k\ell ij}$, it follows that

$$W_{12} = \int_\Omega C_{ijk\ell}\,\varepsilon_{k\ell}^{(1)}\,\varepsilon_{ij}^{(2)}\;d\mathbf{x} = \int_\Omega C_{k\ell ij}\,\varepsilon_{ij}^{(2)}\,\varepsilon_{k\ell}^{(1)}\;d\mathbf{x} = \int_\Omega \sigma_{k\ell}^{(2)}\varepsilon_{k\ell}^{(1)}\;d\mathbf{x} = W_{21}. \tag{7.4.17}$$

Thus, we have established the equality in Eq. (7.4.14). From Eq. (7.4.17), we also have

$$\begin{aligned}\int_\Omega \sigma_{ij}^{(1)}\varepsilon_{ij}^{(2)}\;d\mathbf{x} &= \int_\Omega \sigma_{ij}^{(2)}\varepsilon_{ij}^{(1)}\;d\mathbf{x},\\ \int_\Omega \boldsymbol{\sigma}^{(1)} : \boldsymbol{\varepsilon}^{(2)}\;d\mathbf{x} &= \int_\Omega \boldsymbol{\sigma}^{(2)} : \boldsymbol{\varepsilon}^{(1)}\;d\mathbf{x}.\end{aligned} \tag{7.4.18}$$

Example 7.4.4

(a) Consider a cantilever beam of length L subjected to two different types of loads: a concentrated load F at the free end and a uniformly distributed load of intensity q throughout the span (see Fig. 7.4.5). Verify that the work done by the point load F in moving through the displacement v^q produced by q is equal to the work done by the distributed force q in moving through the displacement v^F produced by the point load F, $W_{12} = W_{21}$.

(b) A load $P = 4000$ lb acting at a point A of a beam produces 0.25 in. at point B and 0.75 in. at point C of the beam. Find the deflection of point A produced by loads 4500 lb and 2000 lb acting at points B and C, respectively.

Solution: (a) The deflection $v^F(x)$ due to the concentrated load alone is

$$v^F(x) = \frac{FL^3}{6EI}\left[2 - 3\,\frac{x}{L} + \left(\frac{x}{L}\right)^3\right],$$

and the deflection equation due to the distributed load is

$$v^q(x) = \frac{qL^4}{24EI}\left[3 - 4\,\frac{x}{L} + \left(\frac{x}{L}\right)^4\right].$$

The work done by load F in moving through the displacement due to the application of the uniformly distributed load q is

$$W_{12} = Fv^q(0) = \frac{FqL^4}{8EI}\,.$$

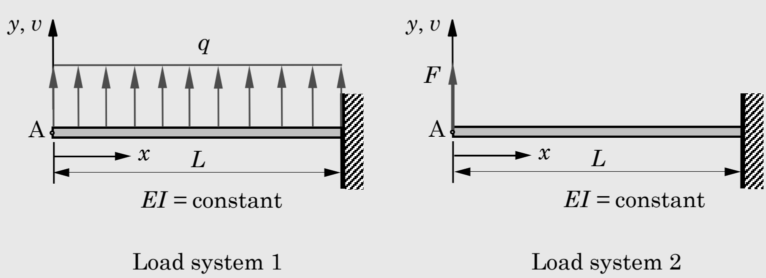

Fig. 7.4.5: A cantilever beam subjected to two different types of loads.

The work done by the uniformly distributed q in moving through the displacement field due to the application of point load F is

$$W_{21} = \int_0^L q\,v^F(x)\,dx = \int_0^L q\,\frac{F}{6EI}\left(x^3 - 3L^2x + 2L^3\right)dx = \frac{FqL^4}{8EI},$$

which is in agreement with W_{12}.

(b) From Betti's reciprocity theorem and the principle of superposition, we have

$$F_{\rm B}\cdot v_{\rm BA} + F_{\rm C}\cdot v_{\rm CA} = F_{\rm A}\cdot v_{\rm AB} + F_{\rm A}\cdot v_{\rm AC} = F_{\rm A}\cdot v_{\rm A}$$

where $v_{\rm A} = v_{\rm AB} + v_{\rm AC}$, $F_{\rm A} = 4,000$ lb, $F_{\rm B} = 4,500$ lb, $F_{\rm C} = 2,000$ lb. We obtain

$$v_{\rm A} = \frac{F_{\rm B}\cdot v_{\rm BA} + F_{\rm C}\cdot v_{\rm CA}}{F_{\rm A}} = \frac{4500\times 0.25 + 2000\times 0.75}{4000} = 0.65625 \text{ in}.$$

7.4.3 Maxwell's Reciprocity Theorem

An important special case of Betti's reciprocity theorem is given by Maxwell's (1831–1879) reciprocity theorem. Maxwell's theorem was given in 1864, whereas Betti's theorem was given in 1872. Therefore, it may be considered that Betti generalized the work of Maxwell.

Consider a linear elastic solid subjected to force $\mathbf{F}^1$ of unit magnitude acting at point 1, and force $\mathbf{F}^2$ of unit magnitude acting at a different point 2 of the body. Let $\mathbf{u}_{12}$ be the displacement of point 1 in the direction of force $\mathbf{F}^1$ produced by unit force $\mathbf{F}^2$, and $\mathbf{u}_{21}$ be the displacement of point 2 in the direction of force $\mathbf{F}^2$ produced by unit force $\mathbf{F}^1$ (see Fig. 7.4.6). From Betti's theorem it follows that

$$\mathbf{F}^1\cdot\mathbf{u}_{12} = \mathbf{F}^2\cdot\mathbf{u}_{21} \quad \text{or} \tag{7.4.19}$$

$$\mathbf{u}_{12} = \mathbf{u}_{21}. \tag{7.4.20}$$

Equation (7.4.19) is a statement of Maxwell's theorem. If $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ denote the unit vectors along forces $\mathbf{F}^1$ and $\mathbf{F}^2$, respectively, Maxwell's theorem states that the displacement of point 1 in the $\hat{\mathbf{e}}_1$ direction produced by unit force acting at point 2 in the $\hat{\mathbf{e}}_2$ direction is equal to the displacement of point 2 in the $\hat{\mathbf{e}}_2$ direction produced by unit force acting at point 1 in the $\hat{\mathbf{e}}_1$ direction. We close this section with several examples of the use of Maxwell's theorem.

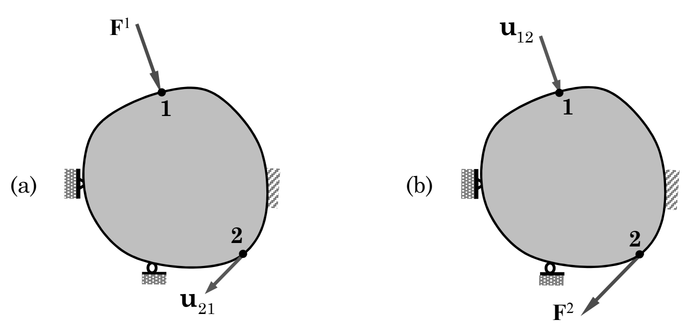

Fig. 7.4.6: Configurations of the body discussed in Maxwell's theorem.

Example 7.4.5

Consider a cantilever beam ($E = 24 \times 10^6$ psi, $I = 120$ in.^{4}) of length 12 ft. subjected to a point load 4000 lb at the free end, as shown in Fig. 7.4.7(a). Use Maxwell's theorem to find the deflection at a point 3 ft. from the free end.

Solution: By Maxwell's theorem, the displacement $v_{\rm BC}$ at point B ($x = 3$ ft.) produced by the 4000-lb load at point C ($x = 0$) is equal to the deflection $v_{\rm CB}$ at point C produced by applying the 4000-lb load at point B. Let $v_{\rm B}$ and $\theta_{\rm B}$ denote the deflection and slope, respectively, at point B owing to load $F = 4000$ lb applied at point B, as shown in Fig. 7.4.7(b). The deflection at point B ($x = b = 3$ ft.) caused by load $F = 4000$ lb at point C ($x = 0$) is ($v_{\rm B} = Fa^3/3EI$ and $\theta_{\rm B} = Fa^2/2EI$)

$$\begin{aligned} v_{\rm BC} = v_{\rm CB} &= v_{\rm B} + (3 \times 12)\theta_{\rm B} \\ &= \frac{4000(9 \times 12)^3}{3EI} + \frac{(3 \times 12)4000(9 \times 12)^2}{2EI} \\ &= \frac{243 \times 6000 \times (12)^3}{24 \times 10^6 \times 120} = 0.8748 \text{ in.} \end{aligned}$$

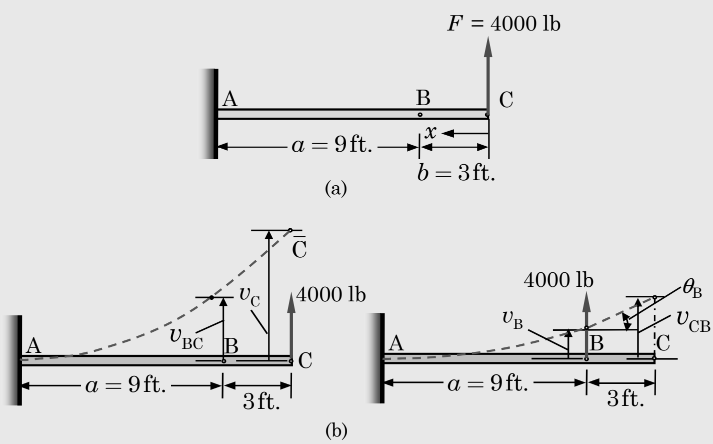

Fig. 7.4.7: The cantilever beam of Example 7.4.5.

Example 7.4.6

Consider a circular plate of radius a with an axisymmetric boundary condition, and subjected to an asymmetric loading of the type (see Fig. 7.4.8)

$$q(r, \theta) = q_0 + q_1 \frac{r}{a} \cos\theta, \tag{7.4.20}$$

where q_0 represents the uniform part of the load for which the solution can be determined for various axisymmetric boundary conditions [see Reddy (2007)]. In particular, the deflection of a clamped circular plate under a point load F_0 at the center is given by

$$v(r) = \frac{F_0 a^2}{16\pi D}\left[1 - \frac{r^2}{a^2} + 2\frac{r^2}{a^2}\ln\left(\frac{r}{a}\right)\right]. \tag{7.4.21}$$

Use the Betti/Maxwell reciprocity theorem to determine the center deflection of a clamped plate under an asymmetric distributed load.

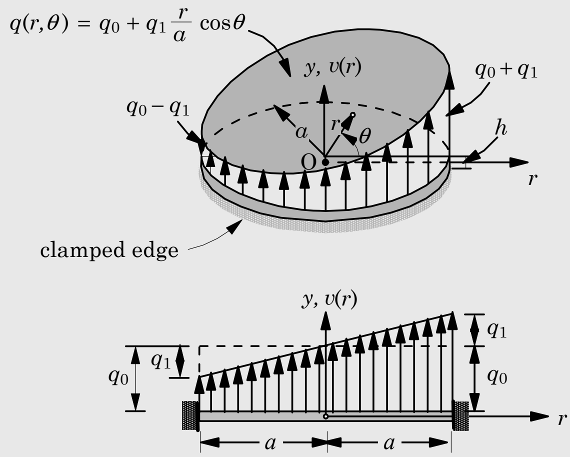

Fig. 7.4.8: A circular plate subjected to an asymmetric loading.

Solution: By Maxwell's theorem, the work done by a point load F_0 at the center of the plate due to the deflection (at the center) v_c caused by the distributed load $q(r,\theta)$ is equal to the work done by the distributed load $q(r,\theta)$ in moving through the displacement $v_0(r)$ caused by the point load F_0 at the center (it is not necessary to make $F_0 = 1$ because it will cancel out from both sides). The center deflection of a clamped circular plate under asymmetric load given in Eq. (7.4.20) is $v_c = v(0)$:

$$F_0\, v_c = \frac{F_0 a^2}{16\pi D} \int_0^{2\pi} \int_0^a \left(q_0 + \frac{q_1}{a} r\cos\theta\right) \left[1 - \frac{r^2}{a^2}\left(1 - 2\ln\frac{r}{a}\right)\right] r\, dr\, d\theta$$

$$v_c = \frac{q_0 a^2}{16\pi D} \int_0^a \left(r - \frac{r^3}{a^2} - \frac{2}{a^2} r^3 \ln\frac{r}{a}\right) dr = \frac{q_0 a^4}{64D}\,, \tag{7.4.22}$$

where the following integral identity is used in arriving at the result:

$$\int r^n \ln(\alpha r)\, dr = \frac{r^{n+1}}{n+1} \ln(\alpha r) - \frac{r^{n+1}}{(n+1)^2}\,, \quad \alpha = \text{ constant.} \tag{7.4.23}$$

7.5 Solution of Two-Dimensional Problems

7.5.1 Introduction

In a class of problems in elasticity, due to geometry, material properties, boundary conditions and external applied loads, the solutions (that is, displacements and stresses) are not dependent on one of the coordinates. Such problems are called *plane elasticity* problems. The plane elasticity problems considered here are grouped into *plane strain* and *plane stress* problems. Both classes of problems are described by a set of two *coupled* partial differential equations expressed in terms of two dependent variables that represent the two components of the displacement vector. The governing equations of plane strain problems differ from those of the plane stress problems only in the coefficients of the differential equations, as shown shortly. The discussion here is limited to isotropic materials.

7.5.2 Plane Strain Problems

Plane strain problems are characterized by the displacement field

$$\mathbf{u} = u_x\,\hat{\mathbf{e}}_x + u_y\,\hat{\mathbf{e}}_y \quad [u_x = u_x(x,y),\ u_y = u_y(x,y), \quad u_z = 0]\,, \tag{7.5.1}$$

where (u_x, u_y, u_z) denote the components of the displacement vector $\mathbf{u}$ in the (x, y, z) coordinate system. An example of a plane strain problem is provided by the long cylindrical member (not necessarily of circular cross section) under external loads that are independent of the z-coordinate, as shown in Fig. 7.5.1. For cross sections sufficiently far from the ends, the displacement u_z is zero and u_x and u_y are independent of z, that is, a state of plane strain exists.

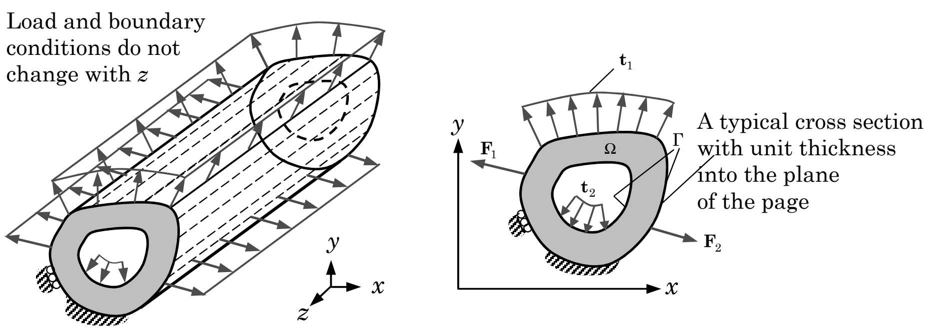

Fig. 7.5.1: An example of a plane strain problem.

The displacement field (7.5.1) results in the following strain field:

$$\begin{gathered}\varepsilon_{xx} = \frac{\partial u_x}{\partial x}\,,\quad 2\varepsilon_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\,,\quad \varepsilon_{yy} = \frac{\partial u_y}{\partial y}\,, \\ \varepsilon_{xz} = \varepsilon_{yz} = \varepsilon_{zz} = 0,\end{gathered}. \tag{7.5.2}$$

The stress components are calculated using the stress–strain relations [see Eq. (7.2.9); also note $\lambda/(\mu + \lambda) = 2\nu$]

$$\begin{aligned}&\sigma_{xx} = (2\mu + \lambda)\varepsilon_{xx} + \lambda\,\varepsilon_{yy}, \quad \sigma_{yy} = (2\mu + \lambda)\varepsilon_{yy} + \lambda\,\varepsilon_{xx}, \quad \sigma_{xy} = 2\mu\,\varepsilon_{xy}, \\ &\sigma_{zz} = \lambda\,(\varepsilon_{xx} + \varepsilon_{yy}) = \nu\,(\sigma_{xx} + \sigma_{yy})\,, \quad \sigma_{xz} = 0, \quad \sigma_{yz} = 0.\end{aligned} \tag{7.5.3}$$

Writing in terms of E and ν directly from Eq. (7.2.10), we have

$$\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & 0\\ \nu & 1-\nu & 0\\ 0 & 0 & \frac{(1-2\nu)}{2}\end{bmatrix}\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ 2\varepsilon_{xy}\end{Bmatrix}. \tag{7.5.4}$$

The equations of equilibrium of three-dimensional linear elasticity, with the body force components

$$f_x = f_x(x,y), \quad f_y = f_y(x,y), \quad f_z = 0, \tag{7.5.5}$$

reduce to the following two equations:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + f_x = 0, \tag{7.5.6}$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + f_y = 0. \tag{7.5.7}$$

The boundary conditions are either the stress type

$$\left.\begin{aligned} t_x \equiv \sigma_{xx} n_x + \sigma_{xy} n_y = \hat{t}_x \\ t_y \equiv \sigma_{xy} n_x + \sigma_{yy} n_y = \hat{t}_y \end{aligned}\right\} \quad \text{on } \Gamma_\sigma, \tag{7.5.8}$$

or the displacement type

$$u_x = \hat{u}_x, \quad u_y = \hat{u}_y \quad \text{on } \Gamma_u. \tag{7.5.9}$$

Here (n_x, n_y) denote the components (or direction cosines) of the unit normal vector on the boundary Γ, Γ_σ and Γ_u are disjoint (i.e., nonoverlapping) portions of the boundary Γ such that their sum is equal to the total boundary

$$\Gamma = \Gamma_\sigma + \Gamma_u, \quad \Gamma_\sigma \cap \Gamma_u = \text{ empty}, \tag{7.5.10}$$

$\hat{t}_x$ and $\hat{t}_y$ are the components of the specified traction vector, and $\hat{u}_x$ and $\hat{u}_y$ are the components of the specified displacement vector. Only one element of each pair, (u_x, t_x) and (u_y, t_y), should be specified at a boundary point.

The preceding discussion can be extended to plane strain problems in cylindrical coordinates. We now consider an example of a plane strain problem.

Example 7.5.1

Consider an isotropic, hollow circular cylinder of internal radius a and outside radius b. The cylinder is held between rigid supports such that $u_z = 0$ at $z = \pm L/2$, pressurized at $r = a$ as well as at $r = b$, and is rotating with a uniform speed of ω about its axis (i.e., the z-axis), as shown in Fig. 7.5.2. Determine the displacements, strains, and stresses in the cylinder.

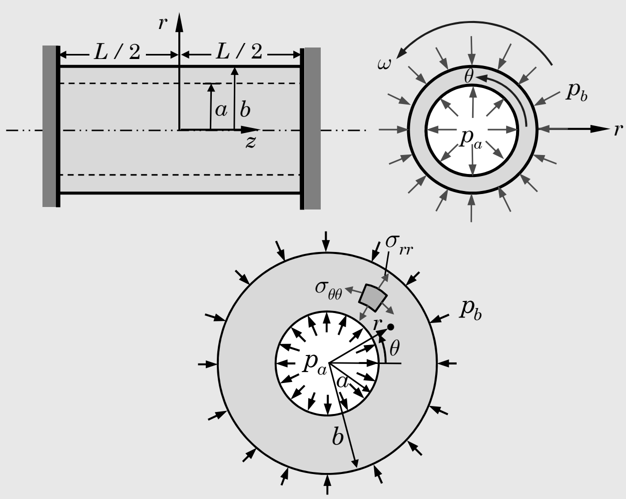

Fig. 7.5.2: Rotating cylindrical pressure vessel.

Solution: Because of the geometry and boundary conditions (material is isotropic), the cylindrical coordinate system (r, θ, z) is most convenient to formulate the problem. The rotation of the cylinder about its own axis generates a radial (centrifugal) force of magnitude $\rho_0\, \omega^2 r$ at a distance r. Thus, the body force vector is $\mathbf{f} = \rho_0\, \omega^2 r\, \hat{\mathbf{e}}_r$. Also, we find that the problem has symmetry about $z = 0$, and the plane $z = 0$ has exactly the same boundary conditions as the plane $z = L/2$. Therefore, we find that the problem has symmetry about $z = L/4$. This way, it is clear that we can consider any section of unit length of the cylinder to determine the displacements, strains, and stresses. In other words, it is a plane strain problem, and the solution is independent of θ (due to the axisymmetric geometry, forces, and material properties) and z. In fact, the only nonzero displacement is u_r, and it is only a function of the radial coordinate r. The problem has only stress boundary conditions (BVP type II),

$$\text{At } r = a: \quad \hat{\mathbf{n}} = -\hat{\mathbf{e}}_r, \quad \mathbf{t} = p_a\, \hat{\mathbf{e}}_r \quad \text{or} \quad \sigma_{rr} = -p_a, \quad \sigma_{r\theta} = 0, \tag{7.5.11}$$

$$\text{At } r = b: \quad \hat{\mathbf{n}} = \hat{\mathbf{e}}_r, \quad \mathbf{t} = -p_b\, \hat{\mathbf{e}}_r \quad \text{or} \quad \sigma_{rr} = -p_b, \quad \sigma_{r\theta} = 0. \tag{7.5.12}$$

We begin with the assumed displacement field

$$u_r = U(r), \quad u_\theta = u_z = 0, \tag{7.5.13}$$

where $U(r)$ is an unknown function to be determined such that the equations of elasticity and boundary conditions of the problem are satisfied. The strains are [see Eq. (7.2.3)]

$$\varepsilon_{rr} = \frac{dU}{dr}, \quad \varepsilon_{\theta\theta} = \frac{U}{r}, \quad \varepsilon_{zz} = \varepsilon_{r\theta} = \varepsilon_{rz} = \varepsilon_{\theta z} = 0. \tag{7.5.14}$$

The stresses are determined using the stress–strain relations in Eq. (7.2.9)

$$\begin{aligned}
\sigma_{rr} &= 2\mu\varepsilon_{rr} + \lambda\,(\varepsilon_{rr} + \varepsilon_{\theta\theta}) = (2\mu + \lambda)\frac{dU}{dr} + \lambda\frac{U}{r},\\
\sigma_{\theta\theta} &= 2\mu\varepsilon_{\theta\theta} + \lambda\,(\varepsilon_{rr} + \varepsilon_{\theta\theta}) = (2\mu + \lambda)\frac{U}{r} + \lambda\frac{dU}{dr},\\
\sigma_{zz} &= \lambda\,(\varepsilon_{rr} + \varepsilon_{\theta\theta}) = \lambda\left(\frac{dU}{dr} + \frac{U}{r}\right).
\end{aligned} \tag{7.5.15}$$

All other stresses, $\sigma_{r\theta}$, σ_{rz}, and $\sigma_{\theta z}$, are zero.

The last two equations of equilibrium, Eq. (7.2.7) without the acceleration terms, are trivially satisfied, and the first equation reduces to

$$\frac{d\sigma_{rr}}{dr} + \frac{1}{r}\,(\sigma_{rr} - \sigma_{\theta\theta}) + \rho_0\omega^2 r = 0,$$

which can be expressed in terms of $U(r)$ using Eq. (7.5.15)

$$(2\mu + \lambda)\frac{d^2U}{dr^2} + \lambda\frac{d}{dr}\left(\frac{U}{r}\right) + \frac{2\mu}{r}\left(\frac{dU}{dr} - \frac{U}{r}\right) + \rho_0\omega^2 r = 0. \tag{7.5.16}$$

Simplifying the expression, we obtain

$$r^2\frac{d^2U}{dr^2} + r\frac{dU}{dr} - U = -\alpha r^3, \quad \alpha = \frac{\rho_0\omega^2}{2\mu + \lambda}. \tag{7.5.17}$$

The linear differential equation (7.5.17) can be transformed to one with constant coefficients by a change of independent variable, $r = e^\xi$ (or $\xi = \ln r$). Using the chain rule of differentiation, we obtain

$$\frac{dU}{dr} = \frac{dU}{d\xi}\frac{d\xi}{dr} = \frac{1}{r}\frac{dU}{d\xi}, \quad \frac{d^2U}{dr^2} = \frac{d}{dr}\left(\frac{1}{r}\frac{dU}{d\xi}\right) = \frac{1}{r^2}\left(-\frac{dU}{d\xi} + \frac{d^2U}{d\xi^2}\right). \tag{7.5.18}$$

Substituting these expressions into Eq. (7.5.17), we obtain

$$\frac{d^2U}{d\xi^2} - U = -\alpha e^{3\xi}. \tag{7.5.19}$$

Seeking a solution in the form, $U(\xi) = e^{m\xi}$, we obtain $m = \pm 1$, and the total solution to the problem is

$$U(\xi) = c_1 e^{\xi} + c_2 e^{-\xi} - \frac{\alpha}{8} e^{3\xi}. \tag{7.5.20}$$

Changing back to the original independent variable r, the radial displacement is

$$u_r(r) = U(r) = c_1 r + \frac{c_2}{r} - \frac{\alpha}{8} r^3, \tag{7.5.21}$$

where the constants c_1 and c_2 are to be determined using the boundary conditions in Eqs. (7.5.11) and (7.5.12). Hence, we must compute σ_{rr},

$$\begin{aligned}\sigma_{rr} &= (2\mu+\lambda)\left(c_1 - \frac{c_2}{r^2} - \frac{3\alpha}{8} r^2\right) + \lambda\left(c_1 + \frac{c_2}{r^2} - \frac{\alpha}{8} r^2\right) \\ &= 2(\mu+\lambda)c_1 - 2\mu\frac{c_2}{r^2} - \frac{(3\mu+2\lambda)\alpha}{4} r^2. \end{aligned} \tag{7.5.22}$$

Applying the stress boundary conditions in (7.5.11) and (7.5.12), we obtain

$$\begin{aligned} 2(\mu+\lambda)c_1 - 2\mu\frac{c_2}{a^2} - \frac{(3\mu+2\lambda)\alpha}{4} a^2 &= -p_a, \\ 2(\mu+\lambda)c_1 - 2\mu\frac{c_2}{b^2} - \frac{(3\mu+2\lambda)\alpha}{4} b^2 &= -p_b. \end{aligned} \tag{7.5.23}$$

Solving for the constants c_1 and c_2,

$$\begin{aligned} c_1 &= \frac{1}{2(\mu+\lambda)}\left[\left(\frac{p_a a^2 - p_b b^2}{b^2 - a^2}\right) + (b^2 + a^2)\frac{(3\mu+2\lambda)}{(2\mu+\lambda)}\frac{\rho_0\omega^2}{4}\right], \\ c_2 &= \frac{a^2 b^2}{2\mu}\left[\left(\frac{p_a - p_b}{b^2 - a^2}\right) + \frac{(3\mu+2\lambda)}{(2\mu+\lambda)}\frac{\rho_0\omega^2}{4}\right]. \end{aligned} \tag{7.5.24}$$

Finally, the displacement u_r and stress σ_{rr} in the cylinder are given by

$$\begin{aligned} u_r &= \frac{1}{2(\mu+\lambda)}\left[\left(\frac{p_a a^2 - p_b b^2}{b^2 - a^2}\right) + (b^2 + a^2)\frac{(3\mu+2\lambda)}{(2\mu+\lambda)}\frac{\rho_0\omega^2}{4}\right] r \\ &\quad + \frac{a^2 b^2}{2\mu}\left[\left(\frac{p_a - p_b}{b^2 - a^2}\right) + \frac{(3\mu+2\lambda)}{(2\mu+\lambda)}\frac{\rho_0\omega^2}{4}\right]\frac{1}{r} - \frac{\rho_0\omega^2}{8(2\mu+\lambda)} r^3, \end{aligned} \tag{7.5.25}$$

$$\begin{aligned} \sigma_{rr} &= \left[\left(\frac{p_a a^2 - p_b b^2}{b^2 - a^2}\right) + (b^2 + a^2)\frac{(3\mu+2\lambda)}{(2\mu+\lambda)}\frac{\rho_0\omega^2}{4}\right] \\ &\quad - \frac{a^2 b^2}{r^2}\left[\left(\frac{p_a - p_b}{b^2 - a^2}\right) + \frac{(3\mu+2\lambda)}{(2\mu+\lambda)}\frac{\rho_0\omega^2}{4}\right] - \frac{(3\mu+2\lambda)\alpha}{4} r^2. \end{aligned} \tag{7.5.26}$$

Similarly, stresses $\sigma_{\theta\theta}$ and σ_{zz} can be computed.

7.5.3 Plane Stress Problems

A state of *plane stress* is one in which the stresses associated with one of the coordinates (z) are zero and the other stresses are functions of the remaining two coordinates (x and y):

$$\begin{aligned} \sigma_{xz} = \sigma_{yz} &= \sigma_{zz} = 0, \\ \sigma_{xx} = \sigma_{xx}(x,y), \quad \sigma_{xy} &= \sigma_{xy}(x,y), \quad \sigma_{yy} = \sigma_{yy}(x,y). \end{aligned} \tag{7.5.27}$$

An example of a plane stress problem is provided by a thin plate subjected to loads in the xy plane that are independent of z, as shown in Fig. 7.5.3. The top and bottom surfaces of the plate are assumed to be traction-free, and $f_z = 0$ and $u_z = 0$.

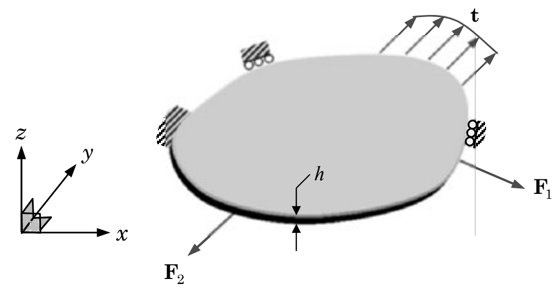

Fig. 7.5.3: A thin plate in a state of plane stress.

The stress–strain relations of a plane stress state for an isotropic material are obtained by inverting the strain–stress relations in Eq. (6.3.23):

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{(1-\nu)}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{Bmatrix}. \tag{7.5.28}$$

The equations of equilibrium as well as boundary conditions of a plane stress problem are the same as those listed in Eqs. (7.5.6)–(7.5.9). Note that the governing equations of plane stress and plane strain differ from each other only on account of the difference in the constitutive equations for the two cases.

Example 7.5.2

Consider a thin, uniform, solid circular disk of radius a, spinning at a constant angular velocity of ω, as shown in Fig. 7.5.4. Use the semi-inverse method to determine the displacements, strains, and stresses in the disk.

Solution: This problem is almost the same as the problem of the rotating cylinder considered in Example 7.5.1. The difference is that the cylinder problem was one of plane strain and the present thin disk problem is one of plane stress. First, we set up the polar cylindrical coordinate system (r, θ), with the origin at the center of the disk, r being the radial coordinate

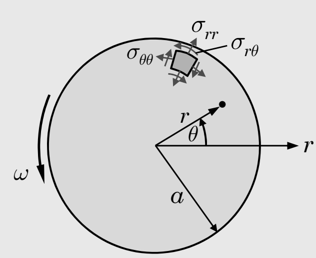

Fig. 7.5.4: Thin, uniform, spinning solid disk.

and θ the circumferential coordinate. The boundary conditions are

$$u_r(0,\theta) = \text{ finite}, \quad \sigma_{rr}(a,\theta) = \sigma_{r\theta}(a,\theta) = 0. \tag{1}$$

Because of the axisymmetry of the geometry, boundary conditions, and material, the disk experiences only a radial displacement field that varies only with r. Using the semi-inverse method, we assume

$$u_r = U(r), \quad u_\theta = 0, \tag{2}$$

where $U(r)$ is an unknown function to be determined such that the equations of elasticity and boundary conditions of the problem are satisfied. The strains associated with the displacement field (2) are

$$\varepsilon_{rr} = \frac{dU}{dr}, \quad \varepsilon_{\theta\theta} = \frac{U}{r}, \quad \varepsilon_{r\theta} = 0. \tag{3}$$

The stresses are determined using the stress–strain relations for *plane stress*, Eq. (7.5.28). We obtain

$$\begin{aligned}\sigma_{rr} &= \frac{E}{1-\nu^2}\left(\varepsilon_{rr} + \nu\varepsilon_{\theta\theta}\right) = \frac{E}{1-\nu^2}\frac{dU}{dr} + \frac{E\nu}{1-\nu^2}\frac{U}{r},\\ \sigma_{\theta\theta} &= \frac{E}{1-\nu^2}\left(\nu\varepsilon_{rr} + \varepsilon_{\theta\theta}\right) = \frac{E\nu}{1-\nu^2}\frac{dU}{dr} + \frac{E}{1-\nu^2}\frac{U}{r}.\end{aligned} \tag{4}$$

The shear stress $\sigma_{r\theta}$ is zero.

The first two equations of equilibrium, Eq. (7.2.7) without the acceleration terms, are trivially satisfied, and the first equation reduces to

$$\frac{d\sigma_{rr}}{dr} + \frac{1}{r}\left(\sigma_{rr} - \sigma_{\theta\theta}\right) + \rho_0\omega^2 r = 0,$$

where $\rho_0 f_r = \rho_0\omega^2 r$. The above equation can be expressed in terms of $U(r)$ using Eq. (4)

$$\frac{E}{1-\nu^2}\left[\frac{d^2U}{dr^2} + \nu\frac{d}{dr}\left(\frac{U}{r}\right) + \frac{(1-\nu)}{r}\left(\frac{dU}{dr} - \frac{U}{r}\right)\right] + \rho_0\omega^2 r = 0. \tag{5}$$

Simplifying the expression, we obtain

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(rU)\right] = -\alpha\, r, \quad \alpha = \left(\frac{1-\nu^2}{E}\right)\rho_0\omega^2, \tag{6}$$

where we have used the identities

$$\frac{1}{r}\left(\frac{dU}{dr} - \frac{U}{r}\right) = \frac{d}{dr}\left(\frac{U}{r}\right), \quad \frac{d}{dr}\left(\frac{dU}{dr} + \frac{U}{r}\right) = \frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(rU)\right]. \tag{7}$$

The solution to Eq. (6) is given by

$$u_r(r) = U(r) = \frac{c_1}{2}r + \frac{c_2}{r} - \frac{\alpha}{8}r^3, \tag{8}$$

where the constants c_1 and c_2 are to be determined using the boundary conditions in Eq. (1). The fact that u_r is finite (i.e., bounded) at $r = 0$ requires $c_2 = 0$. Then we have

$$U(r) = \frac{c_1}{2}r - \frac{\alpha}{8}r^3, \quad \frac{dU}{dr} = -\frac{3}{8}\alpha r^2 + \frac{c_1}{2}, \quad \frac{U}{r} = -\frac{1}{8}\alpha r^2 + \frac{c_1}{2}. \tag{9}$$

Computing σ_{rr} using Eq. (4), we obtain

$$\sigma_{rr} = \frac{E}{1-\nu^2}\left(\frac{dU}{dr} + \nu\frac{U}{r}\right) = \frac{E\alpha}{(1-\nu^2)}\left[-\frac{3+\nu}{8}\alpha r^2 + \frac{1+\nu}{2}c_1\right]. \tag{10}$$

Then $\sigma_{rr}(a,\theta) = 0$ gives

$$c_1 = \frac{1}{4}\left(\frac{3+\nu}{1+\nu}\right)\alpha a^2. \tag{11}$$

Thus, the solution in Eq. (8) becomes

$$u(r) = \frac{1}{4}\left(\frac{3+\nu}{1+\nu}\right)\alpha a^2 r - \frac{\alpha}{8} r^3 = \frac{(1-\nu)}{8E}\left[2(3+\nu)a^2 - (1+\nu)r^2\right]\rho_0\omega^2 r. \tag{12}$$

The stresses σ_{rr} and $\sigma_{\theta\theta}$ are

$$\begin{aligned} \sigma_{rr}(r) &= \frac{(3+\nu)}{8}\left(a^2 - r^2\right)\rho_0\omega^2 \\ \sigma_{\theta\theta}(r) &= \frac{1}{8}\left[(3+\nu)a^2 - (1+3\nu)r^2\right]\rho_0\omega^2 \end{aligned} \tag{13}$$

The values of the maximum displacement and maximum stresses are

$$\begin{aligned} u_{\max} &= u_r(a) = \frac{(1-\nu)(5+\nu)}{8E}\rho_0\omega^2 a^3, \\ \sigma_{\max} &= \sigma_{rr}(0) = \sigma_{\theta\theta}(0) = \frac{(3+\nu)}{8}\rho_0\omega^2 a^2. \end{aligned} \tag{14}$$

7.5.4 Unification of Plane Strain and Plane Stress Problems

The equilibrium equations (7.5.6) and (7.5.7), which are valid for both plane stress and plane strain, can be expressed in index notation as

$$\sigma_{\beta\alpha,\beta} + f_\alpha = 0, \tag{7.5.29}$$

To unify the formulation for plane strain and plane stress, we introduce the parameter s:

$$s = \begin{cases} \frac{1}{1-\nu}, & \text{for plane strain} \\ 1+\nu, & \text{for plane stress.} \end{cases} \tag{7.5.30}$$

Then the constitutive equations of plane stress as well as plane strain can be expressed as

$$\begin{aligned} \sigma_{\alpha\beta} &= 2\mu\left[\varepsilon_{\alpha\beta} + \left(\tfrac{s-1}{2-s}\right)\varepsilon_{\gamma\gamma}\delta_{\alpha\beta}\right], \\ \varepsilon_{\alpha\beta} &= \frac{1}{2\mu}\left[\sigma_{\alpha\beta} - \left(\tfrac{s-1}{s}\right)\sigma_{\gamma\gamma}\delta_{\alpha\beta}\right], \end{aligned} \tag{7.5.31}$$

where α, β, and γ take values of 1 and 2 (or x and y). The compatibility equation (3.7.4) for plane stress and plane strain problems takes the form

$$\varepsilon_{\alpha\alpha,\beta\beta} - \varepsilon_{\alpha\beta,\alpha\beta} = 0 \quad (\alpha, \beta = 1, 2), \tag{7.5.32}$$

or, in terms of stress components,

$$\nabla^2\sigma_{\alpha\alpha} = -s\, f_{\alpha,\alpha}. \tag{7.5.33}$$

Comparing the constitutive equations of plane strain and plane stress, Eqs. (7.5.4) and (7.5.28), it is clear that the plane strain equations can be transformed to corresponding plane stress equations, and vice versa, by a simple change in material parameters, as follows:

$$\begin{aligned} &\text{Plane stress to plane strain:} \quad E \to \tfrac{E}{1-\nu^2} \quad \text{and} \quad \nu \to \tfrac{\nu}{1-\nu}, \\ &\text{Plane strain to plane stress:} \quad E \to \tfrac{(1+2\nu)E}{(1+\nu)^2} \quad \text{and} \quad \nu \to \tfrac{\nu}{1+\nu}. \end{aligned} \tag{7.5.34}$$

7.5.5 Airy Stress Function

Airy stress function is a potential function introduced to identically satisfy the equations of equilibrium, Eqs. (7.5.6) and (7.5.7). First, we assume that the body force vector **f** is derivable from a scalar potential V_f such that

$$\mathbf{f} = -\boldsymbol{\nabla} V_f \quad \text{or} \quad f_x = -\frac{\partial V_f}{\partial x}, \quad f_y = -\frac{\partial V_f}{\partial y}. \tag{7.5.35}$$

When body forces are derivable from a potential V_f, they are said to be *conservative*. Next, we introduce the Airy stress function $\Phi(x, y)$ such that

$$\sigma_{xx} = \frac{\partial^2 \Phi}{\partial y^2} + V_f, \quad \sigma_{yy} = \frac{\partial^2 \Phi}{\partial x^2} + V_f, \quad \sigma_{xy} = -\frac{\partial^2 \Phi}{\partial x \partial y}. \tag{7.5.36}$$

This definition of $\Phi(x, y)$ automatically satisfies the equations of equilibrium (7.5.6) and (7.5.7).

The stresses derived from Eq. (7.5.36) are subject to the compatibility conditions (7.5.33). Substituting for $\sigma_{\alpha\beta}$ in terms of Φ from Eq. (7.5.36) into Eq. (7.5.33), we obtain

$$\nabla^4 \Phi + (2 - s)\nabla^2 V_f = 0, \tag{7.5.37}$$

where $\nabla^4 = \nabla^2 \nabla^2$ is the *biharmonic operator*, which, in two dimensions, has the form

$$\nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}.$$

If the body forces are zero, we have $V_f = 0$ and Eq. (7.5.37) reduces to the *biharmonic equation*

$$\nabla^4 \Phi = 0. \tag{7.5.38}$$

In cylindrical coordinate system, Eqs. (7.5.35) and (7.5.36) take the form

$$f_r = -\frac{\partial V_f}{\partial r}, \quad f_\theta = -\frac{1}{r}\frac{\partial V_f}{\partial \theta}, \tag{7.5.39}$$

$$\sigma_{rr} = \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} + V_f, \quad \sigma_{\theta\theta} = \frac{\partial^2 \Phi}{\partial r^2} + V_f, \quad \sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \Phi}{\partial \theta}\right). \tag{7.5.40}$$

The biharmonic operator $\nabla^4 = \nabla^2 \nabla^2$ can be expressed using the definition of ∇^2 in a cylindrical coordinate system

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}. \tag{7.5.41}$$

In summary, the solution to a plane elastic problem using the Airy stress function involves finding the solution to Eq. (7.5.37) and satisfying the boundary conditions of the problem. The most difficult part is finding a solution to the fourth-order equation (7.5.37) over a given domain. Often the form of the Airy stress function is obtained by either the inverse method or semi-inverse method. Next we consider several examples of the Airy stress function approach. Additional examples can be found in the books by Timoshenko and Goodier (1970) and Slaughter (2002).

Example 7.5.3

Suppose that the Airy stress function is a second-order polynomial (the lowest order that gives a nonzero stress field) of the form

$$\Phi(x, y) = c_1 xy + c_2 x^2 + c_3 y^2. \tag{7.5.42}$$

Assuming that the body force field is zero, determine if the constants c_1, c_2, and c_3 correspond to a possible state of stress for some boundary value problem (the inverse method).

Solution: Clearly, the biharmonic equation is trivially satisfied by Φ in Eq. (7.5.42). The corresponding stress field is

$$\sigma_{xx} = \frac{\partial^2 \Phi}{\partial y^2} = 2c_3, \quad \sigma_{yy} = \frac{\partial^2 \Phi}{\partial x^2} = 2c_2, \quad \sigma_{xy} = -\frac{\partial^2 \Phi}{\partial x \partial y} = -c_1. \tag{7.5.43}$$

Thus, the constants represent a uniform stress state throughout the body, and it is independent of the geometry. Thus, there are infinite number of problems for which the stress field is a solution. In particular, the rectangular domain with the boundary stresses shown in Fig. 7.5.5 is one such problem.

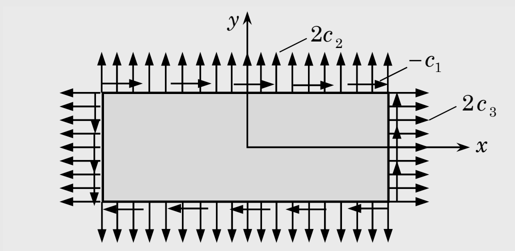

Fig. 7.5.5: A plane problem with uniform stress field.

Example 7.5.4

Take the Airy stress function to be a third-order polynomial of the form

$$\Phi(x, y) = c_1 xy + c_2 x^2 + c_3 y^2 + c_4 x^2 y + c_5 xy^2 + c_6 x^3 + c_7 y^3. \tag{7.5.44}$$

Assuming that the body force field is zero, determine the stress field and identify a possible boundary value problem.

Solution: We note that $\nabla^4 \Phi = 0$ for any c_i. The corresponding stress field is

$$\sigma_{xx} = 2c_3 + 2c_5 x + 6c_7 y, \quad \sigma_{yy} = 2c_2 + 2c_4 y + 6c_6 y, \quad \sigma_{xy} = -c_1 - 2c_4 x - 2c_5 y. \tag{7.5.45}$$

Again, there are an infinite number of problems for which the stress field is a solution. In particular, for $c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = 0$, the solution corresponds to a thin beam in pure bending (see Fig. 7.5.6).

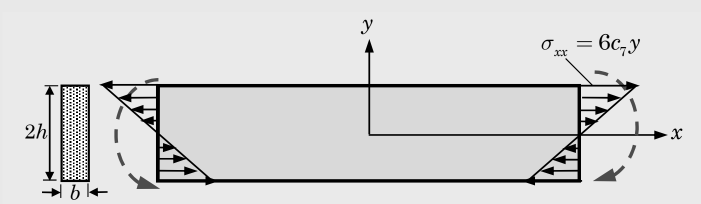

Fig. 7.5.6: A thin beam in pure bending.

Example 7.5.5

Take the Airy stress function to be a fourth-order polynomial of the form (omit terms that were already considered in the last two cases)

$$\Phi(x,y) = c_8x^2y^2 + c_9x^3y + c_{10}xy^3 + c_{11}x^4 + c_{12}y^4, \tag{7.5.46}$$

and determine the stress field and associated boundary value problems.

Solution: Computing $\nabla^4\Phi$ and equating it to zero (body force field is zero) we find that

$$c_8 + 3(c_{11} + c_{12}) = 0.$$

Thus out of five constants only four of them are independent. The corresponding stress field is

$$\begin{aligned}
\sigma_{xx} &= 2c_8x^2 + 6c_{10}xy + 12c_{12}y^2 = -6c_{11}x^2 + 6c_{10}xy + 6c_{12}(2y^2 - x^2)\\
\sigma_{yy} &= 2c_8y^2 + 6c_9xy + 12c_{11}x^2 = 6c_9xy + 6c_{11}(2x^2 - y^2) - 6c_{12}y^2\\
\sigma_{xy} &= -4c_8xy - 3c_9x^2 - 3c_{10}y^2 = 12c_{11}xy + 12c_{12}xy - 3c_9x^2 - 3c_{10}y^2.
\end{aligned} \tag{7.5.47}$$

By suitable adjustment of the constants, we can obtain various loads on rectangular plates. For instance, taking all coefficients except c_{10} equal to zero, we obtain

$$\sigma_{xx} = 6c_{10}xy, \quad \sigma_{yy} = 0, \quad \sigma_{xy} = -3c_{10}y^2.$$

7.5.6 Saint-Venant's Principle

A boundary value problem of elasticity requires the boundary conditions to be known in the form of stresses or displacements [see Eqs. (7.5.8) and (7.5.9)] at *every point* of the boundary. As shown in Example 7.5.3, the boundary forces are distributed as a function of the distance along the boundary. If the boundary forces are distributed in any other form (other than per unit surface area), the boundary conditions cannot be expressed as point wise quantities.

For example, consider the cantilever beam with an end load, as shown in Fig. 7.5.7. At $x = 0$, where $\hat{\mathbf{n}} = -\hat{\mathbf{e}}_x$, we are required to specify $t_x = -\sigma_{xx}$ and $t_y = -\sigma_{xy}$ (because u_x and u_y are clearly not zero there). There is no problem in stating that $\sigma_{xx}(0, y) = 0$, but σ_{xy} is not known point wise. We can possibly say that the integral of $t_y = -\sigma_{xy}$ over the beam cross section must be equal to P:

$$\int_A t_y(0,y)\,dA = -\int_A \sigma_{xy}(0,y)\,dA = P,$$

which is not equal to specifying σ_{xy} point wise. If we state that $\sigma_{xy}(0,y) = -P/A$, where A is the cross-sectional area of the beam, then we have a inconsistency that σ_{xy} is nonzero from the left face and zero from the bottom and top faces of the beam. Thus, there is a stress singularity at points $(x,y) = (0, \pm h)$. We also have a different type of singularity at points $(x,y) = (L, \pm h)$. Strictly speaking, such problems do not admit exact elasticity solutions. We must overcome such singularities by reformulating the problem as one that admits an engineering solution.

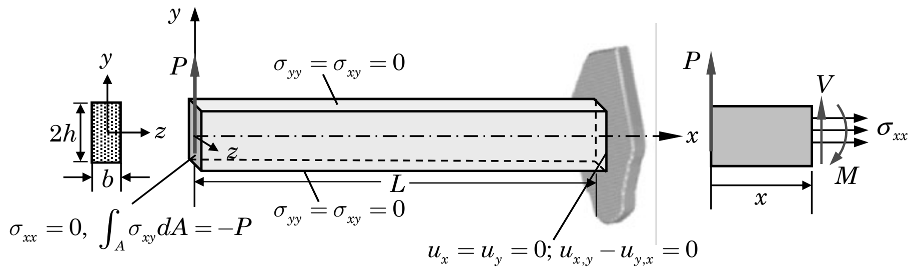

Fig. 7.5.7: A cantilever beam under an end load.

Analytical (approximate) solutions for such problems, when they exist, show that a change in the distribution of the load on the end, without change of the resultant, alters the stress significantly only near the end. *Saint-Venant's principle* says that the effect of the change in the boundary condition from point wise specification to a *statically equivalent* condition (that is, the same net force and moment due to the distributed forces and stresses) is local; that is, the solutions obtained with the two sets of boundary conditions are approximately the same at points sufficiently far from the points where the elasticity boundary conditions are replaced with statically equivalent boundary conditions. Of course, the phrase "sufficiently far" is rather ambiguous. The distance is often taken to be equal to or greater than the length scale of the portion of the boundary where the boundary conditions are replaced. In the case of the beam shown in Fig. 7.5.7, the distance is $2h$ (height of the beam). In the next example, we discuss an engineering solution to the problem shown in Fig. 7.5.7.

Example 7.5.6

Here we consider the problem of a cantilever beam with an end load, as shown in Fig. 7.5.7. The problem can be treated as a plane stress if the beam is of small thickness b compared to the height, $b << h$ (of course, $h << L$ to call it a beam). If the beam is a portion of a very long slab, in the thickness direction, it can be treated as a plane strain problem. Write the boundary conditions and determine the Airy stress function, stresses, and displacements of the problem.

Solution: The boundary conditions are of mixed type (see Fig. 7.5.7): The tractions are specified on the boundaries $x = 0$ and $y = \pm h$, while the displacements are specified on the boundary $x = L$. However, boundary conditions of plane elasticity can be written only on $x = L$ and $y = \pm h$. On $x = 0$, we know only the total force in the y-direction and not the associated stress. Hence, it must be written as an integral condition on stress $\sigma_{xy}(0, y)$. Thus, we have

$$\begin{gathered}\sigma_{xx}(0,y) = 0, \quad \sigma_{xy}(x,-h) = 0, \quad \sigma_{yy}(x,-h) = 0,\\ \sigma_{xy}(x,h) = 0, \quad \sigma_{yy}(x,h) = 0,\end{gathered} \tag{7.5.48}$$

$$u_x(L,y) = 0, \quad u_y(L,y) = 0, \tag{7.5.49}$$

$$b\int_{-h}^{h} \sigma_{xy}(0,y)\, dy = -P. \tag{7.5.50}$$

Due to the boundary condition in Eq. (7.5.50), the resulting boundary value problem is not an exact elasticity problem in the sense that boundary values are not specified point wise. If P is replaced with a shear stress condition $\sigma_{xy}(0, y) = \tau_0$, it is a proper elasticity boundary condition, but even in this case there is a singularity at $x = L$ and $y = h$.

This problem is discussed in most elasticity and continuum mechanics books, despite the fact that it is not a well-posed problem owing to point singularities at the corners of the domain. Therefore, the solution being sought is an approximate solution, which is a reasonable one, by Saint-Venant's principle, away from the isolated points of singularity.

The semi-inverse method allows us to identify the form of the Airy stress function. The knowledge of the stress distributions from the elementary theory of beams provides the needed clue to identify the terms in the Airy stress function. Recall the following stress field from the Euler–Bernoulli beam theory [see Section 7.3.4, Eq. (7.3.31)]:

$$\sigma_{xx} = \frac{M(x)y}{I}, \quad \sigma_{yy} = 0, \quad \sigma_{xy} = \frac{V(x)Q(y)}{Ib}, \tag{7.5.51}$$

where M is the bending moment and V is the shear force [see Eq. (7.5.24)]:

$$M = \int_A y\sigma_{xx}\, dA, \quad V = \int_A \sigma_{xy}\, dA, \tag{7.5.52}$$

I is the moment of inertia about the z-axis, and Q is the first moment of area

$$I = \int_A y^2\, dA = 2bh^3/3, \quad Q(y) = \int_{\bar{A}} y\, dA = b\int_y^h y\, dy. \tag{7.5.53}$$

Here $\bar{A}$ denotes the cross-sectional area between line y and the top of the beam. Clearly, Q is a quadratic function of y. We also note that $M(x)$ is a linear function of x while V is a constant for the problem at hand. Therefore, σ_{xx} is linear in x, σ_{xy} is a quadratic in y, and $\sigma_{yy} = 0$ at $y = \pm h$ for any x (except possibly at $x = L$). Using this qualitative information and definitions (7.5.35), in the absence of body forces (i.e., $V_f = 0$), we take the Airy stress function to be

$$\Phi(x, y) = x(c_1 y + c_2 y^2 + c_3 y^3). \tag{7.5.54}$$

Note that only the first and third terms are dictated by the stress field in a beam. The second term is added to make it a complete quadratic polynomial in y. Also, Φ cannot have terms higher than x because of the boundary condition $\sigma_{yy}(x, \pm h) = 0$. The nonzero stresses are

$$\sigma_{xx} = \frac{\partial^2\Phi}{\partial y^2} = x\left(2c_2 + 6c_3 y\right), \quad \sigma_{yy} = \frac{\partial^2\Phi}{\partial x^2} = 0, \quad \sigma_{xy} = -\frac{\partial^2\Phi}{\partial x\partial y} = -(c_1 + 2c_2 y + 3c_3 y^2). \tag{7.5.55}$$

The choice in (7.5.54) satisfies the biharmonic equation for any values of c_1, c_2, and c_3. We determine the constants c_i using the stress boundary conditions in Eqs. (7.5.48) and (7.5.50). The stress boundary conditions $\sigma_{xx}(0, y) = 0$ and $\sigma_{yy}(x, \pm h) = 0$ are trivially satisfied. We have

$$\sigma_{xy}(x, \pm h) = 0 \;\rightarrow\; c_1 - 2c_2 h + 3c_3 h^2 = 0 \;\text{ and }\; c_1 + 2c_2 h + 3c_3 h^2 = 0,$$

which yield $c_2 = 0$ and $c_1 = -3h^2 c_3$. Lastly, we have

$$b\int_{-h}^{h} \sigma_{xy}(0, y)\, dy = -P \;\rightarrow\; -2hb(c_1 + h^2 c_3) = -P. \tag{7.5.56}$$

Thus, the constants c_i are

$$c_1 = \frac{3P}{4bh}, \quad c_2 = 0, \quad c_3 = -\frac{P}{4bh^3}, \tag{7.5.57}$$

and the Airy stress function becomes

$$\Phi(x, y) = -\frac{Pxy}{6I}\left(y^2 - 3h^2\right). \tag{7.5.58}$$

The stresses from Eq. (7.5.55) are [$I = 2bh^3/3 = Ah^2/3$, where $A = 2bh$ is the area of cross section of the beam]

$$\begin{aligned} \sigma_{xx} &= -\frac{6Pxy}{4bh^3} = -\frac{Pxy}{I}, \quad \sigma_{yy} = 0, \\ \sigma_{xy} &= -\frac{3P}{4bh} + \frac{3Py^2}{4bh^3} = -\frac{P}{2I}\left(h^2 - y^2\right). \end{aligned} \tag{7.5.59}$$

The stresses in Eq. (7.5.59) are exactly those predicted by the classical (i.e., Euler–Bernoulli) beam theory, where $M(x) = -Px$ and $V = -P$. This is not surprising because our choice of terms in the Airy stress function was dictated by the form of the stress field from the classical beam theory. This also indicates that we cannot obtain any better stress field than the elementary beam theory for the boundary conditions (7.5.48)–(7.5.50).

The strain field associated with the stress field in Eq. (7.5.59) is computed using the strain–stress relations in Eq. (6.3.32):

$$\begin{aligned}
\varepsilon_{xx} &= \tfrac{1}{E}\sigma_{xx} = -\tfrac{1}{EI}Pxy,\\
\varepsilon_{yy} &= -\tfrac{\nu}{E}\sigma_{xx} = \tfrac{\nu}{EI}Pxy,\\
\varepsilon_{xy} &= \tfrac{(1+\nu)}{E}\sigma_{xy} = -\tfrac{(1+\nu)}{2EI}P(h^2-y^2),
\end{aligned} \tag{7.5.60}$$

where ν is the Poisson ratio and E is Young's modulus. The strain field in Eq. (7.5.60) is the same as that in Eq. (3.7.12), with $x_1 = x$ and $x_2 = y$. Therefore, the displacements are the same as those determined in Example 3.7.2, namely in Eq. (3.7.22), with $u_1 = u_x$ and $u_2 = u_y$:

$$\begin{aligned}
u_x(x,y) &= \frac{PL^3}{6EI}\frac{y}{L}\left\{3\left[1-\left(\frac{x}{L}\right)^2\right] + (2+\nu)\left(\frac{y}{h}\right)^2\left(\frac{h}{L}\right)^2 - 3(1+\nu)\left(\frac{h}{L}\right)^2\right\},\\
u_y(x,y) &= \frac{PL^3}{6EI}\left\{2 - 3\frac{x}{L}\left[1-\nu\left(\frac{y}{h}\right)^2\left(\frac{h}{L}\right)^2\right] + \frac{x^3}{L^3} + 3(1+\nu)\left(\frac{h}{L}\right)^2\left(1-\frac{x}{L}\right)\right\}.
\end{aligned}$$

As $(h/L)^2 \to 0$, we recover the Euler–Bernoulli beam solution.

Example 7.5.7

Consider a thin rectangular plate of length $2a$, width $2b$, and thickness h that has a circular hole of radius R at the center of the plate. A uniform traction of magnitude σ_0 is applied to the ends of the plate, as shown in Fig. 7.5.8. Determine the stress field in the plate under the assumption that $R << b$.

Solution: The boundary conditions of the problem are

$$\sigma_{xx}(\pm a, y) = \sigma_0,\quad \sigma_{xy}(\pm a, y) = 0,\quad \sigma_{yy}(x, \pm b) = 0,\quad \sigma_{xy}(x,\pm b) = 0, \tag{1}$$

$$\sigma_{rr}(R,\theta) = 0,\quad \sigma_{r\theta}(R,\theta) = 0. \tag{2}$$

Since the hole is assumed to be very small compared to the height of the plate (i.e., $R << b$), we can solve the problem for a stress field inside a circular region of radius $b > c > R$, as shown in Fig. 7.5.8. The stresses at radius c are essentially the same as in the plate without the hole (a consequence of Saint-Venant's principle). We use the cylindrical coordinate system to determine the stress field inside the circle of radius c.

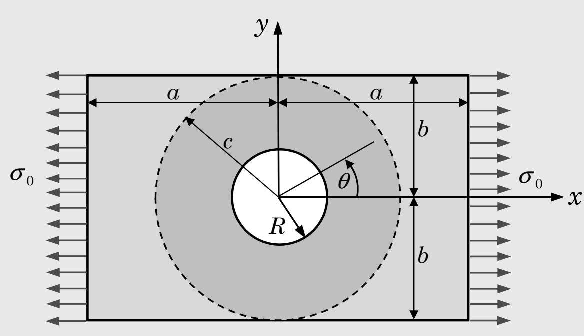

Fig. 7.5.8: A thin rectangular plate with a central hole.

Recall from Eq. (4.3.7) the transformation equations between $(\sigma_{xx}, \sigma_{yy}, \sigma_{xy})$ and $(\sigma_{11} = \sigma_{rr},\ \sigma_{22} = \sigma_{\theta\theta},\ \sigma_{12} = \sigma_{r\theta})$:

$$\begin{aligned}
\sigma_{rr} &= \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + \sigma_{xy}\sin 2\theta,\\
\sigma_{\theta\theta} &= \sigma_{xx}\sin^2\theta + \sigma_{yy}\cos^2\theta - \sigma_{xy}\sin 2\theta,\\
\sigma_{r\theta} &= -\frac{1}{2}\left(\sigma_{xx} - \sigma_{yy}\right)\sin 2\theta + \sigma_{xy}\cos 2\theta.
\end{aligned} \tag{3}$$

Using the transformation equations in Eq. (3), we can write the boundary conditions at $r = c$ for any θ as ($\sigma_{yy} = 0$ and $\sigma_{xy} = 0$):

$$\begin{aligned}
\sigma_{rr}(c,\theta) &= \sigma_0\ \cos^2\theta = \frac{\sigma_0}{2}\big(1 + \cos 2\theta\big),\\
\sigma_{\theta\theta}(c,\theta) &= \sigma_0\ \sin^2\theta = \frac{\sigma_0}{2}\big(1 - \cos 2\theta\big),\\
\sigma_{r\theta}(c,\theta) &= -\frac{\sigma_0}{2}\sin 2\theta.
\end{aligned} \tag{4}$$

The form of the boundary conditions in Eq. (4) suggests that the Airy stress function Φ should be of the form

$$\Phi(r,\theta) = G(r) + F(r)\ \cos 2\theta, \tag{5}$$

with $G(r)$ and $F(r)$ satisfying [because $\nabla^2\nabla^2\Phi = \nabla^2\nabla^2 G(r) + \nabla^2\nabla^2(F\ \cos 2\theta) = 0$ implies that $\tilde{\nabla}^2\tilde{\nabla}^2 G(r) = 0$ and $\hat{\nabla}^2\hat{\nabla}^2 F = 0$]

$$\tilde{\nabla}^2\tilde{\nabla}^2 G = \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)^2 G(r) = 0, \quad \hat{\nabla}^2\hat{\nabla}^2 F = \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{4}{r^2}\right)^2 F(r) = 0. \tag{6}$$

The general solutions to the equations in (6) are of the form

$$F(r) = \frac{c_1}{r^2} + c_2 + c_3 r^2 + c_4 r^4, \quad G(r) = c_5 + c_6\ln r + c_7 r^2 + c_8 r^2\ln r, \tag{7}$$

and we have

$$\begin{aligned}
\frac{dF}{dr} &= -\frac{2c_1}{r^3} + 2c_3 r + 4c_4 r^3, \qquad \frac{d^2F}{dr^2} = \frac{6c_1}{r^4} + 2c_3 + 12c_4 r^2,\\
\frac{dG}{dr} &= \frac{c_6}{r} + 2c_7 r + r c_8(1 + 2\ln r), \quad \frac{d^2G}{dr^2} = -\frac{c_6}{r^2} + 2c_7 + c_8(3 + 2\ln r),\\
\frac{\partial\Phi}{\partial r} &= \left[\frac{c_6}{r} + 2c_7 r + r c_8(1 + 2\ln r)\right] + \left(-\frac{2c_1}{r^3} + 2c_3 r + 4c_4 r^3\right)\cos 2\theta,\\
\frac{\partial^2\Phi}{\partial r^2} &= \left[-\frac{c_6}{r^2} + 2c_7 + c_8(3 + 2\ln r)\right] + \left(\frac{6c_1}{r^4} + 2c_3 + 12c_4 r^2\right)\cos 2\theta,\\
\frac{\partial\Phi}{\partial\theta} &= -2\left(\frac{c_1}{r^2} + c_2 + c_3 r^2 + c_4 r^4\right)\sin 2\theta,\\
\frac{\partial^2\Phi}{\partial\theta^2} &= -4\left(\frac{c_1}{r^2} + c_2 + c_3 r^2 + c_4 r^4\right)\cos 2\theta,\\
\frac{\partial^2\Phi}{\partial\theta\partial r} &= -2\left(-\frac{2c_1}{r^3} + 2c_3 r + 4c_4 r^3\right)\sin 2\theta.
\end{aligned} \tag{8}$$

Substituting the expressions from Eq. (7) into Eq. (5) and using the definition of the stress components, we obtain

$$\begin{aligned}
\sigma_{rr} &= \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} = \frac{c_6}{r^2} + 2c_7 + c_8(1 + 2\ln r) - \left(\frac{6c_1}{r^4} + \frac{4c_2}{r^2} + 2c_3\right)\cos 2\theta,\\
\sigma_{\theta\theta} &= \frac{\partial^2\Phi}{\partial r^2} = -\frac{c_6}{r^2} + 2c_7 + c_8(3 + 2\ln r) + \left(\frac{6c_1}{r^4} + 2c_3 + 12c_4 r^2\right)\cos 2\theta,\\
\sigma_{r\theta} &= -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\right) = \frac{1}{r^2}\frac{\partial\Phi}{\partial\theta} - \frac{1}{r}\frac{\partial^2\Phi}{\partial\theta\partial r} = \left(-\frac{6c_1}{r^4} - \frac{2c_2}{r^2} + 2c_3 + 6c_4 r^2\right)\sin 2\theta.
\end{aligned} \tag{9}$$

Note that c_5 does not enter the calculation of stresses. The boundary conditions in Eqs. (2) and (4) are used to determine the remaining constants. As $r \to \infty$, the expressions for stresses in (8) must approach those in Eq. (4). For this to happen, c_4 and c_8 must be zero and $2c_7 = \sigma_0/2$ and $2c_3 = -\sigma_0/2$. The boundary conditions in Eq. (2) yield the following relations among the remaining constants:

$$\frac{c_6}{R^2} + 2c_7 = 0, \quad \frac{6c_1}{R^4} + \frac{4c_2}{R^2} + 2c_3 = 0, \quad -\frac{6c_1}{R^4} - \frac{2c_2}{R^2} + 2c_3 = 0. \tag{10}$$

Solving these equations, we obtain

$$c_1 = -\frac{\sigma_0 R^4}{4}, \quad c_2 = \frac{\sigma_0 R^2}{2}, \quad c_3 = -\frac{\sigma_0}{4}, \quad c_4 = 0, \quad c_6 = -\frac{\sigma_0 R^2}{2}, \quad c_7 = \frac{\sigma_0}{4}, \quad c_8 = 0. \tag{11}$$

Substituting these values into Eq. (8), we obtain

$$\begin{aligned}
\sigma_{rr} &= \frac{\sigma_0}{2}\left[\left(1 - \frac{R^2}{r^2}\right) + \left(1 + \frac{3R^4}{r^4} - \frac{4R^2}{r^2}\right)\cos 2\theta\right], \\
\sigma_{\theta\theta} &= \frac{\sigma_0}{2}\left[\left(1 + \frac{R^2}{r^2}\right) - \left(1 + \frac{3R^4}{r^4}\right)\cos 2\theta\right], \\
\sigma_{r\theta} &= -\frac{\sigma_0}{2}\left(1 - \frac{3R^4}{r^4} + \frac{2R^2}{r^2}\right)\sin 2\theta.
\end{aligned} \tag{11}$$

The maximum normal stress occurs at $(r,\theta) = (R, \pm 90^\circ)$ and shear stress at $(r,\theta) = (\sqrt{3}R, -45^\circ)$:

$$\sigma_{\max} = \sigma_{\theta\theta}(R, \pm 90^\circ) = 3\sigma_0, \quad \sigma_{r\theta}(\sqrt{3}R, -45^\circ) = \frac{2}{3}\sigma_0. \tag{12}$$

7.5.7 Torsion of Cylindrical Members

The stress function approach used to study a number of plane elasticity problems in the previous sections is also useful in studying torsion of noncircular cylindrical members. However, we cannot use the Airy stress function here because the present problem does not fall into the category of plane elasticity problems. The governing equations for this problem must be developed from basic principles. The problem was first studied by Saint-Venant using the semi-inverse method.

Consider a cylindrical member of noncircular cross-section and length L and subjected to an end torque $\mathbf{T} = T\,\hat{\mathbf{e}}_z$, as shown in Fig. 7.5.9(a). The lateral surface of the cylinder is free of tractions. Saint-Venant studied the problem under the following assumptions:

1. The projection of each cross section onto the xy-plane rotates about the z-axis (taken through the geometric centroid of the cross section) with no in-plane distortion.

2. The amount of rotation of each cross section is proportional to its distance from the end of the cylinder, $\Theta = \theta z$, where Θ is the *twist* and θ is the *twist per unit length.*

3. Each cross section's out-of-plane distortion is the same and its magnitude is proportional to θ.

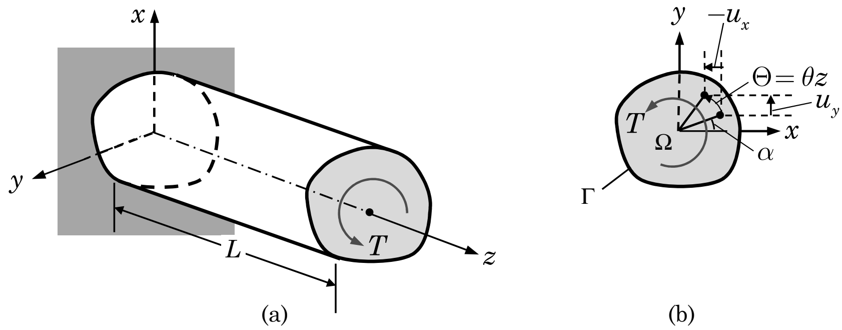

Fig. 7.5.9: (a) Torsion of a cylindrical member. (b) A typical cross section.

In view of the aforementioned assumptions, our attention is focused on a typical cross section of the cylinder; the plane of the cross section is denoted by Ω and its boundary by Γ, as shown in Fig. 7.5.9(b). Our interest is to determine the shear stresses, σ_{xz} and σ_{yz}, produced by the torque, because they are needed in the design of shafts used, for example, in power transmission. There are two different formulations to study the problem. One is based on the *warping function* and the other on *Prandtl stress function*. The details of these two formulations are presented next.

7.5.7.1 Warping function

The displacements of a typical point (r, α) in Ω can be computed as follows [refer to Fig. 7.5.9(b)]:

$$\begin{aligned} u_x &= r\,\cos(\Theta+\alpha) - r\,\cos\alpha = x(\cos\Theta - 1) - y\,\sin\Theta, \\ u_y &= r\,\sin(\Theta+\alpha) - r\,\sin\alpha = x\,\sin\Theta + y(\cos\Theta - 1). \end{aligned} \tag{7.5.61}$$

The third assumption implies that

$$u_z = \theta\psi(x, y), \tag{7.5.62}$$

where ψ denotes the warping function. If $\Theta = \theta\, z$ is very small compared to unity, $\Theta << 1$, the displacement field becomes (because $\cos\Theta \approx 0$ and $\sin\Theta \approx \Theta$)

$$u_x = -\theta yz, \quad u_y = \theta xz, \quad u_z = \theta\,\psi(x, y). \tag{7.5.63}$$

Since we started with an assumed displacement field (a semi-inverse method), we only make sure that the equations of equilibrium are satisfied (and the compatibility equations are automatically met). Toward this end, we compute strains first and then stresses. The linear strain-displacement relations ($\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ give $\varepsilon_{xx} = 0$, $\varepsilon_{yy} = 0$, $\varepsilon_{zz} = 0$, $\varepsilon_{xy} = 0$, and

$$\varepsilon_{xz} = \frac{\theta}{2}\left(\frac{\partial\psi}{\partial x} - y\right), \quad \varepsilon_{yz} = \frac{\theta}{2}\left(\frac{\partial\psi}{\partial y} + x\right). \tag{7.5.64}$$

The stresses are computed using the constitutive equations of an isotropic material, $\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij}$. We find that $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \sigma_{xy} = 0$, and

$$\sigma_{xz} = \mu\theta\left(\frac{\partial\psi}{\partial x} - y\right), \quad \sigma_{yz} = \mu\theta\left(\frac{\partial\psi}{\partial y} + x\right). \tag{7.5.65}$$

Thus, a cross section of the cylinder experiences only the shear stresses σ_{xz} and σ_{yz}; the projected shear traction vector at a point (x, y) of a cross section is $\mathbf{t}(\hat{\mathbf{e}}_z) = \sigma_{xz}\,\hat{\mathbf{e}}_x + \sigma_{yz}\,\hat{\mathbf{e}}_y$.

Assuming that the body forces are negligible, the first two equilibrium equations ($\sigma_{ij,j} = 0$) are trivially satisfied. The third equilibrium equation reduces to

$$\frac{\partial\sigma_{xz}}{\partial x} + \frac{\partial\sigma_{yz}}{\partial y} = 0 \quad\Rightarrow\quad \mu\theta\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right) = 0 \ \text{ in } \ \Omega. \tag{7.5.66}$$

The boundary conditions on the lateral surface of the cylinder, that is, on the boundary Γ, where the unit normal is given by $\hat{\mathbf{n}} = n_x\hat{\mathbf{e}}_x + n_y\hat{\mathbf{e}}_y$, are $t_x = t_y = t_z = 0$. Because all but σ_{xz} and σ_{yz} are zero and $n_z = 0$, the boundary conditions $t_x = t_y = 0$ are trivially satisfied. The remaining boundary conditions $t_z = 0$ yield

$$t_z = \sigma_{xz}n_x + \sigma_{yz}n_y = \mu\theta\left(\frac{\partial\psi}{\partial x} - y\right)n_x + \mu\theta\left(\frac{\partial\psi}{\partial y} + x\right)n_y = 0. \tag{7.5.67}$$

From Fig. 7.5.10, we note that n_x and n_y can be calculated as

$$n_x = \frac{dy}{ds}, \quad n_y = -\frac{dx}{ds}, \quad \hat{\mathbf{n}} = \frac{dy}{ds}\,\hat{\mathbf{e}}_x - \frac{dx}{ds}\,\hat{\mathbf{e}}_y. \tag{7.5.68}$$

Then the boundary condition in Eq. (7.5.67) becomes

$$\left(\frac{\partial\psi}{\partial x} - y\right)\frac{dy}{ds} - \left(\frac{\partial\psi}{\partial y} + x\right)\frac{dx}{ds} = 0 \ \text{ on } \ \Gamma. \tag{7.5.69}$$

Thus, the boundary value problem becomes one of finding ψ such that

$$\nabla^2\psi = 0 \ \text{ in } \ \Omega, \quad \left(\frac{\partial\psi}{\partial x} - y\right)\frac{dy}{ds} - \left(\frac{\partial\psi}{\partial y} + x\right)\frac{dx}{ds} = 0 \ \text{ on } \ \Gamma. \tag{7.5.70}$$

Once $\psi(x, y)$ is known, the shear stresses can be computed from Eq. (7.5.65).

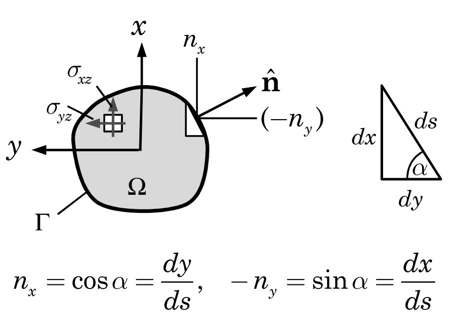

Fig. 7.5.10: Calculation of the direction cosines.

Example 7.5.8

Consider the case in which $\psi = 0$ and use the inverse method to determine the problem (i.e., cross section of the cylinder) for which it corresponds. Also, determine the stresses σ_{xz} and σ_{yz} as well as the projected shear stress magnitude in terms of the shear modulus $\mu = G$ and the applied torque T.

Solution: For $\psi = 0$, $\nabla^2\psi = 0$ is trivially satisfied. The boundary condition in Eq. (7.5.69) becomes

$$y\frac{dy}{ds} + x\frac{dx}{ds} = 0 \quad \rightarrow \quad \frac{d}{ds}\left(x^2 + y^2\right) = 0 \quad \text{or} \quad x^2 + y^2 = \text{constant}, c^2 \tag{1}$$

on the boundary Γ. This equation corresponds to that of a circle with Γ being the boundary of a circle of radius c and Ω being the interior of the circle; that is, the cross section of the cylinder is a circle of radius c.

The stresses are

$$\sigma_{xz} = -\mu\theta y, \quad \sigma_{yz} = \mu\theta x. \tag{2}$$

To express the stresses in terms of the torque, we write the equilibrium of moments about the z-axis. We obtain

$$T = \int_\Omega (x\sigma_{yz} - y\sigma_{xz})\, dx\, dy = \mu\theta \int_\Omega (x^2 + y^2) dx\, dy = \mu\theta c^2 \frac{\pi c^2}{2} = \mu\theta \frac{\pi c^4}{2}, \tag{3}$$

or $\mu\theta = 2T/\pi c^4$. Note that

$$\int_\Omega (x^2 + y^2) dx\, dy \equiv J$$

is the polar moment of inertia. Then the stresses in Eq. (2) can be expressed in terms of T as

$$\sigma_{xz} = -\frac{2T}{\pi c^4} y, \quad \sigma_{yz} = \frac{2T}{\pi c^4} x. \tag{4}$$

The projected shear stress magnitude at any point on the cross section is

$$\tau = |\mathbf{t}(\hat{\mathbf{e}}_z)| = \sqrt{\sigma_{xz}^2 + \sigma_{yz}^2} = \frac{2T}{\pi c^4}\sqrt{(x^2 + y^2)} = \frac{2Tr}{\pi c^4}. \tag{5}$$

Clearly, the maximum shear stress is $\tau_{\max} = \frac{2T}{\pi c^3}$.

The exact solutions of the torsion problem (7.5.70) are possible for elliptical and rectangular cross sections. For geometrically complicated cross sections, one must use numerical methods.

7.5.7.2 Prandtl's stress function

Here we begin with an assumed stress field. We note that the following stresses are identically zero for the problem:

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \sigma_{xy} = 0. \tag{7.5.71}$$

Therefore, only stress equilibrium equation left to be satisfied is

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} = 0. \tag{7.5.72}$$

We choose to satisfy this equation identically by introducing a stress function $\Psi(x, y)$, called the *Prandtl stress function*, such that

$$\sigma_{xz} = \frac{\partial \Psi}{\partial y}, \quad \sigma_{yz} = -\frac{\partial \Psi}{\partial x}. \tag{7.5.73}$$

Since we started with the stress field, the stress function Ψ is subject to satisfying the strain compatibility conditions in Eqs. (3.7.7) and (3.7.8), which can be expressed in terms of the stresses, as given in Eq. (7.2.27). For the case at hand, Eq. (7.2.27) takes the form $\sigma_{3\alpha,\beta\beta} = 0$, for $\alpha, \beta = 1, 2$:

$$\begin{aligned} \frac{\partial^2 \sigma_{xz}}{\partial x^2} + \frac{\partial^2 \sigma_{xz}}{\partial y^2} = 0 \;\; &\Rightarrow \;\; \frac{\partial}{\partial y}\left(\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2}\right) = 0, \\ \frac{\partial^2 \sigma_{yz}}{\partial x^2} + \frac{\partial^2 \sigma_{yz}}{\partial y^2} = 0 \;\; &\Rightarrow \;\; \frac{\partial}{\partial x}\left(\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2}\right) = 0. \end{aligned} \tag{7.5.74}$$

From these two equations it follows that Ψ is governed by the equation

$$\nabla^2 \Psi = c, \tag{7.5.75}$$

where c is a constant. Equation (7.5.75) must be solved subject to the traction-free boundary condition on the lateral surface Γ:

$$\sigma_{xz} n_x + \sigma_{yz} n_y = \frac{\partial \Psi}{\partial y}\frac{dy}{ds} + \frac{\partial \Psi}{\partial x}\frac{dx}{ds} \equiv \frac{d\Psi}{ds} = 0 \;\; \text{on} \;\; \Gamma, \tag{7.5.76}$$

That is, Ψ is a constant, say K, on Γ. For multiply connected cross sections, the constant K on different boundaries, in general, has different values. For simply connected cross sections, we can arbitrarily set the constant to zero, $K =$ because the constant does not contribute to the stress field. In summary, the Prandtl stress function is determined from solving the boundary value problem

$$\nabla^2 \Psi = c \;\; \text{in} \;\; \Omega, \quad \Psi = 0 \;\; \text{on} \;\; \Gamma. \tag{7.5.77}$$

The warping function $\psi(x, y)$ is related to the Prandtl stress function by

$$\frac{\partial \psi}{\partial x} = \frac{1}{\mu\theta}\frac{\partial \Psi}{\partial y} + y, \quad \frac{\partial \psi}{\partial y} = -\frac{1}{\mu\theta}\frac{\partial \Psi}{\partial x} - x. \tag{7.5.78}$$

The two equations in (7.5.78) can be combined by differentiating the first one with respect to y and the second one with respect to x and eliminating ψ to obtain

$$-\nabla^2 \Psi = 2\mu\theta \;\; \text{in} \;\; \Omega, \quad \Psi = 0 \;\; \text{on} \;\; \Gamma. \tag{7.5.79}$$

Once Ψ is known, the stresses can be determined from Eq. (7.5.73).

As in the case of the warping function, exact solutions of the torsion problem (7.5.79) are possible for a few simple cross sections. For geometrically complicated cross sections, one must use numerical methods. In general, solving Eq. (7.5.79) is simpler than solving Eq. (7.5.70) because of the complicated boundary condition in Eq. (7.5.69). To solve Eq. (7.5.79), one assumes Ψ to be in the form $\Psi = Af(x, y)$, where A is a constant and $f(x, y)$ is a sufficiently differentiable (i.e., $\nabla^2 f \neq 0$) function that is identically zero on the boundary. If $-\nabla^2 f$ is a nonzero constant c (so that Ac can be equated to $2\mu\theta$), we solve for the constant A and obtain the complete solution. If $\nabla^2 f$ is not a constant, an exact solution is not possible, although an approximate solution can be obtained. Next we consider an example.

Example 7.5.9

Consider a cylindrical shaft of elliptical cross section, Ω. The boundary Γ is the ellipse with semi-axes a and b:

$$\Gamma = \left\{ (x,y) : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \right\}. \tag{1}$$

Determine the Prandtl stress function and the shear stresses.

Solution: We select Ψ to be

$$\Psi(x,y) = A\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right), \tag{2}$$

where A is a constant to be determined such that Eq. $(7.5.79)_1$ is satisfied. Since the boundary condition $\Psi = 0$ on Γ is satisfied, we substitute Ψ from Eq. (2) into $-\nabla^2\Psi = 2\mu\theta$ and obtain

$$-2A\left(\frac{1}{a^2} + \frac{1}{b^2}\right) = 2\mu\theta \;\Rightarrow\; A = -\frac{\mu\theta a^2 b^2}{a^2+b^2}. \tag{3}$$

The Prandtl stress function is then given by

$$\Psi(x,y) = \frac{\mu\theta a^2 b^2}{a^2+b^2}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right). \tag{4}$$

For solid cylinders of elliptic cross section, the twist per unit length θ can be related to the applied torque T by

$$\begin{aligned} T &= \int_\Omega (x\sigma_{yz} - y\sigma_{xz})\, dx\, dy = -\int_\Omega \left(x\frac{\partial\Psi}{\partial x} + y\frac{\partial\Psi}{\partial y}\right) dx\, dy \\ &= -\int_\Omega \left[\frac{\partial(x\Psi)}{\partial x} + \frac{\partial(y\Psi)}{\partial y}\right] dx\, dy + 2\int_\Omega \Psi(x,y)\; dx\, dy \\ &= -\oint_\Gamma (x\Psi\, dy + y\Psi\, dx) + 2\int_\Omega \Psi(x,y)\; dx\, dy = 2\int_\Omega \Psi(x,y)\; dx\, dy, \end{aligned} \tag{5}$$

where we used the fact that $\Psi = 0$ on Γ of a solid cylinder. For the problem at hand we obtain

$$T = 2\int_\Omega \Psi(x,y)\; dx\, dy = \frac{\pi\mu\theta a^3 b^3}{(a^2+b^2)}. \tag{6}$$

Then the stresses σ_{xz} and σ_{yz} are calculated using Eq. (7.5.73) as

$$\sigma_{xz} = -\frac{2\mu\theta a^2}{a^2+b^2}\, y = -\frac{2T}{\pi a b^3} y, \quad \sigma_{yz} = -\frac{2\mu\theta b^2}{a^2+b^2}\, x = \frac{2T}{\pi a^3 b} x. \tag{7}$$

For $b < a$, the maximum shear stress occurs at $(x,y) = (0, \pm b)$, and the shear stress magnitude is

$$\tau_{\max} = \frac{2\mu\theta a^2 b}{a^2+b^2} = \frac{2T}{\pi a b^4}. \tag{8}$$

For solid circular cylinders, $b = a$, Eqs. (7) and (8) yield the same results as in Example 7.5.7 with $c = a = b$.

The warping function can be determined from Eq. $(7.5.78)_1$ by integrating with respect to x and setting the integration constant to zero:

$$\psi(x,y) = \frac{b^2-a^2}{a^2+b^2}\, xy \;\rightarrow\; u_z = \theta\psi(x,y) = \frac{b^2-a^2}{a^2+b^2}\,\theta xy = -\frac{(a^2-b^2)T}{\mu\pi a^3 b^3}\, xy. \tag{9}$$

7.6 Methods Based on Total Potential Energy

7.6.1 Introduction

In Chapter 5 of this book, laws of physics (or conservation principles) and vector mechanics are used to derive the equations governing continua. These equations, as applied to solid bodies, can also be formulated by means of variational principles. Variational principles have played an important role in solid mechanics. The principle of minimum total potential energy, for example, can be regarded as a substitute for the equations of equilibrium of elastic bodies. Similarly, Hamilton's principle can be used in lieu of the equations governing dynamical systems, and the variational forms presented by Biot replace certain equations in linear continuum thermodynamics.

The use of variational principles makes it possible to concentrate in a single functional all of the intrinsic features of the problem at hand: the governing equations, the boundary conditions, initial conditions, constraint conditions, and even jump conditions. Variational principles can serve to derive not only the governing equations but they also suggest nature of the boundary conditions. Finally, and perhaps most importantly, variational principles provide a natural means for seeking approximate solutions; they are at the heart of the most powerful approximate methods in use in mechanics (e.g., the traditional Ritz and Galerkin methods, and the finite element method). In many cases they can also be used to establish upper and/or lower bounds on approximate solutions.

This section is devoted to the study of the principle of minimum total potential energy and its applications. To keep the scope of the chapter within reasonable limits, only key elements of the principle are presented here. Additional information can be found in the textbook by Reddy (2002).

7.6.2 The Variational Operator

Mathematically speaking, an integral of the form

$$I(u) = \int_{\Omega} F(\mathbf{x}, u, \boldsymbol{\nabla} u)\, d\mathbf{x}$$

whose value is a real number, that is, I is a mapping that transforms functions u from a function space into a real number field, is called a *functional*. Note that $F(\mathbf{x}, u, \boldsymbol{\nabla} u)$ does not qualify as a functional because it is a function and not a real number. An example of a functional is provided by the strain energy U of an elastic body. In particular,

$$U = \frac{EA}{2} \int_0^L \left(\frac{du}{dx}\right)^2 dx$$

is a functional.

As in the case of the minimum of an ordinary function $f(x)$, the minimum of a functional $I(u)$ involves differentiation with respect to the dependent variable(s). The derivative with respect to a dependent variable is known as the *Gâuteax* *derivative,* which is defined as

$$\delta F(u) \equiv \frac{d}{d\epsilon} F(u + \epsilon\, v)\Big|_{\epsilon=0} \tag{7.6.1}$$

and we say that $\delta F(u)$ is the first variation of the function $F(u)$ in the direction of v. The quantity ϵv is denoted as δu, and it is called the *first variation* of u. The operator δ itself is known as the *variational operator*.

The variational operator δ acts much like a total differential operator d, except that it operates with respect to the dependent variable(s) rather than the independent variables, like the coordinate x and time t. Indeed, the laws of *variation* of sums, products, ratios, and powers of functions of a dependent variable u are completely analogous to the corresponding laws of differentiation; that is, the variational calculus (i.e., calculus with δ) resembles the differential calculus. For example, if $F_1 = F_1(u)$ and $F_2 = F_2(u)$ are functions of a dependent variable u, we have

$$\begin{aligned} \delta(F_1 \pm F_2) &= \delta F_1 \pm \delta F_2. \\ \delta(F_1\ F_2) &= \delta F_1\ F_2 + F_1 \delta F_2. \\ \delta\left(\frac{F_1}{F_2}\right) &= \frac{\delta F_1\ F_2 - F_1\ \delta F_2}{F_2^2}. \\ \delta(F_1)^n &= n(F_1)^{n-1}\delta F_1. \end{aligned} \tag{7.6.2}$$

If $G = G(u, v, w)$ is a function of several dependent variables u, v, and w, and possibly their derivatives, the total variation is the sum of partial variations:

$$\delta G = \delta_u G + \delta_v G + \delta_w G, \tag{7.6.3}$$

where, for example, δ_u denotes the partial variation with respect to u. The variational operator can be interchanged with differential and integral operators:

$$\delta(\boldsymbol{\nabla} u) = \boldsymbol{\nabla}(\delta u). \tag{7.6.4}$$

$$\delta\left(\int_\Omega u\, d\mathbf{x}\right) = \int_\Omega \delta u\, d\mathbf{x}. \tag{7.6.5}$$

Equations (7.6.2)–(7.6.5) are valid in multiple dimensions and for functions that depend on more than one dependent variable.

Similar to the necessary and sufficient conditions from the calculus of variations for the minimum of a functional, the conditions for the minimum of a functional are

$$\delta I = 0 \text{ (necessary condition)}, \tag{7.6.7}$$

$$\delta^2 I > 0 \text{ (sufficient condition)}. \tag{7.6.8}$$

When I denotes a certain energy functional in solid mechanics, the necessary condition (7.6.7) yields some associated governing equations, which are equivalent to those derived from the conservation principles of mechanics. However, Eq. (7.6.7) also gives the form of boundary conditions. The equations obtained in Ω from the necessary condition (7.6.7) for equilibrium problems are known as the *Euler equations* (or the *Euler–Lagrange equations* for dynamical systems) and those obtained on Γ (or on a portion of Γ) are known as the *natural boundary conditions*.

The variational principles of solid mechanics can be classified into three categories [see Oden and Reddy (1982) and Reddy (2002)]: (1) variational principles involving (energy) functionals that involve the primary variables such as displacements and temperature are called *primal principles*; (2) variational principles that are based on functionals containing the secondary variables such as stresses and heat flux are called *dual principles*; and (3) variational principles based on functionals that include both primary and secondary variables (e.g., both stresses and displacements, or stresses, strains, and displacements) are called *mixed principles*. In this section we consider the variational principle based on the *total potential energy functional* for linear elastic bodies that contains the displacement field as the dependent variables.

7.6.3 The Principle of the Minimum Total Potential Energy

7.6.3.1 Construction of the total potential energy functional

Recall from Sections 6.2 and 7.5 that for elastic bodies (in the absence of temperature variations) there exists a strain energy density function U_0 (measured per unit volume) such that [see Eq. (6.2.15)]

$$\boldsymbol{\sigma} = \frac{\partial U_0}{\partial \boldsymbol{\varepsilon}} \qquad \left(\sigma_{ij} = \frac{\partial U_0}{\partial \varepsilon_{ij}}\right). \tag{7.6.9}$$

The strain energy density U_0 is a function of strains at a point and is assumed to be positive definite. For linear elastic bodies (that is, obeying the generalized Hooke's law), the strain energy density is given by [see Eq. (6.3.1) or (7.6.9)]

$$U_0 = \frac{1}{2}\boldsymbol{\sigma} : \boldsymbol{\varepsilon} = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} = \frac{1}{2}c_{ijkl}\varepsilon_{ij}\varepsilon_{kl}. \tag{7.6.10}$$

Hence, the total strain energy of the body $\mathcal{B}$ occupying volume Ω is given by

$$U = \int_\Omega U_0(\varepsilon_{ij})\, d\mathbf{x} = \frac{1}{2}\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\varepsilon}\, d\mathbf{x} = \frac{1}{2}\int_\Omega \sigma_{ij}\, \varepsilon_{ij}\, d\mathbf{x}. \tag{7.6.11}$$

The total work done by applied body force $\mathbf{f}$ and surface force $\mathbf{t}$ is given by [see Eq. (7.6.11)]

$$V = -\left[\int_\Omega \mathbf{f}\cdot\mathbf{u}\, d\mathbf{x} + \oint_\Gamma \mathbf{t}\cdot\mathbf{u}\, ds\right], \tag{7.6.12}$$

where the minus sign in the expression for V indicates that the work is expended, whereas U in Eq. (7.6.11) is the available strain energy stored in body $\mathcal{B}$. The total potential energy (functional) of body $\mathcal{B}$ is the sum of the strain energy stored in the body and the work done by external forces

$$\begin{aligned}\Pi = U + V &= \frac{1}{2}\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\varepsilon}\, d\mathbf{x} - \left[\int_\Omega \mathbf{f}\cdot\mathbf{u}\, d\mathbf{x} + \oint_\Gamma \mathbf{t}\cdot\mathbf{u}\, ds\right]\\ &= \frac{1}{2}\int_\Omega \sigma_{ij}\,\varepsilon_{ij}\, d\mathbf{x} - \left[\int_\Omega f_i u_i\, d\mathbf{x} + \oint_\Gamma t_i u_i\, ds\right].\end{aligned} \tag{7.6.13}$$

The *principle of minimum total potential energy* can be stated as follows: If a body is in equilibrium, of all *admissible* displacement fields $\mathbf{u}$ the one $\mathbf{u}_0$ that makes the total potential energy a minimum corresponds to the equilibrium solution:

$$\Pi(\mathbf{u}_0) \leq \Pi(\mathbf{u}). \tag{7.6.14}$$

An admissible displacement is the one that satisfies the geometric constraints of the problem.

7.6.3.2 Euler's equations and natural boundary conditions

Here, we illustrate how the Navier equations of elasticity, Eq. (7.2.17) and the traction boundary conditions in Eq. (7.2.18), can be derived as the Euler equations using the principle of minimum total potential energy. Consider a linear elastic body $\mathcal{B}$ occupying volume Ω with boundary Γ and subjected to body force $\mathbf{f}$ (measured per unit volume) and surface traction $\hat{\mathbf{t}}$ on portion Γ_σ of the surface. We assume that the displacement vector $\mathbf{u}$ is specified to be $\hat{\mathbf{u}}$ on the remaining portion, Γ_u, of the boundary ($\Gamma = \Gamma_u \cup \Gamma_\sigma$). Therefore, $\delta\mathbf{u} = \mathbf{0}$ on Γ_u.

The total potential energy functional is given by (summation on repeated indices is implied throughout this discussion)

$$\Pi(\mathbf{u}) = \int_\Omega \left(\frac{1}{2}\sigma_{ij}\,\varepsilon_{ij} - f_i\,u_i\right) d\mathbf{x} - \int_{\Gamma_\sigma} \hat{t}_i\,u_i\,ds, \tag{7.6.15}$$

The first term under the volume integral represents the strain energy density of the elastic body, the second term represents the work done by the body force $\mathbf{f}$, and the third term represents the work done by the specified traction $\hat{\mathbf{t}}$.

The strain-displacement relations and stress–strain relations for an isotropic elastic body are given by Eqs. (7.2.1) and (7.2.9), respectively. Substituting Eqs. (7.2.1) and (7.2.9) into Eq. (7.6.15), we obtain

$$\begin{aligned}\Pi(\mathbf{u}) = &\int_\Omega \left[\frac{\mu}{4}\left(u_{i,j} + u_{j,i}\right)\left(u_{i,j} + u_{j,i}\right) + \frac{\lambda}{2}u_{i,i}\,u_{k,k} - f_i\,u_i\right] d\mathbf{x}\\ &- \int_{\Gamma_\sigma} \hat{t}_i u_i\,ds.\end{aligned} \tag{7.6.16}$$

Setting the first variation of Π to zero (that is, using the principle of minimum total potential energy), we obtain

$$\begin{aligned}0 = &\int_\Omega \left[\frac{\mu}{2}\left(\delta u_{i,j} + \delta u_{j,i}\right)\left(u_{i,j} + u_{j,i}\right) + \lambda \delta u_{i,i} u_{k,k} - f_i \delta u_i\right] d\mathbf{x}\\ &- \int_{\Gamma_\sigma} \hat{t}_i\,\delta u_i\,ds,\end{aligned} \tag{7.6.17}$$

wherein the product rule of variation is used and similar terms are combined. Next, we use the component form of the gradient theorem to relieve δu_i of any derivative so that we can use the fundamental lemma of variational calculus to set the coefficients of δu_i to zero in Ω and on the portion of Γ where δu_i is arbitrary. Using the gradient theorem, we can write

$$\int_\Omega \delta u_{i,j}\left(u_{i,j}+u_{j,i}\right) d\mathbf{x} = -\int_\Omega \delta u_i \left(u_{i,j}+u_{j,i}\right)_{,j}\, d\mathbf{x} + \oint_\Gamma \delta u_i \left(u_{i,j}+u_{j,i}\right) n_j\, ds,$$

where n_j denotes the jth direction cosine of the unit normal vector to the surface $\hat{\mathbf{n}}$. Using this result in Eq. (7.6.17) we arrive at

$$\begin{aligned} 0 &= \int_\Omega \left[-\frac{\mu}{2}\left(u_{i,j}+u_{j,i}\right)_{,j}\delta u_i - \frac{\mu}{2}\left(u_{i,j}+u_{j,i}\right)_{,i}\delta u_j - \lambda u_{k,ki}\delta u_i - f_i \delta u_i\right] d\mathbf{x} \\ &\quad + \oint_\Gamma \left[\frac{\mu}{2}\left(u_{i,j}+u_{j,i}\right)\left(n_j \delta u_i + n_i \delta u_j\right) + \lambda\, u_{k,k}\, n_i\, \delta u_i\right] ds - \int_{\Gamma_\sigma} \delta u_i \hat{t}_i\, ds \\ &= \int_\Omega \left[-\mu\left(u_{i,j}+u_{j,i}\right)_{,j} - \lambda u_{k,ki} - f_i\right]\delta u_i\, d\mathbf{x} \\ &\quad + \oint_\Gamma \left[\mu\left(u_{i,j}+u_{j,i}\right) + \lambda u_{k,k}\,\delta_{ij}\right] n_j\,\delta u_i\, ds - \int_{\Gamma_\sigma} \delta u_i\, \hat{t}_i\, ds. \end{aligned} \tag{7.6.18}$$

In arriving at the last step, a change of dummy indices is made to combine terms.

Recognizing that the expression inside the square brackets of the closed surface integral is nothing but σ_{ij} and $\sigma_{ij}n_j = t_i$ by Cauchy's formula, we can write

$$\oint_\Gamma \left[\mu\left(u_{i,j}+u_{j,i}\right) + \lambda u_{k,k}\delta_{ij}\right] n_j \delta u_i\, ds = \oint_\Gamma t_i\, \delta u_i\, ds.$$

This boundary expression resulting from the "integration-by-parts" to relieve $\delta\mathbf{u}$ of any derivatives is used to classify the variables of the problem. The coefficient of δu_i is called the *secondary variable*, and the varied quantity itself (without the variational symbol) is called the *primary variable*. Thus, u_i is the primary variable and t_i is the corresponding secondary variable. They always appear in pairs, and only one element of the pair may be specified at any boundary point. Specification of a primary variable is called the *essential boundary condition* and specification of a secondary variable is termed the *natural boundary condition*. They are also known as the geometric and force boundary conditions, respectively. In applied mathematics, they are known as the *Dirichlet boundary condition* and the *Neumann boundary condition*, respectively.

Returning to the boundary integral, it can be expressed as the sum of integrals on Γ_u and Γ_σ:

$$\oint_\Gamma t_i \delta u_i\, ds = \int_{\Gamma_u} t_i \delta u_i\, ds + \int_{\Gamma_\sigma} t_i \delta u_i\, ds = \int_{\Gamma_\sigma} t_i \delta u_i\, ds.$$

The integral over Γ_u is set to zero because $\mathbf{u}$ is specified there, that is, $\delta\mathbf{u} = \mathbf{0}$. Hence, Eq. (7.6.18) becomes

$$0 = \int_\Omega \left[-\mu\left(u_{i,j}+u_{j,i}\right)_{,j} - \lambda u_{k,ki} - f_i\right]\delta u_i\, d\mathbf{x} + \int_{\Gamma_\sigma} \delta u_i \left(t_i - \hat{t}_i\right) ds. \tag{7.6.19}$$

Using the fundamental lemma of calculus of variations, we set the coefficients of δu_i in Ω and δu_i on Γ_σ from Eq. (7.6.19) to zero separately and obtain

$$\mu\, u_{i,jj} + (\mu + \lambda)u_{k,ki} + f_i = 0 \text{ in } \Omega, \tag{7.6.20}$$

$$n_j\sigma_{ij} - \hat{t}_i = 0 \text{ on } \Gamma_\sigma, \tag{7.6.21}$$

for $i = 1, 2, 3$. Equation (7.6.20) represents the Navier equations of elasticity (7.2.17), and the natural boundary conditions (7.6.21) are the same as the traction boundary conditions listed in Eq.(7.2.18).

7.6.3.3 Minimum property of the total potential energy functional

To show that the total potential energy of a linear elasticity body is indeed the minimum at its equilibrium configuration, we consider the total potential energy functional [more general than the one considered in Eq. (7.6.14)]:

$$\Pi(\mathbf{u}) = \int_\Omega \left(\tfrac{1}{2}C_{ijk\ell}\,\varepsilon_{k\ell}\,\varepsilon_{ij} - f_i\,u_i\right)d\mathbf{x} - \int_{\Gamma_\sigma} \hat{t}_i\,u_i\,ds, \tag{7.6.22}$$

where $C_{ijk\ell}$ are the components of the fourth-order elasticity tensor.

Let $\mathbf{u}$ be the true displacement field and $\bar{\mathbf{u}}$ be an arbitrary but admissible displacement field. Then $\bar{\mathbf{u}}$ is of the form

$$\bar{\mathbf{u}} = \mathbf{u} + \alpha\mathbf{v},$$

where α is a real number and $\mathbf{v}$ is a sufficiently differentiable function that satisfies the homogeneous form of the essential boundary condition $\mathbf{v} = \mathbf{0}$ on Γ_u. Then $\Pi(\bar{\mathbf{u}})$ is given by

$$\Pi(\mathbf{u} + \alpha\mathbf{v}) = \int_\Omega \left[\tfrac{1}{2}C_{ijk\ell}\left(\varepsilon_{k\ell} + \alpha g_{k\ell}\right)\left(\varepsilon_{ij} + \alpha g_{ij}\right) - f_i\left(u_i + \alpha v_i\right)\right]d\mathbf{x}$$
$$- \int_{\Gamma_\sigma} \hat{t}_i(u_i + \alpha v_i)ds,$$

where

$$g_{ij} = \tfrac{1}{2}(v_{i,j} + v_{j,i}).$$

Collecting the terms, we obtain (because $C_{ijk\ell} = C_{k\ell ij}$)

$$\Pi(\bar{\mathbf{u}}) = \Pi(\mathbf{u}) + \alpha\left[\int_\Omega \left(-f_i v_i + C_{ijk\ell}\varepsilon_{k\ell}g_{ij} + \tfrac{1}{2}\alpha\, C_{ijk\ell}g_{ij}g_{k\ell}\right)d\mathbf{x} - \int_{\Gamma_\sigma} \hat{t}_i v_i\,ds\right], \tag{7.6.23}$$

Using the equilibrium equations (7.2.5) and the generalized Hooke's law $\sigma_{ij} = C_{ijk\ell}\varepsilon_{k\ell}$ we can write

$$-\int_\Omega f_i v_i\,d\mathbf{x} = \int_\Omega \sigma_{ij,j}v_i\,d\mathbf{x} = \int_\Omega C_{ijk\ell}\,\varepsilon_{k\ell,j}\,v_i\,d\mathbf{x}$$
$$= -\int_\Omega C_{ijk\ell}\varepsilon_{k\ell}v_{i,j}\,d\mathbf{x} + \int_{\Gamma_\sigma} C_{ijk\ell}\,\varepsilon_{k\ell}\,v_i\,n_j\,ds$$
$$= -\int_\Omega C_{ijk\ell}\,\varepsilon_{k\ell}\,g_{ij}\,d\mathbf{x} + \int_{\Gamma_\sigma} \hat{t}_i v_i\,ds, \tag{7.6.24}$$

where the condition $v_i = 0$ on Γ_u is used in arriving at the last step. Substituting Eq. (7.6.24) into Eq. (7.6.23), we arrive at

$$\Pi(\bar{\mathbf{u}}) = \Pi(\mathbf{u}) + \frac{\alpha^2}{2}\int_\Omega C_{ijk\ell}\, g_{ij}\, g_{k\ell}\, d\mathbf{x}. \tag{7.6.25}$$

In view of the nonnegative nature of the second term on the right-hand side of Eq. (7.6.23), it follows that

$$\Pi(\bar{\mathbf{u}}) \geq \Pi(\mathbf{u}), \tag{7.6.26}$$

and $\Pi(\bar{\mathbf{u}}) = \Pi(\mathbf{u})$ only if the quadratic expression $\frac{1}{2}C_{ijk\ell}g_{ij}g_{k\ell}$ is zero. Owing to the positive-definiteness of the strain energy density, the quadratic expression is zero only if $v_i = 0$, which in turn implies $\bar{u}_i = u_i$. Thus, Eq. (7.6.26) implies that of all admissible displacement fields the body can assume, the true one is that which makes the total potential energy a minimum. Next, we consider an example to illustrate the use of the principle of minimum total potential energy.

Example 7.6.1

Consider the bending of a beam according to the Euler–Bernoulli beam theory (see Section 7.3.4). Construct the total potential energy functional and then determine the governing equation and boundary conditions of the problem.

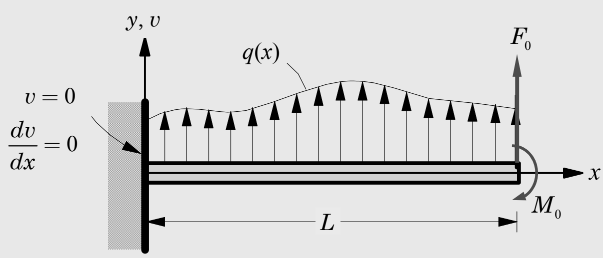

Fig. 7.6.1: A beam with applied loads.

Solution: The total potential energy of a cantilever beam under pure bending by distributed transverse force $q(x)$ and point load F_0 (see Fig. 7.6.1) with the assumption of small strains and displacements for the linear elastic case (i.e., obeys Hooke's law) is given by

$$\Pi(v) = \frac{1}{2}\int_0^L \left[EI\left(\frac{d^2v}{dx^2}\right)^2\right] dx - \left[\int_0^L q(x)v(x)\,dx + F_0 v(L) + M_0\left(-\frac{dv}{dx}\right)_{x=L}\right], \tag{7.6.27}$$

where L is the length, A the cross-sectional area, I moment of inertia about the axis (y) of bending, and E is Young's modulus of the beam. The first term represents the strain energy U (see Example 7.4.3); the second term represents the work done by the applied distributed load $q(x)$ in moving through the deflection $v(x)$; the third terms represents the work done by the point load F_0 in moving through the displacement $v(L)$; and the last term represents the work done by moment M_0 in moving through the rotation $\theta_x(L) = (-\frac{dv}{dx})_{x=L}$.

Applying the principle of minimum total potential energy, $\delta\Pi = 0$, we obtain

$$0 = \delta\Pi = \int_0^L EI\frac{d^2v}{dx^2}\frac{d^2\delta v}{dx^2}\,dx - \left[\int_0^L q\delta v\,dx + F_0\delta v(L) + M_0\left(-\frac{d\delta v}{dx}\right)_{x=L}\right]. \tag{1}$$

Next, we carry out integration-by-parts on the first term to relieve δv of any derivative so that we can use the fundamental lemma of variational calculus to obtain the Euler equation:

$$
\begin{aligned}
0 = &\int_0^L \frac{d^2}{dx^2}\left(EI\frac{d^2v}{dx^2}\right)\delta v\, dx + \left[EI\frac{d^2v}{dx^2}\frac{d\delta v}{dx} - \frac{d}{dx}\left(EI\frac{d^2v}{dx^2}\right)\delta v\right]_0^L \\
&- \left[\int_0^L q\delta v\, dx + F_0\delta v(L) + M_0\left(-\frac{d\delta v}{dx}\right)_{x=L}\right]. \qquad (2)
\end{aligned}
$$

The boundary terms resulting from integration-by-parts allows us to classify the boundary conditions of the problem. The quantities with δ, δv, and $\delta(dv/dx)$ in the boundary terms, indicate that v and (dv/dx) (removing the variational operator from the quantities) are the quantities whose specification constrains the beam geometrically. These variables are called the *primary variables*:

$$
v\,; \qquad \frac{dv}{dx}\,. \tag{7.6.28}
$$

Thus, the deflection v and slope (or rotation) dv/dx are the primary variables of the problem. The expressions that are coefficients of δv and $\delta(dv/dx)$ in the boundary terms are called the *secondary variables*:

$$
\delta w: \quad \frac{d}{dx}\left(EI\frac{d^2v}{dx^2}\right)\,; \quad \delta\left(\frac{dv}{dx}\right): \quad EI\frac{d^2v}{dx^2}\,. \tag{7.6.29}
$$

It is clear that the secondary variables are nothing but the shear force $V(x) = dM/dx$ and bending moment $M(x)$

$$
V(x) = -\frac{d}{dx}\left(EI\frac{d^2v}{dx^2}\right)\,; \quad M(x) = -EI\frac{d^2v}{dx^2}\,. \tag{7.6.30}
$$

Only one element of each of the pairs (v, V) and $(dv/dx, M)$ may be specified at a point. Note that the identification of the primary and secondary variables is unique. Specifying a primary variable constitutes a *geometric* or *essential* boundary condition, and specification of a secondary variable constitutes a *force* or *natural* boundary condition.

Returning to the expression in Eq. (2), first we collect the coefficients of δv in $(0, L)$ together and set them to zero, because δv is arbitrary in $(0, L)$. We obtain the Euler equation

$$
\frac{d^2}{dx^2}\left(EI\frac{d^2v}{dx^2}\right) - q(x) = 0, \quad 0 < x < L. \tag{7.6.31}
$$

Equation (7.6.31) can also be derived from vector mechanics by considering an element of the beam and summing the forces and moments, and then relating the bending moment M to the deflection v, as discussed in Section 7.3.4.

Now considering all boundary terms in Eq. (2), we conclude that

$$
\left[\frac{d}{dx}\left(EI\frac{d^2v}{dx^2}\right)\right]_{x=0}\delta v(0) = 0, \quad \left[-\frac{d}{dx}\left(EI\frac{d^2v}{dx^2}\right) - F_0\right]_{x=L}\delta v(L) = 0, \tag{3}
$$

$$
\left(EI\frac{d^2v}{dx^2}\right)_{x=0}\left(\frac{d\delta v}{dx}\right)_{x=0} = 0, \quad \left(EI\frac{d^2v}{dx^2} + M_0\right)_{x=L}\left(\frac{d\delta v}{dx}\right)_{x=L} = 0. \tag{4}
$$

If either of the quantities δw and $(d\delta v/dx)$ is zero at $x = 0$ or $x = L$, if v or (dv/dx) is specified, the corresponding variations vanish because a specified quantity cannot be varied; the vanishing of the coefficients of δv and $(d\delta v/dx)$ at points where the geometric boundary conditions are not specified provides the natural boundary conditions. Various combinations of one variable from each of the pairs (v, V) and (θ_x, M), where $\theta_x = -(dv/dx)$, define beams with different boundary conditions.

As an example, suppose that the beam is clamped (i.e., fixed or built-in) at $x = 0$ and free at $x = L$ (a cantilever beam), as shown in Fig. 7.6.1. Then, $\delta v(0) = 0$ and $(d\delta v(0)/dx) = 0$, and the corresponding secondary variables, namely the shear force and bending moment, are unknown there. Since the free end, $x = L$, is subjected to an upward transverse force F_0 and clockwise bending moment M_0, the force or natural boundary conditions become

$$\left[-\frac{d}{dx}\left(EI\frac{d^2v}{dx^2}\right) - F_0\right]_{x=L} = 0, \quad \left(EI\frac{d^2v}{dx^2} + M_0\right)_{x=L} = 0. \tag{5}$$

Since the secondary variables are known at $x = L$, we will not know the corresponding primary variables until the problem is solved. Another example is provided by a simply supported (or hinged at both ends) beam without any applied moments at the supports. Then we have the following boundary conditions:

$$v(0) = 0, \quad \left(EI\frac{d^2v}{dx^2}\right)_{x=0} = 0; \quad v(L) = 0, \quad \left(EI\frac{d^2v}{dx^2}\right)_{x=L} = 0. \tag{6}$$

7.6.4 Castigliano's theorem I

Suppose that the displacement field of a solid body can be expressed in terms of the displacements of a finite number of points $\mathbf{x}_i$ ($i = 1, 2, \cdots N$) as

$$\mathbf{u}(\mathbf{x}) = \sum_{i=1}^{N} \mathbf{u}_i \, \phi_i(\mathbf{x}), \tag{7.6.35}$$

where $\mathbf{u}_i$ are unknown displacement parameters, called *generalized displacements*, and ϕ_i are known functions of position, called *interpolation functions* with the property that ϕ_i is unity at the ith point (i.e., $\mathbf{x} = \mathbf{x}_i$) and zero at all other points ($\mathbf{x}_j$, $j \neq i$). Then it is possible to represent the strain energy U and potential energy V due to applied loads in terms of the generalized displacements $\mathbf{u}_i$. Then the principle of minimum total potential energy can be written as

$$\delta\Pi = \delta U + \delta V = 0 \;\Rightarrow\; \delta U = -\delta V \quad \text{or} \quad \frac{\partial U}{\partial \mathbf{u}_i} \cdot \delta\mathbf{u}_i = -\frac{\partial V}{\partial \mathbf{u}_i} \cdot \delta\mathbf{u}_i, \tag{7.6.36}$$

where sum on repeated indices is implied. Since

$$\frac{\partial V}{\partial \mathbf{u}_i} = -\mathbf{F}_i$$

and $\delta\mathbf{u}_i$ are arbitrary, it follows that

$$\left(\frac{\partial U}{\partial \mathbf{u}_i} - \mathbf{F}_i\right) \cdot \delta\mathbf{u}_i = 0 \quad \text{or} \quad \frac{\partial U}{\partial \mathbf{u}_i} = \mathbf{F}_i. \tag{7.6.37}$$

Equation (7.6.37) is known as Castigliano's theorem I.

When applied to a structure loaded by generalized point loads $\mathbf{F}_i$ with associated generalized displacements $\mathbf{u}_i$, both having the same sense, Castigliano's theorem I gives

$$\frac{\partial U}{\partial \mathbf{u}_i} = \mathbf{F}_i. \tag{7.6.38}$$

It is clear from the derivation that Castigliano's theorem I is a special case of the principle of minimum total potential energy.

Application of Castigliano's theorem I to structural members (trusses and frames) can be found in many books [see Reddy (2002) and references therein]. In Example 7.6.2, an application of Castigliano's theorem I to beams is illustrated.

Example 7.6.2

Consider a straight beam of length L and constant bending stiffness EI (modulus E and moment of inertia I about the axis of bending y). If Δ_i are the generalized displacements and Q_i are the generalized point loads at the ends of the beam segment, as shown in Fig. 7.6.2, use Castigliano's theorem I to establish a relationship between the generalized displacements and generalized forces.

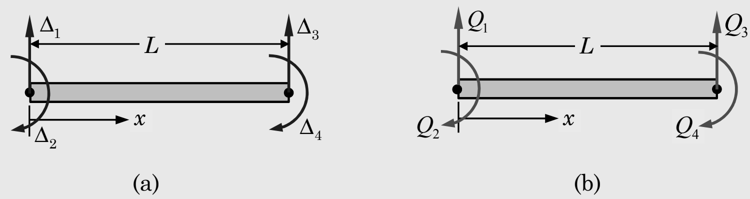

Fig. 7.6.2: (a) Generalized displacements. (b) Generalized forces.

Solution: The equilibrium equation of a beam segment according to the Euler–Bernoulli beam theory (see Example 7.6.1) is

$$EI\frac{d^4v}{dx^4} = 0. \tag{7.6.39}$$

The exact solution to this fourth-order equation is

$$v(x) = c_1 + c_2x + c_3x^2 + c_4x^3, \tag{7.6.40}$$

where c_i $(i = 1, 2, 3, 4)$ are constants of integration, which we express in terms of the deflections and rotations at the two ends of an element beam of length L. Let

$$\begin{aligned}
&\Delta_1 \equiv v(0) = c_1, \quad \Delta_2 \equiv \left(-\frac{dv}{dx}\right)_{x=0} = -c_2,\\
&\Delta_3 \equiv v(L) = c_1 + c_2L + c_3L^2 + c_4L^3,\\
&\Delta_4 \equiv \left(-\frac{dv}{dx}\right)_{x=L} = -c_2 - 2c_3L - 3c_4L^2.
\end{aligned} \tag{7.6.41}$$

Clearly, Δ_1 and Δ_3 are the values of the transverse deflection v at $x = 0$ and $x = L$, respectively, and Δ_2 and Δ_4 are the rotations $-dv/dx$, measured positive clockwise, at $x = 0$ and $x = L$, respectively; see Fig. 7.6.2(b).

The reason for picking two deflection values and two rotations, as opposed to four deflections at four points of the beam, needs to be understood. From Example 7.6.1, it is clear that both v and dv/dx are the primary (kinematic) variables, which must be continuous at every point of the beam. If we were to join two such beam segments (possibly made of different bending stiffness EI), the kinematic variables can be made continuous by equating the like degrees of freedom at the point common to the two segments.

The four equations in Eq. (7.6.41) can be solved for c_i in terms of Δ_i, called *generalized displacements*, which will serve as the generalized coordinates for the application of Castigliano's theorem I. Substituting the result into Eq. (7.6.40) yields

$$v(x) = \phi_1(x)\Delta_1 + \phi_2(x)\Delta_2 + \phi_3(x)\Delta_3 + \phi_4(x)\Delta_4 = \sum_{i=1}^{4} \phi_i(x)\Delta_i, \tag{7.6.42}$$

where $\phi_i(x)$ $(i = 1, 2, 3, 4)$ are known as the *Hermite cubic polynomials*

$$\begin{aligned} \phi_1(x) &= 1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3, \qquad \phi_3(x) = \left(\frac{x}{L}\right)^2\left(3 - 2\frac{x}{L}\right), \\ \phi_2(x) &= -x\left[1 - 2\left(\frac{x}{L}\right) + \left(\frac{x}{L}\right)^2\right], \quad \phi_4(x) = x\frac{x}{L}\left(1 - \frac{x}{L}\right). \end{aligned} \tag{7.6.43}$$

We note that Eq. (7.6.42) has the same form as Eq. (7.6.35).

The strain energy of the beam now can be expressed in terms of the generalized coordinates Δ_i $(i = 1, 2, 3, 4)$ as

$$\begin{aligned} U &= \frac{EI}{2}\int_0^L \left(\frac{d^2v}{dx^2}\right)^2 dx = \frac{EI}{2}\int_0^L \left(\sum_{i=1}^4 \Delta_i \frac{d^2\phi_i}{dx^2}\right)\left(\sum_{j=1}^4 \Delta_j \frac{d^2\phi_j}{dx^2}\right) dx \\ &= \frac{1}{2}\sum_{i=1}^4\sum_{j=1}^4 K_{ij}\Delta_i\Delta_j = \tfrac{1}{2}\{\Delta\}^{\mathrm{T}}[K]\{\Delta\} \end{aligned} \tag{7.6.44}$$

where $[K]$ is known as the stiffness matrix

$$K_{ij} = EI\int_0^L \frac{d^2\phi_i}{dx^2}\frac{d^2\phi_j}{dx^2}\, dx. \tag{7.6.45}$$

Note that $[K]$ is symmetric $(K_{ij} = K_{ji})$. By carrying out the indicated integration, K_{ij} can be evaluated, as will be shown shortly.

Although we assumed that there is no distributed transverse load on the beam, as per Eq. (7.6.39), if there were a distributed load $q(x)$, acting upward, it can be converted to statically equivalent generalized point loads at the end points of the beam segment by

$$q_i = \int_0^L q(x)\phi_i(x)\, dx, \quad i = 1, 2, 3, 4. \tag{7.6.46}$$

The transverse point loads q_1 and q_3 and bending moments q_2 and q_4 together are statically equivalent (that is, satisfy the force and moment equilibrium conditions of the beam) to the distributed load $q(x)$ on the beam, as shown in Fig. 7.6.3(a). We distinguish between q_i and Q_i, because the latter are generalized point loads that are not due to the distributed load, $q(x)$; Q_i are the reactions at the ends of the beam, as shown in Fig. 7.6.3(b). The work done by external loads is

$$V = -\sum_{i=1}^4 (q_i + Q_i)\Delta_i. \tag{7.6.47}$$

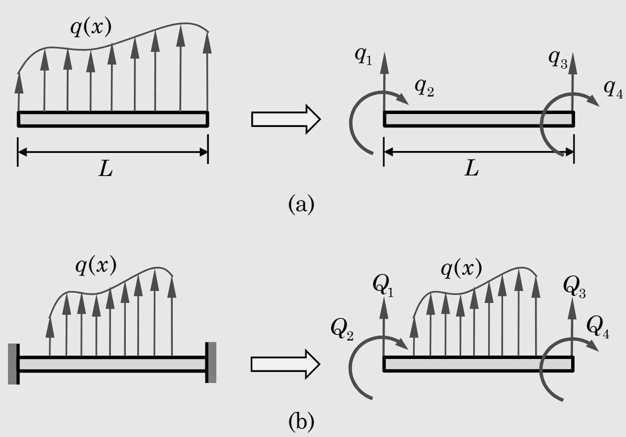

Fig. 7.6.3: (a) Statically equivalent generalized loads q_i due to distributed load $q(x)$ and (b) Generalized reaction forces.

Using Castigliano's theorem I, we obtain the required relations between the generalized displacements $\{\Delta\}$ and generalized forces $\{Q\}$

$$\frac{\partial U}{\partial \Delta_i} = -\frac{\partial V}{\partial \Delta_i} \quad \Rightarrow \quad \sum_{j=1}^{4} K_{ij}\Delta_j = Q_i + q_i \;\text{ or }\; [K]\{\Delta\} = \{q\} + \{Q\},$$

or, in explicit matrix form

$$\frac{2EI}{L^3}\begin{bmatrix} 6 & -3L & -6 & -3L \\ -3L & 2L^2 & 3L & L^2 \\ -6 & 3L & 6 & 3L \\ -3L & L^2 & 3L & 2L^2 \end{bmatrix}\begin{Bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \Delta_4 \end{Bmatrix} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} + \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix}. \tag{7.6.48}$$

It can be verified that the stiffness matrix $[K]$ is singular because the rigid-body motions (that is, rigid-body translation and rotation) of the beam segment are not eliminated.

Example 7.6.3

Consider a beam fixed at $x = 0$ (this geometric condition eliminates the rigid-body motion), supported at $x = L$ by a linear elastic spring with spring constant k, subjected to uniformly distributed load of intensity q_0, and clockwise bending moment M_0 at $x = L$, as shown in Fig. 7.6.4. Determine the elongation $w(L)$ in the spring.

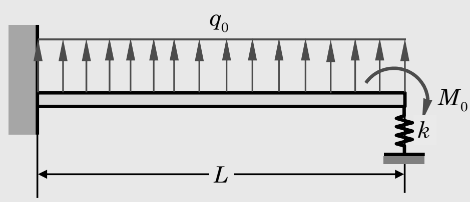

Fig. 7.6.4: A beam fixed at $x = 0$ and supported by a spring at $x = L$.

Solution: The geometric boundary conditions at $x = 0$ require that $\Delta_1 = \Delta_2 = 0$. These conditions remove the rigid-body modes of vertical translation and rotation about the y-axis. The force boundary conditions at $x = L$ require $Q_3 = -F_s = -kv(L) = -k\Delta_3$ and $Q_4 = M_0$. For uniformly distributed load acting upward, $q(x) = q_0$, the load vector $\{q\}$ is given by

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \frac{q_0 L}{12}\begin{Bmatrix} 6 \\ -L \\ 6 \\ L \end{Bmatrix}. \tag{7.6.49}$$

Then we have

$$\frac{2EI}{L^3}\begin{bmatrix} 6 & -3L & -6 & -3L \\ -3L & 2L^2 & 3L & L^2 \\ -6 & 3L & 6 & 3L \\ -3L & L^2 & 3L & 2L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ \Delta_3 \\ \Delta_4 \end{Bmatrix} = \frac{q_0 L}{12}\begin{Bmatrix} 6 \\ -L \\ 6 \\ L \end{Bmatrix} + \begin{Bmatrix} Q_1 \\ Q_2 \\ -k\Delta_3 \\ M_0 \end{Bmatrix}. \tag{7.6.50}$$

Thus, there are four equations in four unknowns, Q_1, Q_2, Δ_3, and Δ_4. Since the last two equations contain Δ_3 and Δ_4 as the only unknowns, we can write

$$\begin{bmatrix} \frac{12EI}{L^3} + k & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}\begin{Bmatrix} \Delta_3 \\ \Delta_4 \end{Bmatrix} = \frac{q_0 L}{12}\begin{Bmatrix} 6 \\ L \end{Bmatrix} + \begin{Bmatrix} 0 \\ M_0 \end{Bmatrix}. \tag{7.6.51}$$

Solving for $\Delta_3 = v(L)$ and $\Delta_4 = -(dv/dx)(L)$, we obtain

$$\begin{aligned}\Delta_3 &= \left(q_0 L^2 - 4M_0\right) \frac{3L^2}{8EI\,(3+\alpha)}, \quad \alpha = \frac{kL^3}{EI}, \\ \Delta_4 &= -\frac{q_0 L^3}{48EI} \frac{(24-\alpha)}{(3+\alpha)} + \frac{M_0 L}{4EI} \frac{(12+\alpha)}{(3+\alpha)}.\end{aligned} \tag{7.6.52}$$

The reactions at the fixed end can be determined using the first two equations in Eq. (7.6.50):

$$\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \frac{2EI}{L^3} \begin{bmatrix} -6 & -3L \\ 3L & L^2 \end{bmatrix} \begin{Bmatrix} \Delta_3 \\ \Delta_4 \end{Bmatrix} - \frac{q_0 L}{12} \begin{Bmatrix} 6 \\ -L \end{Bmatrix}. \tag{7.6.53}$$

The solution obtained in Eqs. (7.6.52) and (7.6.53) is exact because the representation in Eq. (7.6.42) is the exact solution of Eq. (7.6.31) when EI is a constant and the distributed load $q(x)$ is replaced by statically equivalent point forces and moments.

7.6.5 The Ritz Method

7.6.5.1 The variational problem

The Ritz method, named after German engineer W. Ritz (1878–1909), is a numerical method of solving problems posed in terms of solving the *variational problem* of the type: find the function the vector function $u(x)$ from a suitable space $\mathcal{U}$ of functions such that

$$B(u, v) = L(v) \quad \text{for all } v \text{ from } \mathcal{U}. \tag{7.6.54}$$

where $B(u, v)$ is called a *bilinear form* and $L(v)$ is called a *linear form*, with the properties

$$\begin{aligned} B(\alpha u_1 + \beta u_2, v) &= \alpha B(u_1, v) + \beta B(u_2, v) \text{ (linearity in the first argument)} \\ B(u, \alpha v_1 + \beta v_2) &= \alpha B(u, v_1) + \beta B(u, v_2) \text{ (linearity in the second argument)} \\ L(\alpha v_1 + \beta v_2) &= \alpha L(v_1) + \beta L(v_2), \end{aligned} \tag{7.6.55}$$

for any real numbers α and β and dependent variables u, u_1, u_2, v, v_1, and v_2. The bilinear form is said to be symmetric if $B(u, v) = B(v, u)$ (that is, u and v can be interchanged without changing the value of B).

Some space and mathematical concepts from functional analysis are required to formally introduce the properties of the space $\mathcal{U}$, though these would distract the reader from the focus of the book. However, it suffices to say that the space $\mathcal{U}$ possesses properties of an inner product space, that is, functions u from $\mathcal{U}$ are sufficiently differentiable to a certain order as dictated by the functional $I(u)$, u and its various derivatives are square-integrable in the sense that

$$\int_\Omega |u(x)|^2\, dx < \infty, \quad \int_\Omega |\nabla u(x)|^2\, dx < \infty, \quad \text{and so on,}$$

and an inner product can be defined between any two elements u and v of the space $\mathcal{U}$.

Whenever $B(\cdot,\cdot)$ is bilinear and symmetric and $L(\cdot)$ is linear, a quadratic functional can be defined [see Reddy (2002)]:

$$I(u) = \tfrac{1}{2}B(u,u) - L(u), \tag{7.6.56}$$

such that $\delta I = 0$ gives the variational problem in Eq. (7.6.54):

$$0 = \delta I(u) = \tfrac{1}{2}\left[B(\delta u, u) + B(u, \delta u)\right] - L(\delta u) = B(\delta u, u) - L(\delta u),$$

which is the same as Eq. (7.6.54) with $\delta u = v$.

As an example of $I(u)$ in linearized elasticity, we consider the axial deformation of a uniform bar with an end spring. The governing equation is

$$-\frac{d}{dx}\left(EA\frac{du}{dx}\right) = f(x), \quad 0 < x < L, \tag{7.6.57}$$

and the boundary conditions are

$$u(0) = 0, \quad \left[\left(EA\frac{du}{dx}\right) + ku(x)\right]_{x=L} = P, \tag{7.6.58}$$

where $E = E(x)$ is Young's modulus, $A = A(x)$ is the cross-sectional area, L is the length, k is the spring constant, $f(x)$ is the distributed axial load, and P is the axial load at $x = L$, as shown in Fig. 7.6.5. Equations (7.6.57) and (7.6.58) are equivalent to minimizing the total potential energy functional $I(u) = \Pi(u)$:

$$\begin{aligned}\Pi(u) &= \int_0^L \frac{EA}{2}\left(\frac{du}{dx}\right)^2 dx + \frac{k}{2}\left[u(L)\right]^2 - \left[\int_0^L f u\, dx + Pu(L)\right] \\ &= \tfrac{1}{2}B(u,u) - L(u),\end{aligned} \tag{7.6.59}$$

subjected to the geometric boundary condition, $u(0) = 0$. The bilinear and linear forms in this case are

$$\begin{aligned}B(u,v) &= \int_0^L EA\frac{du}{dx}\frac{dv}{dx}\, dx + k\, u(L)v(L), \\ L(v) &= \int_0^L f v\, dx + Pv(L).\end{aligned} \tag{7.6.60}$$

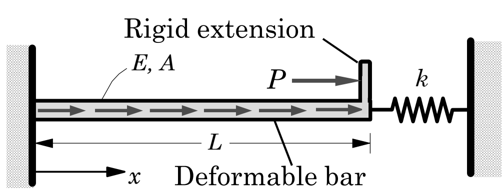

Fig. 7.6.5: Axial deformation of a uniform bar with an end spring.

Another example of the functional I is provided by the total potential energy functional Π in Eq. (7.6.27) associated with the bending of straight beams according to the Euler–Bernoulli beam theory. The bilinear and linear forms in this case are

$$\begin{aligned} B(v,w) &= \int_0^L EI\,\frac{d^2v}{dx^2}\frac{d^2w}{dx^2}\,dx, \\ L(w) &= \int_0^L qw\,dx + F_0 w(L) + M_0\alpha(L), \end{aligned} \tag{7.6.61}$$

where $\alpha = -(dw/dx)$.

7.6.5.2 Description of the method

In the Ritz method, we seek an approximation $U_N(x)$ of $u(x)$, for a fixed and preselected N, in the form

$$u(x) \approx U_N(x) = c_i\phi_i(x) + \phi_0(x), \tag{7.6.62}$$

where summation on repeated index i is implied (over the range of 1 to N), $\phi_i(x)$ are appropriately selected approximation functions, and c_i are unknown parameters. In view of the fact that the natural boundary conditions of the problem are included in the functional $I(u)$, we require the approximate solution U_N to satisfy only the geometric boundary conditions. In order that U_N satisfies the geometric boundary conditions *for any* c_i, it is convenient to choose the approximation in the form (7.6.62) and require $\phi_0(x)$ to satisfy the actual specified geometric boundary conditions. For instance, if $u(x)$ is specified to be $\hat{u}$ at $x = 0$, we require $\phi_0(x)$ be such that $\phi_0(0) = \hat{u}$, while ϕ_i are required to satisfy the homogeneous form of the geometric boundary condition, $\phi_i(0) = 0$. This follows from

$$0 = U_N(0) = \sum_{i=1}^{N} c_i\phi_i(0) + \phi_0(0) = \sum_{i=1}^{N} c_i\phi_i(0) + \hat{u}.$$

Because $U_N(0) = \hat{u}$, it follows that

$$\sum_{i=1}^{N} c_i\phi_i(0) = 0 \;\rightarrow\; \phi_i(0) = 0 \text{ for all } i = 1, 2, \ldots, N.$$

Thus, $\phi_i(x)$ must satisfy the *homogeneous form* of specified essential boundary conditions, and they must be sufficiently differentiable as required by the functional $I(U)$. The parameters c_i are determined by the condition that $I(U_N)$ is the minimum, that is, $\delta I(U) = 0$.

The *approximation functions* ϕ_0 and ϕ_i should be such that the substitution of Eq. (7.6.54) into $\delta\Pi$ results in N linearly independent equations for the parameters c_j ($j = 1, 2, \ldots, N$) so that the system has a solution. To ensure that the algebraic equations resulting from the Ritz approximation have a solution, and the approximate solution $U_N(x)$ converges to the true solution $u(x)$ of the problem as the value of N is increased, ϕ_i ($i = 1, 2, \ldots, N$) and ϕ_0 must satisfy certain requirements, as outlined next.

(1) ϕ_0 must satisfy the *specified* geometric boundary conditions. It is identically zero if all of the specified essential boundary conditions are homogeneous, that is, $\phi_0(x) = 0$. $(7.6.63)_1$

(2) ϕ_i $(i = 1, 2, \ldots, N)$ must satisfy the following three conditions: (a) be continuous, as required by the quadratic functional $I(u)$; (b) satisfy the *homogeneous form* of the specified essential boundary conditions; and (c) the set $\{\phi_i\}$ be linearly independent and complete; completeness refers to the property that all lower order terms up to the highest desired term must be included. $(7.6.63)_2$

Substituting $U_N(c_1, c_2, \cdots, c_N)$ into the total potential energy functional Π in Eq. (7.6.56), we obtain Π as a function of the parameters c_1, c_2, $\cdots$, c_N (after carrying out the indicated integration with respect to x):

$$\begin{aligned}\Pi &= \tfrac{1}{2}B(c_j\phi_j + \phi_0, c_k\phi_k + \phi_0) - L(c_k\phi_k + \phi_0)\\ &= \tfrac{1}{2}\left[c_jc_kB(\phi_j,\phi_k) + 2c_jB(\phi_j,\phi_0) + B(\phi_0,\phi_0)\right] - c_kL(\phi_k) - L(\phi_0).\end{aligned}$$

Then c_i are determined (or adjusted) such that $\delta\Pi = 0$; in other words, we minimize Π with respect to c_i, $i = 1, 2, \cdots, N$:

$$0 = \delta\Pi = \frac{\partial\Pi}{\partial c_1}\delta c_1 + \frac{\partial\Pi}{\partial c_2}\delta c_2 + \ldots + \frac{\partial\Pi}{\partial c_N}\delta c_N = \sum_{j=1}^{N}\frac{\partial\Pi}{\partial c_j}\delta c_j.$$

Because the set $\{c_i\}$ is linearly independent, it follows that

$$\begin{aligned}0 = \frac{\partial\Pi}{\partial c_i} &= \tfrac{1}{2}\left[c_kB(\phi_i,\phi_k) + c_jB(\phi_j,\phi_i) + 2B(\phi_i,\phi_0)\right] - L(\phi_i)\\ &= c_jB(\phi_i,\phi_j) + B(\phi_i,\phi_0) - L(\phi_i) \quad \text{for} \quad i = 1, 2, \cdots, N.\end{aligned}$$

or

$$\mathbf{Bc} = \mathbf{R}, \tag{7.6.64}$$

where

$$B_{ij} = B(\phi_i,\phi_j), \quad R_i = L(\phi_i) - B(\phi_i,\phi_0). \tag{7.6.65}$$

Equation (7.6.64) consists of N linear algebraic equations among N parameters, c_1, $c_2, \cdots$, c_N. Once the parameters are determined from Eq. (7.6.64), the solution U_N to the problem is given by Eq. (7.6.62). We consider couple of examples next.

Example 7.6.4

Formulate the N-parameter Ritz solution $U_N(x)$ of the bar problem described by Eqs. (7.6.57) and (7.6.58) for $AE(x) = a_0(2 - \frac{x}{L})$, $k = 0$, and $f(x) = f_0$, and determine the Ritz solutions for $N = 1$ and $N = 2$ [see Reddy (2002)].

Solution: First, we must select the approximation functions ϕ_0 and ϕ_i. Apart from the guidelines given in (7.6.63), the selection of the coordinate functions is largely arbitrary. As a general rule, the coordinate functions ϕ_i should be selected from an admissible set (that is, those meeting the two conditions listed earlier), from the lowest order to a desirable order, without missing any intermediate terms (i.e., the completeness property). Also, ϕ_0 should be any lowest order (including zero) that satisfied the specified essential boundary conditions of the problem; $\phi_0(x)$ has no continuity (differentiability) requirement.

For the problem at hand, $\phi_0 = 0$ because the specified essential boundary condition is homogeneous, $u(0) = 0$; this homogeneous essential boundary condition requires us to find $\phi_1(x)$ such that $\phi_1(0) = 0$ and it is differentiable at least once with respect to x because $\Pi(u)$ involves the first derivatives of $u \approx U_N$. If an algebraic polynomial is to be selected, the lowest order polynomial that has a nonzero first derivative is

$$\phi_1(x) = a + bx,$$

where a and b are constants. The condition $\phi_1(0) = 0$ gives $a = 0$. Since b is arbitrary, we take it to be unity (any nonzero constant will be absorbed into c_1). When $N > 1$, property 2(c) in Eq. (7.6.63)$_2$ requires that ϕ_i, $i > 1$, should be selected such that the set $\{\phi_i\}_{i=1}^N$ is linearly independent and makes the set complete. In the present case, this is done by choosing ϕ_2 to be x^2. Clearly, $\phi_2(x) = x^2$ meets the conditions $\phi_2(0) = 0$, linearly independent of $\phi_1(x) = x$ (i.e., ϕ_2 is not a constant multiple of ϕ_1), and the set $\{x, x^2\}$ is complete (i.e., no other admissible term up to quadratic is omitted). In other words, in selecting coordinate functions of a given degree, one should not omit any lower-order terms that are admissible. Otherwise, the approximate solution will never converge to the exact solution, no matter how many terms are used in the Ritz approximation, as the exact solution may have those lower order terms that were omitted in the approximate solution. We choose

$$U_2(x) = c_1\phi_1 + c_2\phi_2 + \cdots + c_N\phi_N(x) = c_i\,\phi_i(x). \tag{1}$$

For the choice of algebraic polynomials, the N-parameter Ritz coefficients B_{ij} are [see Eq. (7.6.60)]

$$\begin{aligned} B_{ij} = B(\phi_i, \phi_j) &= \int_0^L a_0\left(1 - \frac{x}{L}\right)\frac{d\phi_i}{dx}\frac{d\phi_j}{dx}\,dx + k\,\phi_i(L)\phi_j(L) \\ &= a_0\,ij\int_0^L \left(1 - \frac{x}{L}\right)x^{i+j-2}\,dx + k(L)^{i+j} \\ &= a_0\frac{ij(1+i+j)}{(i+j-1)(i+j)}(L)^{i+j-1} + k(L)^{i+j}, \end{aligned} \tag{2}$$

$$R_i = \int_0^L f\phi_i\,dx + P\phi_i(L) = \frac{f_0}{i+1}(L)^{i+1} + P(L)^i. \tag{3}$$

Note that $k = 0$ for the problem at hand.

For one-term approximation ($N = 1$), we have

$$a_{11} = \frac{3}{2}a_0L, \quad b_1 = \frac{1}{2}f_0L^2 + PL,$$

$$c_1 = \frac{b_1}{a_{11}} = \frac{6}{9a_0L}\left(\frac{3}{6}f_0L^2 + PL\right) = \frac{f_0L + 2P}{3a_0},$$

and the one-parameter Ritz solution is

$$U_1(x) = \frac{f_0L + 2P}{3a_0}x. \tag{4}$$

For $N = 2$, we have

$$a_{11} = \frac{3}{2}a_0L, \quad a_{12} = a_{21} = \frac{4}{3}a_0L^2, \quad a_{22} = \frac{5}{3}a_0L^3,$$
$$b_1 = \frac{1}{2}f_0L^2 + P_0L, \quad b_2 = \frac{1}{3}f_0L^3 + P_0L^2.$$

The Ritz equations can be written in matrix form as

$$\frac{a_0L}{6}\begin{bmatrix} 9 & 8L \\ 8L & 10L^2 \end{bmatrix}\begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = \frac{f_0L^2}{6}\begin{Bmatrix} 3 \\ 2L \end{Bmatrix} + PL\begin{Bmatrix} 1 \\ L \end{Bmatrix},$$

whose solution by is

$$c_1 = \frac{1}{a_0}\left(\frac{7}{13}f_0 + \frac{6}{13}P\right), \quad c_2 = \frac{3}{13a_0 L}\left(-f_0 L + P\right).$$

Hence, the two-parameter Ritz solution is

$$U_2(x) = \frac{7f_0L + 6P}{13a_0}x + \frac{3(P - f_0L)}{13a_0L}x^2. \tag{5}$$

The exact solution of Eqs. (7.6.57) and (7.6.58) with $u(0) = 0$, $k = 0$, $EA = a_0[2-(x/L)]$, and $f = f_0$ is

$$u(x) = \frac{f_0L}{a_0}x + \frac{(f_0L - P)L}{a_0}\log\left(1 - \frac{x}{2L}\right) \tag{6}$$

$$\approx \frac{f_0L + P}{2a_0}x + \frac{P - f_0L}{8a_0L}x^2 + \frac{P - f_0L}{24a_0L^2}x^3 + \ldots \tag{7}$$

Table 7.6.1 contains a comparison of the Ritz coefficients c_i for $N = 1, 2, \ldots, 8$ with the exact coefficients in Eq. (7) for $L = 10$ ft., $a_0 = 180 \times 10^6$ lb, $f_0 = 0$, and $P = 10 \times 10^6$ lb. Clearly the Ritz coefficients c_i converge to the exact ones as N goes from 1 to 8.

Table 7.6.1: The Ritz coefficients* for the axial deformation of an isotropic elastic bar subjected to axial force.

n	$\bar{c}_1$	$\bar{c}_2$	$\bar{c}_3$	$\bar{c}_4$	$\bar{c}_5$	$\bar{c}_6$	$\bar{c}_7$	$\bar{c}_8$
1	37.037							
2	25.641	12.821						
3	28.219	4.409	4.879					
4	27.691	7.788	0.000	3.029				
5	27.794	6.701	3.389	−1.040	1.664			
6	27.775	7.009	1.904	2.012	−1.142	0.952		
7	27.778	6.929	2.453	0.320	1.447	−0.980	0.560	
8	27.778	6.948	2.272	1.094	−0.287	1.136	−0.769	0.336
Exact	27.778	6.944	2.315	0.868	0.347	0.145	0.062	0.027

* $\bar{c}_i = c_i \times 10^{5+i}$.

Example 7.6.5

Consider a simply supported beam of length L and constant bending stiffness EI, subjected to uniformly distributed transverse load q_0. Determine the transverse displacement $v(x)$ of the beam using the Ritz method with $N = 2$ and $N = 3$.

Solution: The Ritz equations are obtained from Eq. (7.6.64) and (7.6.65), where the bilinear and linear forms defined by Eq. (7.6.61) with $F_0 = M_0 = 0$. We choose an N-parameter Ritz approximation of the form

$$v(x) \approx V_2(x) = c_1\phi_1(x) + c_2\phi_2(x) + \cdots + c_N\phi_N(x) + \phi_0(x).$$

Noting that the essential boundary conditions are $v(0) = v(L) = 0$, we select ϕ_0 to be zero. As far as ϕ_i are concerned, we choose them to vanish at $x = 0$ and $x = L$. Thus, we can choose

$$\phi_1 = x(L - x), \quad \phi_2 = x^2(L - x), \quad \phi_i(x) = x^i(L - x), \quad \cdots, \quad \phi_N(x) = x^N(L - x), \tag{1}$$

and

$$\frac{d\phi_i}{dx} = iLx^{i-1} - (i+1)x^i, \quad \frac{d^2\phi_i}{dx^2} = i(i-1)Lx^{i-2} - (i+1)ix^i. \tag{2}$$

Substituting these expressions into B_{ij} and R_i, we obtain

$$B_{ij} = \int_0^L EI \frac{d^2\phi_i}{dx^2}\frac{d^2\phi_j}{dx^2}\,dx$$
$$= \int_0^L EI \left[i(i-1)Lx^{i-2} - (i+1)ix^i\right]\left[j(j-1)Lx^{j-2} - (j+1)jx^j\right] dx \tag{2}$$
$$R_i = \int_0^L q_0\phi_i\,dx = \int_0^L q_0(x^iL - x^{i+1})\,dx.$$

For $N = 2$, we have

$$EIL\begin{bmatrix} 4 & 2L \\ 2L & 4L^2 \end{bmatrix}\begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = \frac{q_0L^3}{12}\begin{Bmatrix} 2 \\ L \end{Bmatrix}. \tag{3}$$

The solution of these equations yields the result $c_1 = q_0L^2/24EI$ and $c_2 = 0$, and the two-parameter Ritz solution becomes

$$V_2(x) = \frac{q_0L^4}{24EI}\left(\frac{x}{L} - \frac{x^2}{L^2}\right). \tag{4}$$

For $N = 3$ we obtain

$$EIL\begin{bmatrix} 4 & 2L & 2L^2 \\ 2L & 4L^2 & 4L^3 \\ 2L^2 & 4L^3 & 4.8L^4 \end{bmatrix}\begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix} = \frac{q_0L^3}{12}\begin{Bmatrix} 2 \\ L \\ 0.6L^2 \end{Bmatrix}, \tag{5}$$

and we obtain $c_1 = c_2L = -c_3L^2 = q_0L^2/24EI$. Hence, the three-parameter Ritz solution is

$$V_3(x) = \frac{q_0L^4}{24EI}\left(\frac{x}{L} - 2\frac{x^3}{L^3} + \frac{x^4}{L^4}\right), \tag{6}$$

which coincides with the exact solution of the beam problem.

Example 7.6.6

Consider the Poisson equation governing the Prandtl stress function Ψ, Eq. (7.5.79):

$$-\nabla^2\Psi = 2\mu\theta \ \text{ in } \ \Omega, \quad \Psi = 0 \ \text{ on } \ \Gamma. \tag{1}$$

If the cross section of the cylinder is a square, $\Omega = \{-a \le x,\ y \le a\}$, as shown in Fig. 7.6.6, determine the stress function and compute the shear stresses σ_{xz} and σ_{yz} using a one-parameter Ritz approximation.

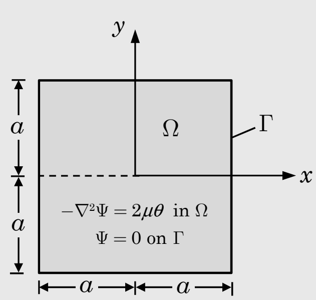

Fig. 7.6.6: Torsion of a cylinder of square cross section.

Solution: The functional associated with Eq. (1) is

$$\Pi(\Psi) = \frac{1}{2}\int_{-a}^{a}\int_{-a}^{a}\left[\left(\frac{\partial \Psi}{\partial x}\right)^2 + \left(\frac{\partial \Psi}{\partial y}\right)^2\right] dx\, dy - 2\mu\theta \int_{-a}^{a}\int_{-a}^{a} \Psi\, dx\, dy. \tag{2}$$

and the bilinear and linear forms are

$$B(\Psi,\Phi) = \int_{-a}^{a}\int_{-a}^{a}\left(\frac{\partial \Psi}{\partial x}\frac{\partial \Phi}{\partial x} + \frac{\partial \Psi}{\partial y}\frac{\partial \Phi}{\partial y}\right) dx\, dy, \quad L(\Phi) = 2\mu\theta \int_{-a}^{a}\int_{-a}^{a} \Phi\, dx\, dy. \tag{3}$$

For $N = 1$ we choose the function

$$\phi_1 = (a^2 - x^2)(a^2 - y^2), \tag{4}$$

and obtain

$$B_{ij} = \int_{-a}^{a}\int_{-a}^{a}\left(\frac{\partial \phi_i}{\partial x}\frac{\partial \phi_j}{\partial x} + \frac{\partial \phi_i}{\partial y}\frac{\partial \phi_j}{\partial y}\right) dx\, dy = \frac{256}{45},$$
$$R_i = 2\mu\theta \int_{-a}^{a}\int_{-a}^{a} \phi_i\, dx\, dy = 2\mu\theta\frac{16}{9a^2}, \tag{5}$$

and the one-parameter solution is given by

$$\Psi_1(x,y) = \frac{5\mu\theta a^2}{8}\left(1 - \frac{x^2}{a^2}\right)\left(1 - \frac{y^2}{a^2}\right). \tag{6}$$

$$\sigma_{xz} = -\frac{5\mu\theta a}{4}\frac{x}{a}\left(1 - \frac{y^2}{a^2}\right), \quad \sigma_{yz} = \frac{5\mu\theta a}{4}\frac{y}{a}\left(1 - \frac{y^2}{a^2}\right).$$

For $N = 2$ with

$$\phi_1 = (a^2 - x^2)(a^2 - y^2), \quad \phi_2 = (x^2 + y^2)\phi_1, \cdots. \tag{7}$$

we obtain

$$a^8\begin{bmatrix} \frac{256}{45} & \frac{1024}{525}a^2 \\ \frac{1024}{525}a^2 & \frac{11264}{4725}a^4 \end{bmatrix}\begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = 2\mu\theta a^6 \begin{Bmatrix} \frac{16}{9} \\ \frac{32}{45}a^2 \end{Bmatrix}, \tag{8}$$

whose solution yields

$$c_1 = \frac{1295}{2216a^2}\mu\theta, \quad c_2 = \frac{525}{4432a^4}\mu\theta. \tag{9}$$

The two-parameter Ritz solution is given by

$$\Psi_2(x,y) = \frac{\mu\theta a^2}{4432}[2590 + 525(\bar{x}^2 + \bar{y}^2)](1 - \bar{x}^2)(1 - \bar{y}^2), \tag{10}$$

where $\bar{x} = x/a$ and $\bar{y} = y/a$.

The exact solution to Eq. (1) can be obtained using the separation of variables method, and it is given by

$$\Psi(x,y) = \frac{32\mu\theta a^2}{\pi^3}\sum_{n=1,3,5\ldots}^{\infty}\frac{1}{n^3}(-1)^{(n-1)/2}\left[1 - \frac{\cosh(n\pi y/2a)}{\cosh(n\pi/2)}\right]\cos\frac{n\pi x}{2a}. \tag{11}$$

The exact value of Ψ at the center of the region is

$$\Psi(0,0) = 0.5884\mu\theta a^2,$$

whereas the two-parameter Ritz solution is $0.5844\mu\theta a^2$, which has an error of only 0.68%.

Although the problem is presented here as one governing the Prandtl stress function for the torsion of a cylindrical member, the equation arises, among others, in connection with the transverse deflection of a membrane fixed on all sides and subjected to uniform pressure f_0 (in place of $2\mu\theta$) and in conduction heat transfer in a square region with internal heat generation of f_0 unit area. The function u denotes the deflection u in the case of a membrane and the temperature T in the case of conduction heat transfer. Thus the results obtained can also be interpreted for these two problems.

7.7 Hamilton's Principle

7.7.1 Introduction

The principle of total potential energy discussed in the previous section can be generalized to initial value problems, that is, problems involving time, and the principle is known as *Hamilton's principle.* In Hamilton's principle the system under consideration is assumed to be characterized by two energy functions: the *kinetic energy* K and the total *potential energy* Π. For *discrete* systems (i.e., systems with a finite number of degrees of freedom), these energies can be described in terms of a finite number of generalized coordinates and their derivatives with respect to time t. For *continuous* systems (that is, systems that are described by an infinite number of generalized coordinates), the energies can be expressed in terms of the dependent variables of the problem that are functions of position and time.

7.7.2 Hamilton's Principle for a Rigid Body

To gain a simple understanding of Hamilton's principle, consider a single particle or a rigid-body (which is a collection of particles, the distance between which is unaltered at all times) of mass m moving under the influence of a force (see Reddy, 2002) $\mathbf{F} = \mathbf{F}(\mathbf{r}, t)$. The path $\mathbf{r}(t)$ followed by the particle is related to the force $\mathbf{F}$ and mass m by the principle of balance of linear momentum (i.e., Newton's second law of motion):

$$\mathbf{F}(\mathbf{r}, t) = \frac{d}{dt}\left(m\frac{d\mathbf{r}}{dt}\right). \tag{7.7.1}$$

A path that differs from the actual path is expressed as $\mathbf{r} + \delta\mathbf{r}$, where $\delta\mathbf{r}$ is the variation of the path for any arbitrarily *fixed* time t. We suppose that the actual path $\mathbf{r}$ and the *varied* path differ except at two distinct times t_1 and t_2, that is, $\delta\mathbf{r}(t_1) = \delta\mathbf{r}(t_2) = \mathbf{0}$. Taking the scalar product of Eq. (7.7.1) with the variation $\delta\mathbf{r}$, and integrating with respect to time between t_1 and t_2, we obtain

$$\int_{t_1}^{t_2}\left[\frac{d}{dt}\left(m\frac{d\mathbf{r}}{dt}\right) - \mathbf{F}(\mathbf{r}, t)\right]\cdot\delta\mathbf{r}\; dt = 0. \tag{7.7.2}$$

Integration-by-parts of the first term in Eq. (7.7.2) yields

$$-\int_{t_1}^{t_2}\left[m\frac{d\mathbf{r}}{dt}\cdot\frac{d\delta\mathbf{r}}{dt} + \mathbf{F}(\mathbf{r}, t)\cdot\delta\mathbf{r}\right]dt + \left(m\frac{d\mathbf{r}}{dt}\cdot\delta\mathbf{r}\right)\bigg|_{t_1}^{t_2} = 0. \tag{7.7.3}$$

The last term in Eq. (7.7.3) vanishes because $\delta\mathbf{r}(t_1) = \delta\mathbf{r}(t_2) = \mathbf{0}$. Also, note that

$$m\frac{d\mathbf{r}}{dt}\cdot\frac{d\delta\mathbf{r}}{dt} = \delta\left[\frac{m}{2}\frac{d\mathbf{r}}{dt}\cdot\frac{d\mathbf{r}}{dt}\right] \equiv \delta K, \tag{7.7.4}$$

where K is the kinetic energy of the particle or a rigid-body

$$K = \frac{1}{2}m\frac{d\mathbf{r}}{dt}\cdot\frac{d\mathbf{r}}{dt} = \frac{1}{2}m\mathbf{v}\cdot\mathbf{v}, \tag{7.7.5}$$

and δK is called the *virtual kinetic energy.* The expression $\mathbf{F}(\mathbf{r}, t) \cdot \delta\mathbf{r}$ is called the *virtual work done by external forces* and denoted by

$$\delta W_E = -\mathbf{F}(\mathbf{r}, t) \cdot \delta\mathbf{r}. \tag{7.7.6}$$

The minus sign indicates that the work is done by external force $\mathbf{F}$ *on* the body in moving through the displacement $\delta\mathbf{r}$. Equation (7.7.3) now takes the form

$$\int_{t_1}^{t_2} (\delta K - \delta W_E)\, dt = 0, \tag{7.7.7}$$

which is known as the *general form of Hamilton's principle* for a single particle or rigid body. Note that a particle or a rigid-body has no strain energy Π because the distance between the particles is unaltered.

Suppose that the force $\mathbf{F}$ is conservative (i.e., the sum of the potential and kinetic energies is conserved) such that it can be replaced by the gradient of a potential

$$\mathbf{F} = -\text{grad } V, \tag{7.7.8}$$

where $V = V(\mathbf{r}, t)$ is the *potential energy due to the loads* on the body. Then Eq. (7.7.7) can be expressed in the form

$$\delta \int_{t_1}^{t_2} (K - V)\ dt = 0, \tag{7.7.9}$$

because ($\mathbf{r} = x_i \hat{\mathbf{e}}_i$)

$$\text{grad } V \cdot \delta\mathbf{r} = \frac{\partial V}{\partial x_i} \delta x_i = \delta V(\mathbf{x}).$$

The difference between the kinetic and potential energies is called the *Lagrangian* function

$$L \equiv K - V. \tag{7.7.10}$$

Equation (7.7.9) is known as Hamilton's principle for the conservative motion of a particle (or a rigid body). The principle can be stated as follows: *The motion of a particle acted on by conservative forces between two arbitrary instants of time t_1 and t_2 is such that the line integral over the Lagrangian function is an extremum for the path motion.* Stated in other words, of all possible paths that the particle could travel from its position at time t_1 to its position at time t_2, its actual path will be one for which the integral

$$I \equiv \int_{t_1}^{t_2} L\ dt \tag{7.7.11}$$

is an extremum (i.e., a minimum, maximum, or an inflection).

If the path $\mathbf{r}$ can be expressed in terms of the generalized coordinates $q_i (i = 1, 2, 3)$, the Lagrangian function can be written in terms of q_i and their time derivatives

$$L = L(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3). \tag{7.7.12}$$

Then the condition for the extremum of I in (7.7.11) results in the equation (note that $\delta q_i = 0$ at t_1 and t_2)

$$\begin{aligned}\delta I &= \delta \int_{t_1}^{t_2} L(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3) dt = 0 \\ &= \int_{t_1}^{t_2} \sum_{i=1}^{3} \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \right) \right] \delta q_i \; dt. \end{aligned} \tag{7.7.13}$$

When all q_i are linearly independent (i.e., no constraints among q_i), the variations δq_i are independent of each other for all t, except that all $\delta q_i = 0$ at t_1 and t_2. Therefore, the coefficients of $\delta q_1, \delta q_2$, and δq_3 vanish separately:

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \right) = 0, \quad i = 1, 2, 3. \tag{7.7.14}$$

These equations are called the *Lagrange equations of motion.* Recall that in Section 7.6 (for a static case) these equations were also called the Euler equations. For the dynamic case involving deformable solids, these equations will be called the *Euler–Lagrange equations*.

When the forces are not conservative, we must deal with the general form of Hamilton's principle in Eq. (7.7.7). In this case, there exists no functional I that must be an extremum. If the virtual work can be expressed in terms of the generalized coordinates q_i by

$$\delta W_E = -\left(F_1 \, \delta q_1 + F_2 \, \delta q_2 + F_3 \, \delta q_3 \right), \tag{7.7.15}$$

where F_i are the *generalized forces*, then we can write Eq. (7.7.14) as

$$\int_{t_1}^{t_2} \sum_{i=1}^{3} \left[\frac{\partial K}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_i} \right) + F_i \right] \delta q_i \; dt = 0, \tag{7.7.16}$$

and the Euler-Lagrange equations for the nonconservative forces are given by

$$\delta q_i : \quad \frac{\partial K}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_i} \right) + F_i = 0, \quad i = 1, 2, 3. \tag{7.7.17}$$

Example 7.7.1

Consider the planar motion of a pendulum that consists of a mass m attached at the end of a rigid massless rod of length L that pivots about a fixed point O, as shown in Fig. 7.7.1. Determine the equation of motion.

Solution: The position of the mass can be expressed in terms of the generalized coordinates $q_1 = l$ and $q_2 = \theta$, measured from the vertical position. Because l is a constant, we have $\dot{q}_1 = 0$ and θ is the only independent generalized coordinate. The force $\mathbf{F}$ acting on the mass m is the component of the gravitational force,

$$\mathbf{F} = mg \left(\cos\theta \, \hat{\mathbf{e}}_r - \sin\theta \, \hat{\mathbf{e}}_\theta \right) \equiv F_r \, \hat{\mathbf{e}}_r + F_\theta \, \hat{\mathbf{e}}_\theta. \tag{7.7.18}$$

The component along $\hat{\mathbf{e}}_r$ does no work because $q_1 = l$ is a constant. The second component, F_θ, is derivable from the potential ($\boldsymbol{\nabla} V = -F_\theta \, \hat{\mathbf{e}}_\theta$):

$$V(\theta) = -\left[-mgl(1 - \cos\theta) \right] = mgl(1 - \cos\theta), \tag{7.7.19}$$

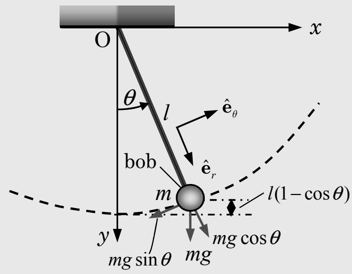

Fig. 7.7.1: Planar motion of a pendulum.

where V represents the potential energy of the mass m at any instant of time with respect to the static equilibrium position $\theta = 0$, and $\boldsymbol{\nabla}$ is the gradient operator in the polar coordinate system:

$$\boldsymbol{\nabla} = \hat{\mathbf{e}}_r \frac{\partial}{\partial r} + \frac{\hat{\mathbf{e}}_\theta}{r} \frac{\partial}{\partial \theta}. \tag{7.7.20}$$

Thus the kinetic energy and the potential energy due to external load are given by

$$\begin{aligned} K &= \frac{m}{2}(\ell\dot{\theta})^2, \quad V = mgl(1-\cos\theta), \\ \delta K &= ml^2\dot{\theta}\,\delta\dot{\theta}, \quad \delta V = mgl\sin\theta\,\delta\theta = -F_\theta(l\,\delta\theta). \end{aligned} \tag{7.7.21}$$

Therefore, the Lagrangian function L is a function of θ and $\dot{\theta}$. The Euler–Lagrange equation is given by

$$\delta q_2 = \delta\theta : \quad \frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0,$$

which yields

$$-mgl\sin\theta - \frac{d}{dt}(ml^2\,\dot{\theta}) = 0 \quad \text{or} \quad \ddot{\theta} + \frac{g}{l}\sin\theta = 0 \quad (F_\theta = ml\,\ddot{\theta}). \tag{7.7.22}$$

Equation (7.7.22) represents a second-order nonlinear differential equation governing θ. For small angular motions, Eq. (7.7.22) can be linearized by replacing $\sin\theta \approx \theta$:

$$\ddot{\theta} + \frac{g}{\ell}\theta = 0. \tag{7.7.23}$$

Now suppose that the mass experiences a resistance force $\mathbf{F}^*$ proportional to its speed (e.g., the mass m is suspended in a medium with viscosity μ). According to Stokes's law,

$$\mathbf{F}^* = -6\pi\mu a l\dot{\theta}\,\hat{\mathbf{e}}_\theta, \tag{7.7.24}$$

where μ is the viscosity of the surrounding medium, a is the radius of the bob, and $\hat{\mathbf{e}}_\theta$ is the unit vector tangential to the circular path. The resistance of the massless rod supporting the bob is neglected. The force $\mathbf{F}^*$ is not derivable from a potential function (i.e., nonconservative). Thus, we have one part of the force (i.e., gravitational force) conservative and the other (i.e., viscous force) nonconservative. Hence, we use Hamilton's principle expressed by Eq. (7.7.14) or Eq. (7.7.17) with

$$\delta W_E = \delta V - \mathbf{F}^* \cdot (l\delta\theta\,\hat{\mathbf{e}}_\theta) = \left(mgl\,\sin\theta + 6\pi\mu a l^2\,\dot{\theta}\right)\delta\theta \equiv -F_\theta l\,\delta\theta.$$

Then the equation of motion is given by $[K = K(\dot{\theta})]$:

$$-\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{\theta}}\right) + F_\theta l = 0 \quad \text{or} \quad \ddot{\theta} + \frac{g}{l}\,\sin\theta + \frac{6\pi a\mu}{m}\,\dot{\theta} = 0. \tag{7.7.25}$$

The coefficient $c = 6\pi a\mu/m$ is called the *damping* coefficient.

7.7.3 Hamilton's Principle for a Continuum

Hamilton's principle for a continuous body $\mathcal{B}$ occupying configuration κ with volume Ω and boundary Γ can be derived following essentially the same ideas as discussed for a particle or a rigid body. In contrast to a rigid body, a continuum is characterized by strain (or internal) energy U, in addition to the kinetic energy K. Newton's second law of motion for a continuous body can be written in general terms as

$$\mathbf{F} - \frac{\partial}{\partial t}\left(m\frac{\partial \mathbf{v}}{\partial t}\right) = \mathbf{0}, \tag{7.7.26}$$

where m is the mass, $\mathbf{v}(\mathbf{x},t) = \partial \mathbf{u}/\partial t$ is the velocity vector, $\mathbf{u}(\mathbf{x},t)$ is the displacement vector, and $\mathbf{F}$ is the resultant of *all* forces acting on the body $\mathcal{B}$. The actual path $\mathbf{u}$ followed by a material particle in position $\mathbf{x}$ in the body is varied, consistent with kinematic (essential) boundary conditions on Γ, to $\mathbf{u}+\delta\mathbf{u}$, where $\delta\mathbf{u}$ is the admissible variation (or virtual displacement) of the path. We assume that the varied path differs from the actual path except at initial and final times, t_1 and t_2, respectively. Thus, an admissible variation $\delta\mathbf{u}$ satisfies the conditions,

$$\begin{aligned} \delta\mathbf{u}(\mathbf{x},t) &= \mathbf{0} \text{ on } \Gamma_u \text{ for all } t, \\ \delta\mathbf{u}(\mathbf{x},t_1) = \delta\mathbf{u}(\mathbf{x},t_2) &= \mathbf{0} \text{ for all } \mathbf{x}, \end{aligned} \tag{7.7.27}$$

where Γ_u denotes the portion of the boundary Γ of the body where the displacement vector $\mathbf{u}$ is specified.

The work done on body $\mathcal{B}$ at time t by the resultant force $\mathbf{F}$, which consists of body force $\mathbf{f}$ and specified surface traction $\hat{\mathbf{t}}$ in moving through respective virtual displacements $\delta\mathbf{u}$, is given by

$$\int_\Omega \mathbf{f}\cdot\delta\mathbf{u}\, d\mathbf{x} + \int_{\Gamma_\sigma} \hat{\mathbf{t}}\cdot\delta\mathbf{u}\, ds - \int_\Omega \boldsymbol{\sigma} : \delta\boldsymbol{\varepsilon}\, d\mathbf{x}, \tag{7.7.28}$$

where $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ are the stress and strain tensors, and Γ_σ is the portion of the boundary Γ on which tractions are specified ($\Gamma = \Gamma_u \cup \Gamma_\sigma$). The last term in Eq. (7.7.28) is known as the *virtual work stored in the body* due to deformation. The strains $\delta\boldsymbol{\varepsilon}$ are assumed to be compatible in the sense that the strain-displacement relations (7.2.1) are satisfied. The work done by the inertia force $\partial(m\mathbf{v})/\partial t$ in moving through the virtual displacement $\delta\mathbf{u}$ is given by

$$\int_\Omega \frac{\partial}{\partial t}\left(\rho_0\frac{\partial \mathbf{u}}{\partial t}\right)\cdot\delta\mathbf{u}\, d\mathbf{x}, \tag{7.7.29}$$

where ρ_0 is the mass density of the medium ($m = \rho_0\, d\mathbf{x}$). We have, analogous to Eq. (7.7.2) for a rigid body, the result

$$\int_{t_1}^{t_2}\left\{\int_\Omega \frac{\partial}{\partial t}\left(\rho_0\frac{\partial \mathbf{u}}{\partial t}\right)\cdot\delta\mathbf{u}\, d\mathbf{x} - \left[\int_\Omega (\mathbf{f}\cdot\delta\mathbf{u} - \boldsymbol{\sigma}:\delta\boldsymbol{\varepsilon})\, d\mathbf{x} + \int_{\Gamma_\sigma}\hat{\mathbf{t}}\cdot\delta\mathbf{u}\, ds\right]\right\}dt = 0,$$

or

$$-\int_{t_1}^{t_2}\left[\int_\Omega \rho_0\frac{\partial \mathbf{u}}{\partial t}\cdot\frac{\partial \delta\mathbf{u}}{\partial t}\, d\mathbf{x} + \int_\Omega (\mathbf{f}\cdot\delta\mathbf{u} - \boldsymbol{\sigma}:\delta\boldsymbol{\varepsilon})\, d\mathbf{x} + \int_{\Gamma_\sigma}\hat{\mathbf{t}}\cdot\delta\mathbf{u}\, ds\right]dt = 0. \tag{7.7.30}$$

In arriving at the expression in Eq. (7.7.30), integration-by-parts is used on the first term; the integrated terms vanish because of the initial and final conditions in Eq. (7.7.27). Equation (7.7.30) is known as the general form of Hamilton's principle for a continuous medium – conservative or not, and elastic or not.

For an elastic body, we recall from the previous sections that the forces $\mathbf{f}$ and $\mathbf{t}$ are conservative,

$$\delta V = -\left(\int_\Omega \mathbf{f}\cdot\delta\mathbf{u}\,d\mathbf{x} + \int_{\Gamma_\sigma} \hat{\mathbf{t}}\cdot\delta\mathbf{u}\,ds\right), \tag{7.7.31}$$

and that there exists a strain energy density function $U_0 = U_0(\boldsymbol{\varepsilon})$ such that

$$\boldsymbol{\sigma} = \frac{\partial U_0}{\partial \boldsymbol{\varepsilon}}. \tag{7.7.32}$$

Substituting Eqs. (7.7.31) and (7.7.32) into Eq. (7.7.30), we obtain

$$\delta \int_{t_1}^{t_2} [K - (V+U)]dt = 0, \tag{7.7.33}$$

where K and U are the kinetic and strain energies:

$$K = \int_\Omega \frac{\rho_0}{2}\frac{\partial\mathbf{u}}{\partial t}\cdot\frac{\partial\mathbf{u}}{\partial t}\,d\mathbf{x},\quad U = \int_\Omega U_0\,d\mathbf{x}. \tag{7.7.34}$$

Equation(7.7.33) represents Hamilton's principle for an elastic body. Recall that the sum of the strain energy U and potential energy V of external forces, $U+V$, is called the total potential energy, Π, of the body. For bodies involving no motion (that is, forces are applied sufficiently slowly such that the motion is independent of time, and the inertia forces are negligible), Hamilton's principle (7.7.33) reduces to the principle of virtual displacements. Equation (7.7.33) may be viewed as the dynamics version of the principle of virtual displacements.

The Euler–Lagrange equations associated with the Lagrangian, $L = K - \Pi$, can be obtained from Eq. (7.7.33):

$$\begin{aligned} 0 &= \delta \int_{t_1}^{t_2} L(\mathbf{u}, \boldsymbol{\nabla}\mathbf{u}, \dot{\mathbf{u}})\,dt \\ &= \int_{t_1}^{t_2} \left[\int_\Omega \left(\rho_0\frac{\partial^2\mathbf{u}}{\partial t^2} - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} - \mathbf{f}\right)\cdot\delta\mathbf{u}\,d\mathbf{x} + \int_{\Gamma_\sigma} (\mathbf{t}-\hat{\mathbf{t}})\cdot\delta\mathbf{u}\,ds\right]dt, \end{aligned} \tag{7.7.35}$$

where $\mathbf{t} = \boldsymbol{\sigma}\cdot\hat{\mathbf{n}}$. In arriving at Eq. (7.7.35) from Eq. (7.7.33), we have used integration-by-parts, gradient theorems, and Eqs. $(7.7.27)_1$. Since $\delta\mathbf{u}$ is arbitrary for t, $t_1 < t < t_2$, and for $\mathbf{x}$ in Ω and also on Γ_σ, it follows that

$$\rho_0\frac{\partial^2\mathbf{u}}{\partial t^2} - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} - \mathbf{f} = \mathbf{0} \quad \text{in } \Omega \text{ for } t > 0, \tag{7.7.36}$$

$$\mathbf{t}(s,t) - \hat{\mathbf{t}}(s,t) = \mathbf{0} \quad \text{on } \Gamma_\sigma \text{ for } t > 0. \tag{7.7.37}$$

Equations (7.7.36) are the Euler–Lagrange equations for an elastic body. Equations (7.7.36) are also subject to initial conditions of the form

$$\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}), \quad \dot{\mathbf{u}}(\mathbf{x},0) = \mathbf{v}_0(\mathbf{x}), \tag{7.7.38}$$

where $\mathbf{u}_0$ and $\dot{\mathbf{v}}_0$ are the initial displacement and initial velocity vectors, respectively.

Example 7.7.2

The displacement field for pure bending (i.e., omit the axial displacement u) of a beam according to the Euler–Bernoulli beam theory is (see Section 7.3.4)

$$u_1(x,z,t) = -y\frac{\partial v}{\partial x}, \quad u_2 = 0, \quad u_3(x,t) = v(x,t), \tag{1}$$

where v is the transverse displacement. Determine the equations of motion of the Euler–Bernoulli beam theory.

Solution: The Lagrange function associated with the dynamics of the Euler–Bernoulli beam is given by $L = K - (U + V)$, where

$$\begin{aligned}
K &= \int_0^L \int_A \left[\frac{\rho_0}{2}\left(-z\frac{\partial^2 v}{\partial x \partial t}\right)^2 + \frac{\rho_0}{2}\left(\frac{\partial w}{\partial t}\right)^2\right] dA\, dx \\
&= \int_0^L \left[\frac{\rho_0 I}{2}\left(\frac{\partial^2 v}{\partial x \partial t}\right)^2 + \frac{\rho_0 A}{2}\left(\frac{\partial v}{\partial t}\right)^2\right] dx, && (2)\\
U &= \int_0^L \int_A \frac{E}{2}\left(-y\frac{\partial^2 v}{\partial x^2}\right)^2 dA\, dx \\
&= \int_0^L \frac{EI}{2}\left(\frac{\partial^2 v}{\partial x^2}\right)^2 dx, && (3)\\
V &= -\int_0^L q\, v\, dx, && (4)
\end{aligned}$$

where q is the transverse distributed load. In arriving at the expressions for K and U, we have used the fact that the x-axis coincides with the geometric centroidal axis, $\int_A y\, dA = 0$.

The Hamilton principle gives

$$\begin{aligned}
0 &= \delta \int_0^T (K - U - V)\, dt \\
&= \int_0^T \int_0^L \left[\rho_0 I \frac{\partial \dot{v}}{\partial x}\frac{\partial \delta \dot{v}}{\partial x} + \rho_0 A \dot{v}\, \delta\dot{v} - EI\frac{\partial^2 v}{\partial x^2}\frac{\partial^2 \delta v}{\partial x^2} + q\, \delta v\right] dx\, dt. && (5)
\end{aligned}$$

The Euler–Lagrange equation obtained from Eq. (5) is the equation of motion governing the Euler–Bernoulli beam theory

$$\frac{\partial^2}{\partial x \partial t}\left(\rho_0 I \frac{\partial^2 v}{\partial x \partial t}\right) - \frac{\partial}{\partial t}\left(\rho_0 A \frac{\partial v}{\partial t}\right) - \frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 v}{\partial x^2}\right) + q = 0, \tag{7.7.39}$$

for $0 < x < L$ and $t > 0$. The first term is the contribution due to rotary inertia. The boundary and initial conditions associated with Eq. (7.7.39) are

$$\begin{aligned}
&\text{Boundary conditions:} \quad \text{specify: } v \text{ or } \frac{\partial}{\partial x}\left(EI\frac{\partial^2 v}{\partial x^2}\right) + \rho_0 I \frac{\partial^3 v}{\partial t^2 \partial x}, \\
&\qquad\qquad\qquad\qquad\quad \text{specify: } \frac{\partial v}{\partial x} \text{ or } EI\frac{\partial^2 v}{\partial x^2}, && (7.7.40)\\
&\text{Initial conditions:} \quad\quad \text{specify: } v(x,0) \text{ and } \dot{v}(x,0).
\end{aligned}$$

Example 7.7.3

Suppose that the Euler–Bernoulli beam of Example 7.7.2 experiences two types of viscous (velocity-dependent) damping: (1) viscous resistance to transverse displacement of the beam and (2) a viscous resistance to straining of the beam material. If the resistance to transverse velocity is denoted by $c(x)$, the corresponding damping force is given by $q_D(x,t) = c(x)\dot{v}$. If the resistance to strain velocity is c_s, the damping stress is $\sigma_{xx}^D = c_s\dot{\varepsilon}_{xx}$. Derive the equations of motion of the beam with both types of damping.

Solution: We must add the following terms due to damping to the expression in Eq. (5) of Example 7.7.2:

$$\begin{aligned}
&-\int_0^T \left[\int_\Omega \sigma_D \delta\varepsilon \, d\mathbf{x} + \int_0^L q_D \delta v \; dx\right] dt \\
&= -\int_0^T \left[\int_0^L \int_A c_s \left(-z\frac{\partial^3 v}{\partial x^2 \partial t}\right)\left(-z\frac{\partial^2 \delta v}{\partial x^2}\right) \, dA\, dx + \int_0^L q_D \delta v \; dx\right] dt \\
&= -\int_0^T \int_0^L \left(Ic_s \frac{\partial^3 v}{\partial x^2 \partial t}\frac{\partial^2 \delta v}{\partial x^2} + c\frac{\partial v}{\partial t}\delta v\right) dx\, dt. \qquad (1)
\end{aligned}$$

Then the expression in Eq. (5) of Example 7.7.2 becomes

$$\begin{aligned}
0 = &\int_0^T \int_0^L \left[\rho_0 I \frac{\partial \dot{v}}{\partial x}\frac{\partial \delta\dot{v}}{\partial x} + \rho_0 A \dot{v}\,\delta\dot{v} - EI\frac{\partial^2 v}{\partial x^2}\frac{\partial^2 \delta v}{\partial x^2} + q\,\delta v\right] dx\, dt \\
&- \int_0^T \int_0^L \left(Ic_s \frac{\partial^3 v}{\partial x^2 \partial t}\frac{\partial^2 \delta v}{\partial x^2} + c\frac{\partial w}{\partial t}\delta v\right) dx\, dt \qquad (2)
\end{aligned}$$

and the Euler–Lagrange equation is

$$\begin{aligned}
\frac{\partial^2}{\partial x \partial t}\left(\rho_0 I \frac{\partial^2 v}{\partial x \partial t}\right) - \frac{\partial}{\partial t}\left(\rho_0 A \frac{\partial v}{\partial t}\right) - \frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 v}{\partial x^2}\right) \\
- \frac{\partial^2}{\partial x^2}\left(Ic_s \frac{\partial^3 v}{\partial x^2 \partial t}\right) - c\frac{\partial v}{\partial t} + q = 0. \qquad (3)
\end{aligned}$$

The boundary and initial conditions for this case are

$$\begin{aligned}
&\text{Boundary conditions:} \quad \text{specify: } v \text{ or } \frac{\partial}{\partial x}\left(EI\frac{\partial^2 v}{\partial x^2} + Ic_s\frac{\partial^3 v}{\partial x^2 \partial t}\right) + \rho_0 I \frac{\partial^3 v}{\partial t^2 \partial x}\,, \\
&\qquad\qquad\qquad\qquad\quad \text{specify: } \frac{\partial v}{\partial x} \text{ or } EI\frac{\partial^2 v}{\partial x^2} + Ic_s\frac{\partial^3 v}{\partial x^2 \partial t}\,, \qquad (4) \\
&\text{Initial conditions:} \quad \text{specify: } v(x,0) \text{ and } \dot{v}(x,0).
\end{aligned}$$

7.8 Summary

This is a very comprehensive chapter on linearized elasticity. Beginning with a summary of the linearized elasticity equations that include the Navier equations and the Beltrami–Michell equations of elasticity, the three types of boundary value problems and the principle of superposition were discussed. The Clapeyron theorem and Betti and Maxwell reciprocity theorems and their applications were also presented. Various methods of solutions, namely, the inverse method, the semi-inverse method, the method of potentials, and variational methods are discussed. The two-dimensional elasticity problems, plane strain and plane stress, are formulated, and their solutions by the inverse method and the Airy stress function method are presented. Analytical solutions of a number of standard boundary value problems of elasticity using the Airy stress function are discussed. Torsion of cylindrical members is also presented. The principle of minimum total potential energy and its special case, the Castigliano theorem I, are discussed. The Ritz method is introduced as a general method of solving problems formulated as variational problems of finding u such that $B(u, v) = L(v)$ holds for all v. Lastly, Hamilton's principle for problems of dynamics is presented. A number of examples are included throughout the chapter.

Solution of elasticity problems discussed in this chapter requires an understanding of the problem from the aspect of suitable boundary conditions; existence of solution symmetries, if any; and the qualitative nature of the solution. Only then one may choose a solution method that suits its solution strategy. An insight into the problem is necessary for the use of the semi-inverse method. If one makes assumptions on the basis of a qualitative understanding of the problem and solves the boundary value problem, then the assumptions are likely to be correct. If not, the assumptions need to be modified. Also, most real-world problems do not admit exact or analytical solutions, and approximate solutions are the only alternative. The theoretical formulation of a problem based on the principles of mechanics is a necessary first and most important step even when one considers its solution by a numerical method. Therefore, a course on continuum mechanics or elasticity helps in correctly formulating the governing equations of boundary value problems of mechanics.

Problems

Strains, Stresses, and Strain Energy

7.1 Define the *deviatoric* components of stress and strain as follows:

$$s_{ij} \equiv \sigma_{ij} - \tfrac{1}{3}\sigma_{kk}\delta_{ij}, \quad e_{ij} \equiv \varepsilon_{ij} - \tfrac{1}{3}\varepsilon_{kk}\delta_{ij}.$$

Determine the constitutive relation between s_{ij} and e_{ij} for an isotropic material.

7.2 For each of the displacement fields given below, sketch the displaced positions in the x_1x_2-plane of the points initially on the sides of the square shown in Fig. P7.2.

$$\text{(a)}\ \ \mathbf{u} = \tfrac{\alpha}{2}\left(x_2\,\hat{\mathbf{e}}_1 + x_1\,\hat{\mathbf{e}}_2\right). \quad \text{(b)}\ \ \mathbf{u} = \tfrac{\alpha}{2}\left(-x_2\,\hat{\mathbf{e}}_1 + x_1\,\hat{\mathbf{e}}_2\right). \quad \text{(c)}\ \ \mathbf{u} = \alpha\, x_1\,\hat{\mathbf{e}}_2.$$

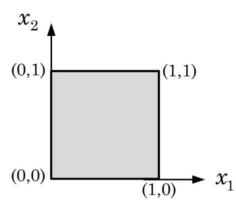

Fig. P7.2

7.3 For each of the displacement fields in Problem **7.2**, determine the components of (a) the Green–Lagrange strain tensor $\mathbf{E}$, (b) the infinitesimal strain tensor $\boldsymbol{\varepsilon}$, (c) the infinitesimal rotation tensor $\boldsymbol{\Omega}$, and (d) the infinitesimal rotation vector $\boldsymbol{\omega}$ (see Sections 3.4 and 3.5 for the definitions).

7.4 Similar to Cauchy's formula for a stress tensor, one can think of a similar formula for the strain tensor,

$$\boldsymbol{\varepsilon}_n = \boldsymbol{\varepsilon} \cdot \hat{\mathbf{n}},$$

where $\boldsymbol{\varepsilon}_n$ represents the strain vector in the direction of the unit normal vector, $\hat{\mathbf{n}}$. Determine the longitudinal strain corresponding to the displacement field $\mathbf{u} = \frac{\alpha}{2}(x_2\,\hat{\mathbf{e}}_1 + x_1\,\hat{\mathbf{e}}_2)$ in the direction of the vector $\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2$.

7.5 For the displacement vector given in the cylindrical coordinate system

$$\mathbf{u} = Ar\,\hat{\mathbf{e}}_r + Brz\,\hat{\mathbf{e}}_\theta + C\sin\theta\,\hat{\mathbf{e}}_z,$$

where A, B, and C are constants, determine the infinitesimal strain components in the cylindrical coordinate system.

7.6 The displacement vector at a point referred to the basis $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$ is $\mathbf{u} = 2\hat{\mathbf{e}}_1 + 2\hat{\mathbf{e}}_2 - 4\hat{\mathbf{e}}_3$. Determine $\bar{u}_i$ with respect to the basis $(\hat{\bar{\mathbf{e}}}_1, \hat{\bar{\mathbf{e}}}_2, \hat{\bar{\mathbf{e}}}_3)$, where $\hat{\bar{\mathbf{e}}}_1 = (2\hat{\mathbf{e}}_1 + 2\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3)/3$ and $\hat{\bar{\mathbf{e}}}_2 = (\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2)/\sqrt{2}$.

7.7 Express Navier's equations of elasticity (7.2.17) in the cylindrical coordinate system.

7.8 An isotropic body (E = 210 GPa and $\nu = 0.3$) with two-dimensional state of stress experiences the following displacement field (in mm):

$$u_1 = 3x_1^2 - x_1^3 x_2 + 2x_2^3, \quad u_2 = x_1^3 + 2x_1 x_2,$$

where x_i are in meters. Determine the stresses and rotation of the body at point $(x_1, x_2) = (0.05, 0.02)$ m.

7.9 A two-dimensional state of stress exists in a body with the following components of stress:

$$\sigma_{11} = c_1 x_2^3 + c_2 x_1^2 x_2 - c_3 x_1, \quad \sigma_{22} = c_4 x_2^3 - c_5, \quad \sigma_{12} = c_6 x_1 x_2^2 + c_7 x_1^2 x_2 - c_8,$$

where c_i are constants. Assuming that the body forces are zero, determine the conditions on the constants so that the stress field is in equilibrium and satisfies the compatibility equations.

7.10 Express the strain energy for a linear isotropic body in terms of the (a) strain components and (b) stress components.

7.11 A rigid uniform member ABC of length L, pinned at A and supported by linear elastic springs, each of stiffness k, at B and C, is shown in Fig. P7.11. Find the total strain energy of the system when the point C is displaced vertically down by the amount u_C.

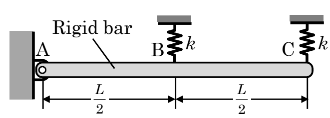

Fig. P7.11

7.12 Repeat Problem **7.11** when the springs are nonlinearly elastic, with the force deflection relationship, $F = ku^2$, where k is a constant.

7.13 Consider the equations of motion of 2-D elasticity (in the x- and z-coordinates) in the absence of body forces:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = \rho_0 \frac{\partial^2 u_x}{\partial t^2}$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = \rho_0 \frac{\partial^2 u_y}{\partial t^2}$$

For a beam of uniform height h and width b, integrate the preceding equations with respect to y from $-h/2$ to $h/2$, and express the results in terms of the stress resultants N and V defined in Eq. (7.3.28). Use the following boundary conditions:

$$\sigma_{xy}(x, h/2) - \sigma_{xy}(x, -h/2) = f(x)/b, \quad \sigma_{xy}(x, h/2) + \sigma_{xy}(x, -h/2) = 0,$$

$$\sigma_{yy}(x, -h/2) = 0, \quad b\sigma_{yy}(x, h/2) = q$$

Next, multiply the first equation of motion with y and integrate it with respect to y from $-h/2$ to $h/2$, and express the results in terms of the stress resultants M and V defined in Eq. (7.3.28).

7.14 For the plane elasticity problems shown in Figs. P7.14(a)-(d), write the boundary conditions and classify them into type I, type II, or type III.

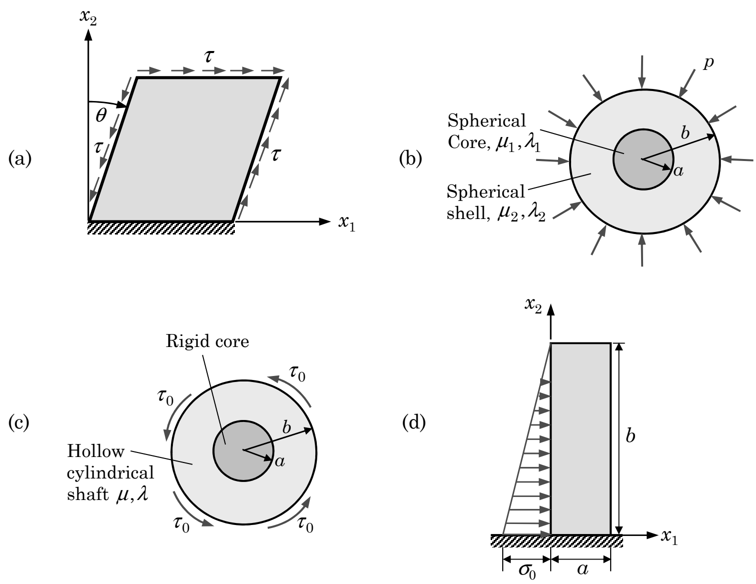

Fig. P7.14

Clapeyron's, Betti's, and Maxwell's Theorems

7.15 Consider a cantilever beam of length L, constant bending stiffness EI, and with right end ($x = L$) fixed, as shown in Fig. P7.15. If the left end ($x = 0$) is subjected to a moment M_0, use Clapeyron's theorem to determine the rotation (in the direction of the moment) at $x = 0$.

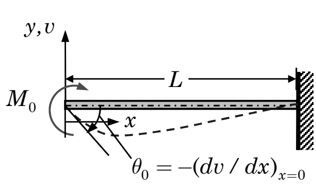

Fig. P7.15

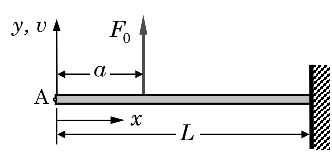

Fig. P7.16

7.16 Consider a cantilever beam of length L, constant bending stiffness EI, and with the right end fixed, as shown in Fig. P7.16. If a point load F_0 is applied at a distance a from the free end, determine the deflection $v(a)$ using Clapeyron's theorem.

7.17 Determine the deflection at the midspan of a cantilever beam subjected to a uniformly distributed load q_0 throughout the span and a point load F_0 at the free end, as shown in Fig. P7.17. Use Maxwell's theorem and superposition.

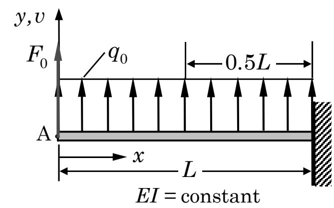

Fig. P7.17

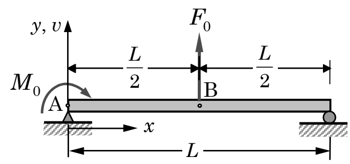

Fig. P7.18

7.18 Consider a simply supported beam of length L subjected to a concentrated load F_0 at the midspan and a bending moment M_0 at the left end, as shown in Fig. P7.18. Verify that Betti's theorem holds.

7.19 Use the reciprocity theorem to determine the deflection $v_c = v(0)$ at the center of a simply supported circular plate under asymmetric loading (see Fig. P7.19):

$$q(r, \theta) = q_0 + q_1 \frac{r}{a} \cos\theta.$$

The deflection $v(r)$ due to a point load F_0 at the center of a simply supported circular plate is

$$v(r) = \frac{F_0 a^2}{16\pi D}\left[\left(\frac{3+\nu}{1+\nu}\right)\left(1 - \frac{r^2}{a^2}\right) + 2\left(\frac{r}{a}\right)^2 \log\left(\frac{r}{a}\right)\right],$$

where $D = Eh^3/[12(1-\nu^2)]$ and h is the plate thickness.

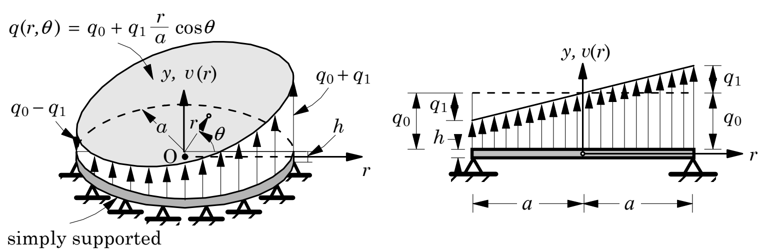

Fig. P7.19

7.20 Use the reciprocity theorem to determine the center deflection $v_c = v(0)$ of a simply supported circular plate under loading $q(r) = q_0(1 - r/a)$.

7.21 Use the reciprocity theorem to determine the center deflection $v_c = v(0)$ of a clamped circular plate under loading $q(r) = q_0(1 - r/a)$. The deflection due to a point load F_0 at the center of a clamped circular plate is given in Eq. (7.4.21).

7.22 Determine the center deflection $v_c = v(0)$ of a clamped circular plate subjected to a point load F_0 at a distance b from the center (and for some θ) using the reciprocity theorem.

7.23 Rewrite Eq. (7.3.10) in a form suitable for direct integration and obtain the solution given in Eq. (7.3.13). *Hint:* Note that $\frac{dU}{dR} + \frac{2U}{R} = \frac{1}{R^2}\frac{d}{dR}\left(R^2 U\right)$.

Solution of Elasticity Problems

7.24 Verify that the compatibility equation (3.7.4) takes the form

$$\varepsilon_{\alpha\alpha,\beta\beta} - \varepsilon_{\alpha\beta,\alpha\beta} = 0 \quad (\alpha, \beta = 1, 2), \tag{1}$$

or, in terms of stress components for the plane stress case,

$$\nabla^2 \sigma_{\alpha\alpha} = -(1+\nu)\, f_{\alpha,\alpha}. \tag{2}$$

7.25 Rewrite Eq. (7.5.17) in a form suitable for direct integration and obtain the solution given in Eq. (7.5.21). *Hint:* Note that $\frac{1}{r}\frac{dU}{dr} - \frac{U}{r^2} = \frac{d}{dr}\left(\frac{U}{r}\right)$.

7.26 Show that the solution to the differential equation for $G(r)$ in Eq. (6) is indeed given by the first equation in Eq. (7) of Example 7.5.7. *Hint*: Note that (verify to yourself)

$$\frac{d^2G}{dr^2} + \frac{1}{r}\frac{dG}{dr} = \frac{1}{r}\frac{d}{dr}\left(r\frac{dG}{dr}\right); \quad \int r\,\ln r\,dr = \frac{r^2}{2}\left(\ln r - \frac{1}{2}\right).$$

7.27 Show that the solution to the differential equation for $F(r)$ in Eq. (6) is indeed given by the second equation in Eq. (7) of Example 7.5.7. *Hint*: Note that (verify to yourself)

$$\frac{d^2F}{dr^2} + \frac{1}{r}\frac{dF}{dr} - \frac{4F}{r^2} = \frac{1}{r^3}\frac{d}{dr}\left(r^3\frac{dF}{dr} - 2r^2F\right).$$

7.28 The only nonzero stress in a prismatic bar of length L, made of an isotropic material (E and ν), is $\sigma_{11} = -M_0 x_3/I$, where M_0 is the bending moment and I is the moment inertia about the x_2-axis, respectively. Determine the three-dimensional displacement field. Eliminate the rigid-body translations and rotations requiring that $\mathbf{u} = \mathbf{0}$ and $\mathbf{\Omega} = \mathbf{0}$ at $\mathbf{x} = \mathbf{0}$.

7.29 A solid circular cylindrical body of radius a and height h is placed between two rigid plates, as shown in Fig. P7.29. The plate at B is held stationary and the plate at A is subjected to a downward displacement of δ. Using a suitable coordinate system, write the boundary conditions for the two cases: (a) When the cylindrical object is bonded to the plates at A and B. (b) When the plates at A and B are frictionless.

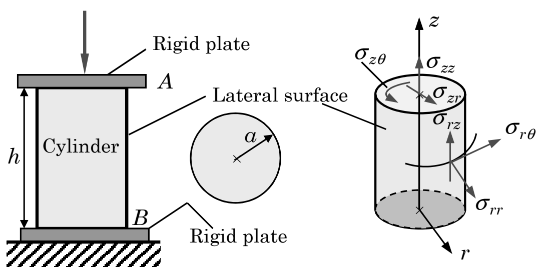

Fig. P7.29

7.30 The lateral surface of a homogeneous, isotropic, solid circular cylinder of radius a, length L, and mass density ρ is bonded to a rigid surface. Assuming that the ends of the cylinder at $z = 0$ and $z = L$ are traction-free (see Fig. P7.30), determine the displacement and stress fields in the cylinder due to its own weight.

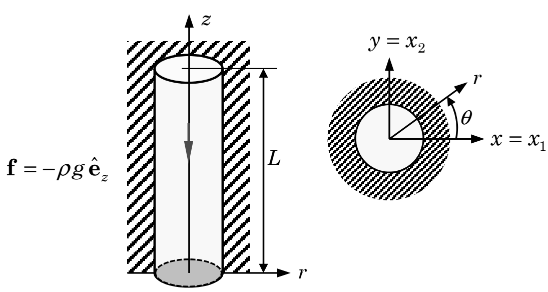

Fig. P7.30

7.31 An external hydrostatic pressure of magnitude p is applied to the surface of a spherical body of radius b with a concentric *rigid* spherical inclusion of radius a, as shown in Fig. P7.31. Determine the displacement and stress fields in the spherical body. Using the stress field obtained, determine the stresses at the surface of a rigid inclusion in an infinite elastic medium.

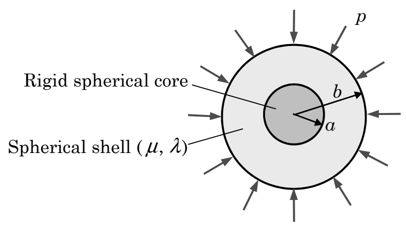

Fig. P7.31

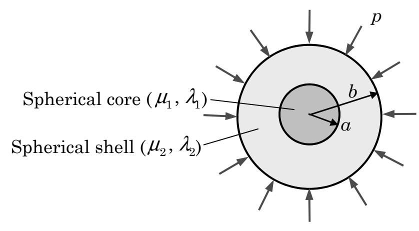

Fig. P7.32

7.32 Consider the concentric spheres shown in Fig. P7.32. Suppose that the core is elastic and the outer shell is subjected to external pressure p (both are linearly elastic). Assuming Lamé constants of μ_1 and λ_1 for the core and μ_2 and λ_2 for the outer shell, and that the interface is perfectly bonded at $r = a$, determine the displacements of the core as well as for the shell.

7.33 Consider a long hollow circular shaft with a *rigid* internal core (a cross section of the shaft is shown in Fig. P7.33). Assuming that the inner surface of the shaft at $r = a$ is perfectly bonded to the rigid core and the outer boundary at $r = b$ is subjected to a uniform *shearing* traction of magnitude τ_0, find the displacement and stress fields in the problem.

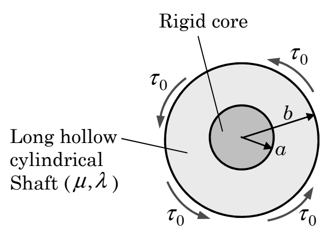

Fig. P7.33

Airy Stress Function

7.34 For the plane stress field

$$\sigma_{xx} = cxy, \quad \sigma_{xy} = 0.5c(h^2 - y^2), \quad \sigma_{yy} = 0,$$

where c and h are constants, (a) show that it is in equilibrium under a zero body force, and (b) find an Airy stress function $\Phi(x, y)$ corresponding to it.

7.35 In cylindrical coordinates, we assume that the body force vector $\mathbf{f}$ is derivable from the scalar potential $V_f(r, \theta)$:

$$\mathbf{f} = -\boldsymbol{\nabla} V_f \quad \left(f_r = -\frac{\partial V_f}{\partial r}, \quad f_\theta = -\frac{1}{r}\frac{\partial V_f}{\partial \theta} \right), \tag{1}$$

and define the Airy stress function $\Phi(r,\theta)$ such that

$$\begin{aligned}\sigma_{rr} &= \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} + V_f,\\ \sigma_{\theta\theta} &= \frac{\partial^2 \Phi}{\partial r^2} + V_f,\\ \sigma_{r\theta} &= -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \Phi}{\partial \theta}\right).\end{aligned} \tag{2}$$

Show that this choice trivially satisfies the equations of equilibrium

$$\begin{aligned}\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) + f_r = 0\,,\\ \frac{\partial \sigma_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{2\sigma_{r\theta}}{r} + f_\theta = 0.\end{aligned} \tag{3}$$

The tensor form of the compatibility condition in Eq. (7.5.33) is invariant.

7.36 Interpret the stress field obtained with the Airy stress function in Eq. (7.5.42) when all constants except c_3 are zero. Use the domain shown in Fig. 7.5.6 to sketch the stress field.

7.37 Interpret the following stress field obtained in Example 7.5.5 using the domain shown in Fig. 7.5.6:

$$\sigma_{xx} = 6c_{10}xy, \quad \sigma_{yy} = 0, \quad \sigma_{xy} = -3c_{10}y^2.$$

Assume that c_{10} is a positive constant.

7.38 Compute the stress field associated with the Airy stress function

$$\Phi(x,y) = Ax^5 + Bx^4y + Cx^3y^2 + Dx^2y^3 + Exy^4 + Fy^5.$$

Interpret the stress field for the case in which constants A, B, and C are zero. Use the rectangular domain shown in Fig. P7.38 to sketch the stress field on its boundaries.

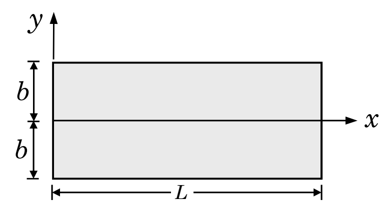

Fig. P7.38

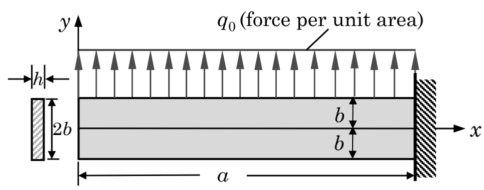

Fig. P7.39

7.39 Determine the Airy stress function for the stress field of the beam shown in Fig. P7.39 and evaluate the stress field.

7.40 Investigate what problem is solved by the Airy stress function

$$\Phi = \frac{3A}{4b}\left(xy - \frac{xy^3}{3b^2}\right) + \frac{B}{4b}y^2,$$

where A and B are constants. Use the domain in Fig. P7.38 to sketch the stress field.

7.41 Show that the Airy stress function

$$\Phi(x,y) = \frac{q_0}{8b^3}\left[x^2\left(y^3 - 3b^2y + 2b^3\right) - \frac{1}{5}y^3\left(y^2 - 2b^2\right)\right]$$

satisfies the compatibility condition. Determine the stress field and find what problem it corresponds to when applied to the region $-b \le y \le b$ and $x = 0, L$ (see Fig. P7.38).

7.42 The thin cantilever beam shown in Fig. P7.42 is subjected to a uniform shearing traction of magnitude τ_0 along its upper surface. Determine if the Airy stress function

$$\Phi(x,y) = \frac{\tau_0}{4}\left(xy - \frac{xy^2}{b} - \frac{xy^3}{b^2} + \frac{ay^2}{b} + \frac{ay^3}{b^2}\right)$$

satisfies the compatibility condition and stress boundary conditions of the problem.

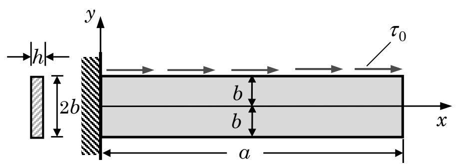

Fig. P7.42

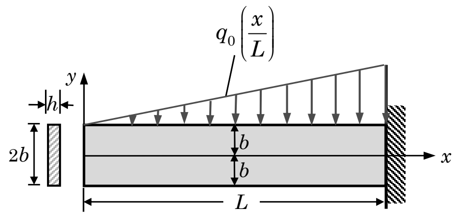

Fig. P7.43

7.43 Consider the problem of a cantilever beam carrying a uniformly varying distributed transverse load, as shown in Fig. P7.43. The following Airy stress function is suggested (explain the terms to yourself):

$$\Phi(x,y) = Axy + Bx^3 + Cx^3y + Dxy^3 + Ex^3y^3 + Fxy^5.$$

Determine each of the constants and find the stress field.

7.44 The curved beam shown in Fig. P7.44 is curved along a circular arc. The beam is fixed at the upper end and it is subjected at the lower end to a distribution of tractions statically equivalent to a force per unit thickness $\mathbf{P} = -P\hat{\mathbf{e}}_1$. Assume that the beam is in a state of plane strain/stress. Show that an Airy stress function of the form

$$\Phi(r) = \left(Ar^3 + \frac{B}{r} + C\,r\log r\right)\sin\theta$$

provides an approximate solution to this problem and solve for the values of the constants A, B, and C.

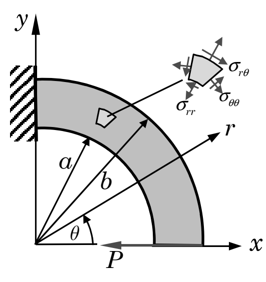

Fig. P7.44

7.45 Determine the stress field in a semi-infinite plate due to a normal load, f_0 force/unit length, acting on its edge, as shown in Fig. P7.45. Use the following Airy stress function (that satisfies the compatibility condition $\nabla^4\Phi = 0$):

$$\Phi(r,\theta) = A\theta + Br^2\theta + Cr\theta\sin\theta + Dr\theta\cos\theta,$$

where A, B, C, and D are constants [see Eq. (7.5.40) for the definition of stress components in terms of the Airy stress function Φ]. Neglect the body forces (i.e., $V_f = 0$). *Hint:* Stresses must be single-valued. Determine the constants using the boundary conditions of the problem.

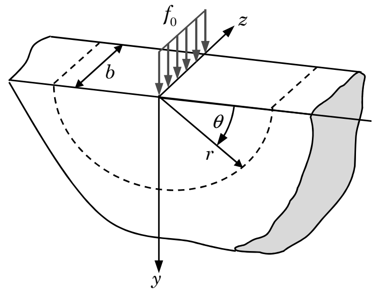

Fig. P7.45

Torsion of Cylindrical Members

7.46 Show that the resultant forces in the three coordinate directions on the end surface (i.e., $z = L$ face) are zero. Also show that the resultant moments about the x- and y-axes on the end surface are also zero.

7.47 Use the warping function $\psi(x, y) = kxy$, where k is a constant, to determine the cross section for which it is the solution. Determine the value of k in terms of the geometric parameters of the cross section and evaluate stresses in terms of these parameters and μ.

7.48 Consider a cylindrical member with the equilateral triangular cross section shown in Fig. P7.48. Show that the exact solution for the problem can be obtained and that the twist per unit length θ and stresses σ_{xz} and σ_{yz} are given by

$$\theta = \frac{5\sqrt{3}T}{27\mu a^4}, \quad \sigma_{xz} = \frac{\mu\theta}{a}(x - a)y, \quad \sigma_{yz} = \frac{\mu\theta}{2a}(x^2 + 2ax - y^2).$$

Hint: First write the equations for the three sides of the triangle (that is, $y = m\,x + c$, where m denotes the slope and c denotes the intercept), with the coordinate system shown in the figure, and then take the product of the three equations to construct the stress function. Also note that

$$\int_\Omega F(x, y)\,dx\,dy = \int_{-2a}^{a} \int_{-\frac{x+2a}{\sqrt{3}}}^{\frac{x+2a}{\sqrt{3}}} F(x, y)\,dy\,dx.$$

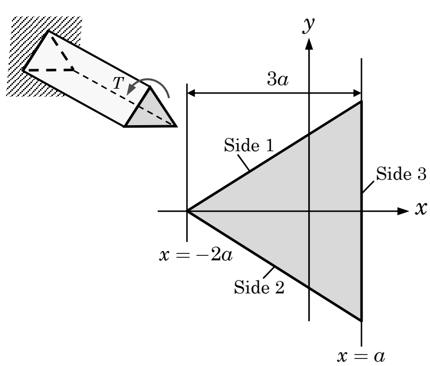

Fig. P7.48

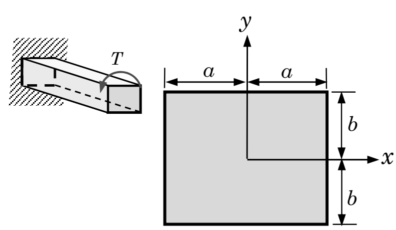

Fig. P7.49

7.49 Consider torsion of a cylindrical member with the rectangular cross section shown in Fig. P7.46. Determine if a function of the form

$$\Psi = A\left(\frac{x^2}{a^2} - 1\right)\left(\frac{y^2}{b^2} - 1\right),$$

where A is a constant, can be used as a Prandtl stress function.

7.50 From Example 7.5.8, we know that for circular cylindrical members we have $\psi = 0$. Use the cylindrical coordinate system to show that $\sigma_{zr} = 0$ and $\sigma_{z\alpha} = Tr/J$, where J is the polar moment of inertia.

Energy and Variational Methods

7.51 *Timoshenko beam theory.* Consider the displacement field

$$u_1(x,y) = y\phi(x), \quad u_2(x,y) = v(x), \quad u_3 = 0, \tag{1}$$

where $v(x)$ is the transverse deflection and ϕ is the rotation about the z-axis. Follow the developments of Section 7.3.4 and Example 7.6.1 (see Fig. 7.6.1) to develop the total potential energy functional

$$\Pi(u,w,\phi) = \frac{1}{2}\int_0^L \left[EI\left(\frac{d\phi}{dx}\right)^2 + GA\left(\frac{dv}{dx}+\phi\right)^2 - qv\right] dx - F_0 v(L) - M_0\phi(L),$$

where EI is the bending stiffness and GA is the shear stiffness (E and G are Young's modulus and shear modulus, respectively, A is the cross-sectional area, and I is the moment of inertia). Then derive the Euler equations and the natural boundary conditions of the Timoshenko beam theory.

7.52 Identify the bilinear and linear forms associated with the quadratic functional of the Timoshenko beam theory in Problem **7.51**.

7.53 The total potential energy functional for a membrane stretched over domain $\Omega \in \Re^2$ is given by

$$\Pi(u) = \int_\Omega \left\{\frac{T}{2}\left[\left(\frac{\partial u}{\partial x_1}\right)^2 + \left(\frac{\partial u}{\partial x_2}\right)^2\right] - fu\right\} d\mathbf{x},$$

where $u = u(x_1, x_2)$ denotes the transverse deflection of the membrane, T is the tension in the membrane, and $f = f(x_1, x_2)$ is the transversely distributed load on the membrane. Determine the governing differential equation and the permissible boundary conditions for the problem (that is, identify the essential and natural boundary conditions of the problem) using the principle of minimum total potential energy.

7.54 Use the results of Example 7.6.2 to obtain the deflection at the center of a clamped-clamped beam (length $2L$ and EI = constant) under uniform load of intensity q_0 and supported at the center by a linear elastic spring (k).

7.55 Use the results of Example 7.6.2 to obtain the deflection $v(L)$ and slopes $(-dv/dx)(L)$ and $(-dv/dx)(2L)$ under a point load F_0 for the beam shown in Fig. P7.55. It is sufficient to set up the three equations for the three unknowns.

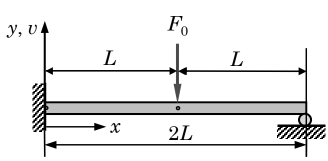

Fig. P7.55

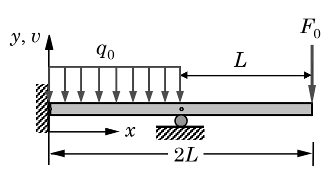

Fig. P7.56

7.56 Use the results of Example 7.6.2 to obtain the deflection $v(2L)$ and slopes at $x = L$ and $x = 2L$ for the beam shown in Fig. P7.56. It is sufficient to set up the three equations for the three unknowns.

7.57 Consider an arbitrary triangular, plane elastic domain Ω of thickness h and made of orthotropic material. Suppose that the body is free of body forces but subjected to tractions on its sides, as shown in Fig. P7.57. Use Catigliano's theorem I and derive a relationship between the point displacements and the corresponding forces at the vertices of the triangle.

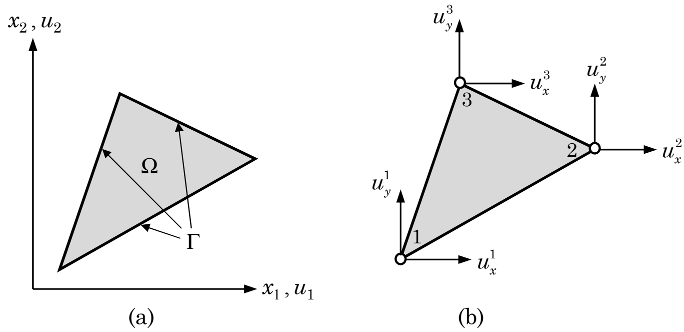

Fig. P7.57

7.58 Find a two-parameter Ritz approximation of the transverse deflection of a simply supported beam (constant EI) on an elastic foundation (modulus k) that is subjected to a uniformly distributed load, q_0. Use (a) algebraic and (b) trigonometric polynomials.

7.59 Establish the total potential energy functional in Eq. (2) of Example 7.6.6.

7.60 Determine a one-parameter Ritz approximation $U_1(x)$ of $u(x)$, which is governed by the equation (like the equation governing the Prandtl stress function over square cross section of 2 units)

$$-\Big(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\Big) = f_0 \quad \text{in a unit square,}$$

subjected to the boundary conditions

$$u(1,y) = u(x,1) = 0, \quad \frac{\partial u}{\partial x}\Big|_{(0,y)} = \frac{\partial u}{\partial y}\Big|_{(x,0)} = 0.$$

Take the origin of the coordinate system at the lower left corner of the unit square.

Hamilton's Principle

7.61 Find Beltrami–Michell equations for dynamic elasticity.

7.62 Extend Clapeyron's Theorem to the dynamic case by starting with the expression

$$\int_0^T (U - K)\, d\mathbf{x},$$

where K is the kinetic energy.

7.63 Consider a pendulum of mass m_1 with a flexible suspension, as shown in Fig. P7.63. The hinge of the pendulum is in a block of mass m_2, which can move up and down between the frictionless guides. The block is connected by a linear spring (of spring constant k) to an immovable support. The coordinate x is measured from the position of the block in which the system remains stationary. Derive the Euler–Lagrange equations of motion for the system.

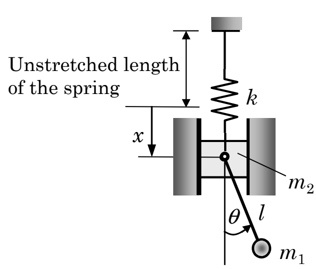

Fig. P7.63

7.64 A chain of total length L and mass m per unit length slides down from the edge of a smooth table. Assuming that the chain is rigid, find the equation of motion governing the chain (see Example 5.3.3).

7.65 Consider a cantilever beam supporting a lumped mass M at its end (J is the mass moment of inertia), as shown in Fig. P7.65. Derive the equations of motion and natural boundary conditions for the problem using the Euler–Bernoulli beam theory.

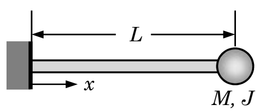

Fig. P7.65

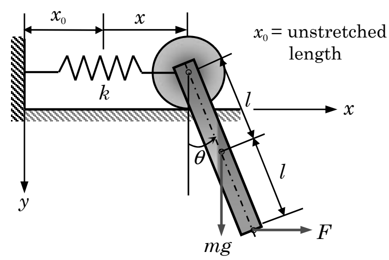

Fig. P7.66

7.66 Derive the equations of motion of the system shown in Fig. P7.66. Assume that the mass moment of inertia of the link about its mass center is $J = m\Omega^2$, where Ω is the radius of gyration.

7.67 Derive the equations of motion of the *Timoshenko beam theory*, starting with the displacement field (including the axial displacement, u):

$$u_1(x, y, t) = u(x, t) + y\phi(x, t), \quad u_2 = v(x, t), \quad u_3 = 0.$$

Assume that the beam is subjected to distributed axial load $f(x, t)$ and transverse load $q(x, t)$, and that the x-axis coincides with the geometric centroid.

7.68 Derive the equations of motion of the third-order *Reddy beam theory* based on the displacement field

$$\begin{aligned} u_1(x, y, t) &= u(x, t) + y\phi(x, t) - c_1 y^3\left(\phi + \frac{\partial v}{\partial x}\right) \\ u_2(x, y, t) &= v(x, t), \quad u_3 = 0, \end{aligned} \tag{1}$$

where $c_1 = 4/(3h^2)$. Assume that the beam is subjected to distributed axial load $f(x, t)$ and transverse load $q(x, t)$, and that the x-axis coincides with the geometric centroid.

7.69 Consider a uniform cross-sectional bar of length L, with the left end fixed and the right end connected to a rigid support via a linear elastic spring (with spring constant k), as shown in Fig. P7.69. Determine the first two natural axial frequencies of the bar using the Ritz method. *Hint:* The kinetic energy K and the strain energy U associated with the axial motion of the member are given by

$$K = \int_0^L \frac{\rho A}{2}\left(\frac{\partial u}{\partial t}\right)^2 dx, \quad U = \int_0^L \frac{EA}{2}\left(\frac{\partial u}{\partial x}\right)^2 dx + \frac{k}{2}[u(L, t)]^2. \tag{1}$$

Use Hamilton's principle to obtain the variational equation, and for periodic motion assume

$$u(x, t) = u_0(x)e^{i\omega t}, \quad i = \sqrt{-1}, \tag{2}$$

where ω is the frequency of natural vibration, and $u_0(x)$ is the amplitude, to reduce the variational statement to

$$0 = \int_0^L \left(\rho A\omega^2 u_0 \delta u_0 - EA\frac{du_0}{dx}\frac{d\delta u_0}{dx}\right) dx - ku_0(L)\delta u_0(L). \tag{3}$$

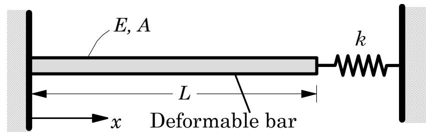

Fig. P7.69

7.70 Consider the equation

$$-\nabla^2 u = \lambda u, \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \Gamma, \tag{1}$$

where Ω is the triangular domain shown in Fig. P7.48 and Γ is its boundary. Equation (1) describes a nondimensional form of the equation governing the natural vibration of a triangular membrane of side a; mass density ρ; and tension T ($\lambda = \rho a^2 \omega^2 / T$, ω being the natural frequency of vibration). Determine the fundamental frequency (that is, determine λ) of vibration by using a one-parameter Ritz approximation of the problem.