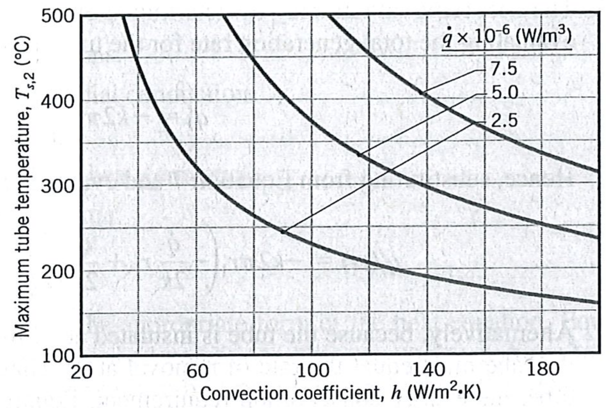


For each generation rate, the minimum value of h needed to maintain $T_{s,2} \leq 350°C$ may be determined from the figure.

3. The temperature distribution, Equation 7, may also be obtained by using the results of Appendix C. Applying a surface energy balance at $r = r_1$, with $q(r) = -\dot{q}\pi(r_2^2 - r_1^2)L$, $(T_{s,2} - T_{s,1})$ may be determined from Equation C.8 and the result substituted into Equation C.2 to eliminate $T_{s,1}$ and obtain the desired expression.

3.6 *Heat Transfer from Extended Surfaces*

The term *extended surface* is commonly used to depict an important special case involving heat transfer by conduction within a solid and heat transfer by convection (and/or radiation) from the boundaries of the solid. Until now, we have considered heat transfer from the boundaries of a solid to be in the same direction as heat transfer by conduction in the solid. In contrast, for an extended surface, the direction of heat transfer from the boundaries is perpendicular to the principal direction of heat transfer in the solid.

Consider a strut that connects two walls at different temperatures and across which there is fluid flow (Figure 3.12). With $T_1 > T_2$, temperature gradients in the x-direction sustain heat transfer by conduction in the strut. However, with $T_1 > T_2 > T_\infty$, there is concurrent heat

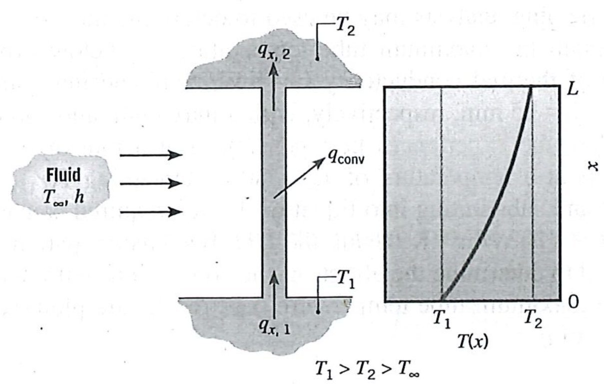


FIGURE 3.12 Combined conduction and convection in a structural element.

transfer by convection to the fluid, causing q_x, and hence the magnitude of the temperature gradient, $|dT/dx|$, to decrease with increasing x.

Although there are many different situations that involve such combined conduction–convection effects, the most frequent application is one in which an extended surface is used specifically to *enhance* heat transfer between a solid and an adjoining fluid. Such an extended surface is termed a *fin.*

Consider the plane wall of Figure 3.13*a* . If T_s is fixed, there are two ways in which the heat transfer rate may be increased. The convection coefficient h could be increased by increasing the fluid velocity, and/or the fluid temperature T_∞ could be reduced. However, there are many situations for which increasing h to the maximum possible value is either insufficient to obtain the desired heat transfer rate or the associated costs are prohibitive. Such costs are related to the blower or pump power requirements needed to increase h through increased fluid motion. Moreover, the second option of reducing T_∞ is often impractical. Examining Figure 3.13*b* , however, we see that there exists a third option. That is, the heat transfer rate may be increased by increasing the surface area across which the convection occurs. This may be done by employing *fins* that *extend* from the wall into the surrounding fluid. The thermal conductivity of the fin material can have a strong effect on the temperature distribution along the fin and therefore influences the degree to which the heat transfer rate is enhanced. Ideally, the fin material should have a large thermal conductivity to minimize temperature variations from its base to its tip. In the limit of infinite thermal conductivity, the entire fin would be at the temperature of the base surface, thereby providing the maximum possible heat transfer enhancement.

Examples of fin applications are easy to find. Consider the arrangement for cooling engine heads on motorcycles and lawn mowers or for cooling electric power transformers. Consider also the tubes with attached fins used to promote heat exchange between air and the working fluid of an air conditioner. Two common finned-tube arrangements are shown in Figure 3.14.

Different fin configurations are illustrated in Figure 3.15. A *straight fin* is any extended surface that is attached to a *plane wall.* It may be of uniform cross-sectional area, or its cross-sectional area may vary with the distance x from the wall. An *annular fin* is one that is circumferentially attached to a cylinder, and its cross section varies with radius from the wall of the cylinder. The foregoing fin types have rectangular cross sections, whose area may be expressed as a product of the fin thickness t and the width w for straight fins or the circumference $2\pi r$ for annular fins. In contrast a *pin fin,* or *spine,* is an extended surface of circular cross section. Pin fins may also be of uniform or nonuniform cross section. In any

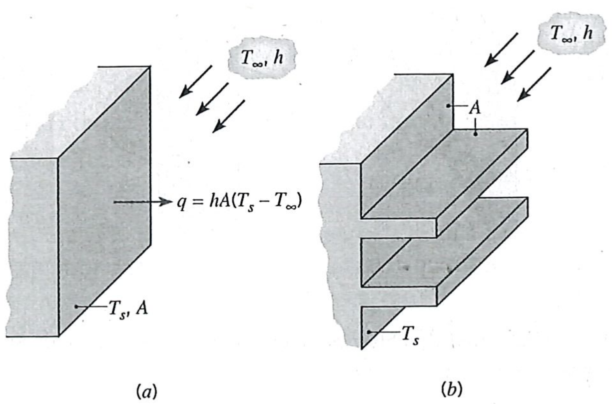


FIGURE 3.13 Use of fins to enhance heat transfer from a plane wall. (*a*) Bare surface. (*b*) Finned surface.

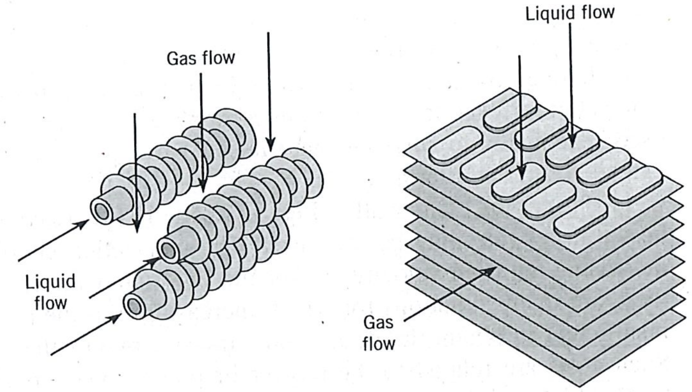


FIGURE 3.14 Schematic of typical finned-tube heat exchangers.

application, selection of a particular fin configuration may depend on space, weight, manufacturing, and cost considerations, as well as on the extent to which the fins reduce the surface convection coefficient and increase the pressure drop associated with flow over the fins.

3.6.1 A General Conduction Analysis

As engineers we are primarily interested in knowing the extent to which particular extended surfaces or fin arrangements could improve heat transfer from a surface to the surrounding fluid. To determine the heat transfer rate associated with a fin, we must first obtain the temperature distribution along the fin. As we have done for previous systems, we begin by performing an energy balance on an appropriate differential element. Consider the extended surface of Figure 3.16. The analysis is simplified if certain assumptions are made. We choose to assume one-dimensional conditions in the longitudinal (x-) direction, even though conduction within the fin is actually two-dimensional. The rate at which energy is convected to the fluid from any point on the fin surface must be balanced by the net rate at which energy reaches that point due to conduction in the transverse (y-, z-) direction. However, in practice the fin is thin, and temperature changes in the transverse

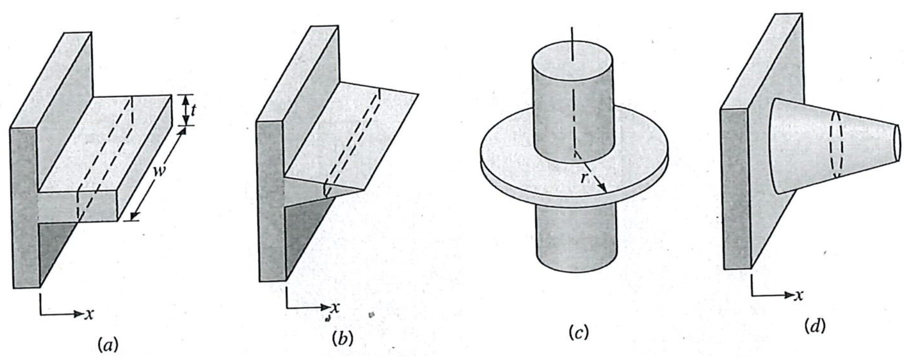


FIGURE 3.15 Fin configurations. (*a*) Straight fin of uniform cross section. (*b*) Straight fin of nonuniform cross section. (*c*) Annular fin. (*d*) Pin fin.

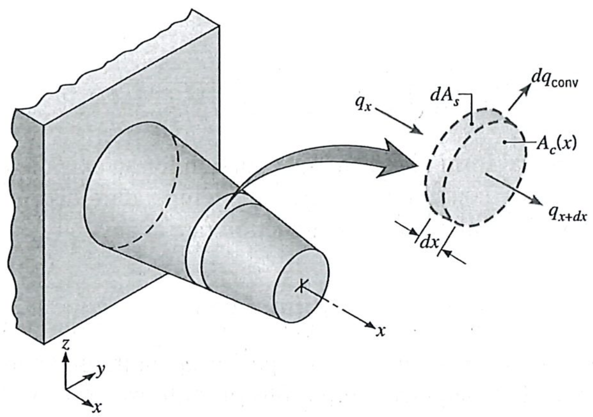


FIGURE 3.16 Energy balance for an extended surface.

direction within the fin are small compared with the temperature difference between the fin and the environment. Hence, we may assume that the temperature is uniform across the fin thickness, that is, it is only a function of x. We will consider steady-state conditions and also assume that the thermal conductivity is constant, that radiation from the surface is negligible, that heat generation effects are absent, and that the convection heat transfer coefficient h is uniform over the surface.

Applying the conservation of energy requirement, Equation 1.12c, to the differential element of Figure 3.16, we obtain

$$q_x = q_{x+dx} + dq_{\text{conv}} \tag{3.61}$$

From Fourier's law we know that

$$q_x = -kA_c \frac{dT}{dx} \tag{3.62}$$

where A_c is the *cross-sectional* area, which may vary with x. Since the conduction heat rate at $x + dx$ may be expressed as

$$q_{x+dx} = q_x + \frac{dq_x}{dx} dx \tag{3.63}$$

it follows that

$$q_{x+dx} = -kA_c \frac{dT}{dx} - k \frac{d}{dx}\left(A_c \frac{dT}{dx}\right) dx \tag{3.64}$$

The convection heat transfer rate may be expressed as

$$dq_{\text{conv}} = h\,dA_s(T - T_\infty) \tag{3.65}$$

where dA_s is the *surface* area of the differential element. Substituting the foregoing rate equations into the energy balance, Equation 3.61, we obtain

$$\frac{d}{dx}\left(A_c \frac{dT}{dx}\right) - \frac{h}{k}\frac{dA_s}{dx}(T - T_\infty) = 0$$

or

$$\frac{d^2T}{dx^2} + \left(\frac{1}{A_c}\frac{dA_c}{dx}\right)\frac{dT}{dx} - \left(\frac{1}{A_c}\frac{h}{k}\frac{dA_s}{dx}\right)(T - T_\infty) = 0 \tag{3.66}$$

This result provides a general form of the energy equation for an extended surface. Its solution for appropriate boundary conditions provides the temperature distribution, which may be used with Equation 3.62 to calculate the conduction rate at any x.

3.6.2 Fins of Uniform Cross-Sectional Area

To solve Equation 3.66 it is necessary to be more specific about the geometry. We begin with the simplest case of straight rectangular and pin fins of uniform cross section (Figure 3.17). Each fin is attached to a base surface of temperature $T(0) = T_b$ and extends into a fluid of temperature T_∞.

For the prescribed fins, A_c is a constant and $A_s = Px$, where A_s is the surface area measured from the base to x and P is the fin perimeter. Accordingly, with $dA_c/dx = 0$ and $dA_s/dx = P$, Equation 3.66 reduces to

$$\frac{d^2T}{dx^2} - \frac{hP}{kA_c}(T - T_\infty) = 0 \tag{3.67}$$

To simplify the form of this equation, we transform the dependent variable by defining an *excess temperature* θ as

$$\theta(x) \equiv T(x) - T_\infty \tag{3.68}$$

where, since T_∞ is a constant, $d\theta/dx = dT/dx$. Substituting Equation 3.68 into Equation 3.67, we then obtain

$$\frac{d^2\theta}{dx^2} - m^2\theta = 0 \tag{3.69}$$

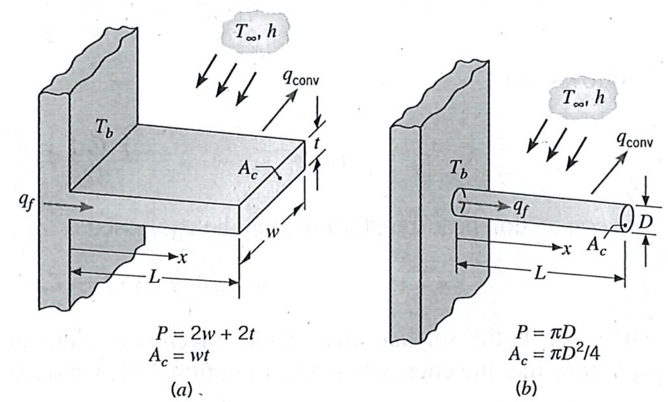


FIGURE 3.17 Straight fins of uniform cross section. (*a*) Rectangular fin. (*b*) Pin fin.

where

$$m^2 \equiv \frac{hP}{kA_c} \tag{3.70}$$

Equation 3.69 is a linear, homogeneous, second-order differential equation with constant coefficients. Its general solution is of the form

$$\theta(x) = C_1 e^{mx} + C_2 e^{-mx} \tag{3.71}$$

By substitution it may readily be verified that Equation 3.71 is indeed a solution to Equation 3.69.

To evaluate the constants C_1 and C_2 of Equation 3.71, it is necessary to specify appropriate boundary conditions. One such condition may be specified in terms of the temperature at the *base* of the fin ($x = 0$)

$$\theta(0) = T_b - T_\infty \equiv \theta_b \tag{3.72}$$

The second condition, specified at the fin tip ($x = L$), may correspond to one of four different physical situations.

The first condition, Case A, considers convection heat transfer from the fin tip. Applying an energy balance to a control surface about this tip (Figure 3.18), we obtain

$$hA_c[T(L) - T_\infty] = -kA_c \left.\frac{dT}{dx}\right|_{x=L}$$

or

$$h\theta(L) = -k \left.\frac{d\theta}{dx}\right|_{x=L} \tag{3.73}$$

That is, the rate at which energy is transferred to the fluid by convection from the tip must equal the rate at which energy reaches the tip by conduction through the fin. Substituting Equation 3.71 into Equations 3.72 and 3.73, we obtain, respectively,

$$\theta_b = C_1 + C_2 \tag{3.74}$$

and

$$h(C_1 e^{mL} + C_2 e^{-mL}) = km(C_2 e^{-mL} - C_1 e^{mL})$$

Solving for C_1 and C_2, it may be shown, after some manipulation, that

$$\frac{\theta}{\theta_b} = \frac{\cosh m(L - x) + (h/mk) \sinh m(L - x)}{\cosh mL + (h/mk) \sinh mL} \tag{3.75}$$

The form of this temperature distribution is shown schematically in Figure 3.18. Note that the magnitude of the temperature gradient decreases with increasing x. This trend is a consequence of the reduction in the conduction heat transfer $q_x(x)$ with increasing x due to continuous convection losses from the fin surface.

We are particularly interested in the amount of heat transferred from the entire fin. From Figure 3.18, it is evident that the fin heat transfer rate q_f may be evaluated in two

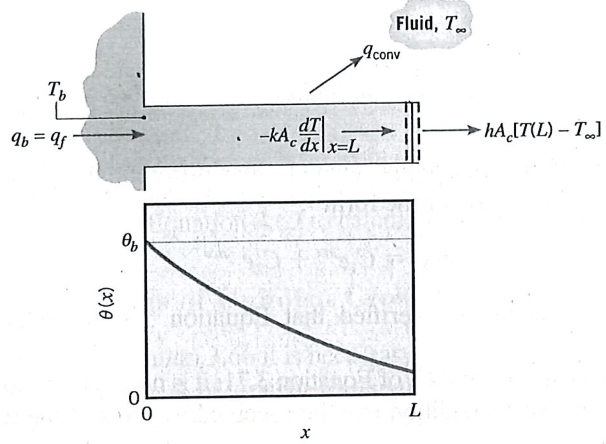


FIGURE 3.18 Conduction and convection in a fin of uniform cross section.

alternative ways, both of which involve use of the temperature distribution. The simpler procedure, and the one that we will use, involves applying Fourier's law at the fin base. That is,

$$q_f = q_b = -kA_c \frac{dT}{dx}\bigg|_{x=0} = -kA_c \frac{d\theta}{dx}\bigg|_{x=0} \tag{3.76}$$

Hence, knowing the temperature distribution, $\theta(x)$, q_f may be evaluated, giving

$$q_f = \sqrt{hPkA_c}\,\theta_b \frac{\sinh mL + (h/mk)\cosh mL}{\cosh mL + (h/mk)\sinh mL} \tag{3.77}$$

However, conservation of energy dictates that the rate at which heat is transferred by convection from the fin must equal the rate at which it is conducted through the base of the fin. Accordingly, the alternative formulation for q_f is

$$q_f = \int_{A_f} h[T(x) - T_\infty]\, dA_s$$

$$q_f = \int_{A_f} h\theta(x)\, dA_s \tag{3.78}$$

where A_f is the *total,* including the tip, *fin surface area.* Substitution of Equation 3.75 into Equation 3.78 would yield Equation 3.77.

The second tip condition, Case B, corresponds to the assumption that the convective heat loss from the fin tip is negligible, in which case the tip may be treated as adiabatic and

$$\frac{d\theta}{dx}\bigg|_{x=L} = 0 \tag{3.79}$$

Substituting from Equation 3.71 and dividing by m, we then obtain

$$C_1 e^{mL} - C_2 e^{-mL} = 0$$

Using this expression with Equation 3.74 to solve for C_1 and C_2 and substituting the results into Equation 3.71, we obtain

$$\frac{\theta}{\theta_b} = \frac{\cosh m(L - x)}{\cosh mL} \tag{3.80}$$

Using this temperature distribution with Equation 3.76, the fin heat transfer rate is then

$$q_f = \sqrt{hPkA_c}\,\theta_b \tanh mL \tag{3.81}$$

In the same manner, we can obtain the fin temperature distribution and heat transfer rate for Case C, where the temperature is prescribed at the fin tip. That is, the second boundary condition is $\theta(L) = \theta_L$, and the resulting expressions are of the form

$$\frac{\theta}{\theta_b} = \frac{(\theta_L/\theta_b)\sinh mx + \sinh m(L - x)}{\sinh mL} \tag{3.82}$$

$$q_f = \sqrt{hPkA_c}\,\theta_b \frac{\cosh mL - \theta_L/\theta_b}{\sinh mL} \tag{3.83}$$

The *very long fin,* Case D, is an interesting extension of these results. In particular, as $L \rightarrow \infty$, $\theta_L \rightarrow 0$ and it is easily verified that

$$\frac{\theta}{\theta_b} = e^{-mx} \tag{3.84}$$

$$q_f = \sqrt{hPkA_c}\,\theta_b \tag{3.85}$$

The foregoing results are summarized in Table 3.4. A table of hyperbolic functions is provided in Appendix B.1.

TABLE 3.4 Temperature distribution and heat loss for fins of uniform cross section

Case	Tip Condition ($x = L$)	Temperature Distribution θ/θ_b	Fin Heat Transfer Rate q_f
A	Convection heat transfer: $h\theta(L) = -kd\theta/dx\vert_{x=L}$	$\dfrac{\cosh m(L-x) + (h/mk)\sinh m(L-x)}{\cosh mL + (h/mk)\sinh mL}$ (3.75)	$M\dfrac{\sinh mL + (h/mk)\cosh mL}{\cosh mL + (h/mk)\sinh mL}$ (3.77)
B	Adiabatic: $d\theta/dx\vert_{x=L} = 0$	$\dfrac{\cosh m(L-x)}{\cosh mL}$ (3.80)	$M \tanh mL$ (3.81)
C	Prescribed temperature: $\theta(L) = \theta_L$	$\dfrac{(\theta_L/\theta_b)\sinh mx + \sinh m(L-x)}{\sinh mL}$ (3.82)	$M\dfrac{(\cosh mL - \theta_L/\theta_b)}{\sinh mL}$ (3.83)
D	Infinite fin ($L \rightarrow \infty$): $\theta(L) = 0$	e^{-mx} (3.84)	M (3.85)

$\theta \equiv T - T_\infty$ $\quad m^2 \equiv hP/kA_c$

$\theta_b = \theta(0) = T_b - T_\infty$ $\quad M \equiv \sqrt{hPkA_c}\,\theta_b$

EXAMPLE 3.9

A very long rod 5 mm in diameter has one end maintained at 100°C. The surface of the rod is exposed to ambient air at 25°C with a convection heat transfer coefficient of 100 W/m$^2\cdot$K.

1. Determine the temperature distributions along rods constructed from pure copper, 2024 aluminum alloy, and type AISI 316 stainless steel. What are the corresponding heat losses from the rods?
2. Estimate how long the rods must be for the assumption of *infinite length* to yield an accurate estimate of the heat loss.

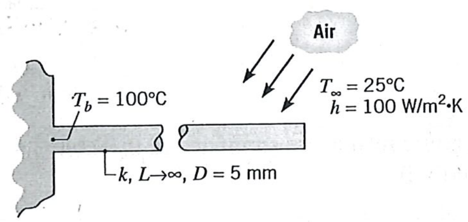
SOLUTION

Known: A long circular rod exposed to ambient air.

Find:

1. Temperature distribution and heat loss when rod is fabricated from copper, an aluminum alloy, or stainless steel.
2. How long rods must be to assume infinite length.

Schematic:



Assumptions:

1. Steady-state conditions.
2. One-dimensional conduction along the rod.
3. Constant properties.
4. Negligible radiation exchange with surroundings.
5. Uniform heat transfer coefficient.
6. Infinitely long rod.

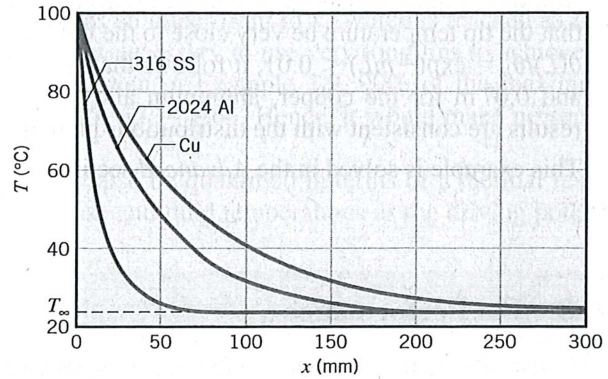
Properties: Table A.1, copper $[T = (T_b + T_\infty)/2 = 62.5°\text{C} \approx 335\ \text{K}]$: $k = 398$ W/m·K. Table A.1, 2024 aluminum (335 K): $k = 180$ W/m·K. Table A.1, stainless steel, AISI 316 (335 K): $k = 14$ W/m·K.

Analysis:

1. Subject to the assumption of an infinitely long fin, the temperature distributions are determined from Equation 3.84, which may be expressed as

$$T = T_\infty + (T_b - T_\infty)e^{-mx}$$

where $m = (hP/kA_c)^{1/2} = (4h/kD)^{1/2}$. Substituting for h and D, as well as for the thermal conductivities of copper, the aluminum alloy, and the stainless steel, respectively, the values of m are 14.2, 21.2, and 75.6 m^{-1}. The temperature distributions may then be computed and plotted as follows:



From these distributions, it is evident that there is little additional heat transfer associated with extending the length of the rod much beyond 50, 200, and 300 mm, respectively, for the stainless steel, the aluminum alloy, and the copper.

From Equation 3.85, the heat loss is

$$q_f = \sqrt{hPkA_c}\,\theta_b$$

Hence for copper,

$$q_f = \Big[100\ \text{W/m}^2\cdot\text{K} \times \pi \times 0.005\ \text{m}$$

$$\times\ 398\ \text{W/m}\cdot\text{K} \times \frac{\pi}{4}(0.005\ \text{m})^2\Big]^{1/2}(100 - 25)°\text{C}$$

$$= 8.3\ \text{W} \qquad \triangleleft$$

Similarly, for the aluminum alloy and stainless steel, respectively, the heat rates are $q_f = 5.6$ W and 1.6 W.

2. Since there is no heat loss from the tip of an infinitely long rod, an estimate of the validity of this approximation may be made by comparing Equations 3.81 and 3.85. To a satisfactory approximation, the expressions provide equivalent results if tanh $mL \geq 0.99$ or $mL \geq 2.65$. Hence a rod may be assumed to be infinitely long if

$$L \geq L_\infty \equiv \frac{2.65}{m} = 2.65\left(\frac{kA_c}{hP}\right)^{1/2}$$

For copper,

$$L_\infty = 2.65\left[\frac{398\ \text{W/m}\cdot\text{K} \times (\pi/4)(0.005\ \text{m})^2}{100\ \text{W/m}^2\cdot\text{K} \times \pi(0.005\ \text{m})}\right]^{1/2} = 0.19\ \text{m} \qquad \triangleleft$$

Results for the aluminum alloy and stainless steel are $L_\infty = 0.13$ m and $L_\infty = 0.04$ m, respectively.

Comments:

1. The foregoing results suggest that the fin heat transfer rate may accurately be predicted from the infinite fin approximation if $mL \geq 2.65$. However, if the infinite fin approximation is to accurately predict the temperature distribution $T(x)$, a larger value of mL would be required. This value may be inferred from Equation 3.84 and the requirement that the tip temperature be very close to the fluid temperature. Hence, if we require that $\theta(L)/\theta_b = \exp(-mL) < 0.01$, it follows that $mL > 4.6$, in which case $L_\infty \approx 0.33$, 0.23, and 0.07 m for the copper, aluminum alloy, and stainless steel, respectively. These results are consistent with the distributions plotted in part 1.
2. This example is solved in the *Advanced* section of *IHT*.

3.6.3 Fin Performance

Recall that fins are used to increase the heat transfer from a surface by increasing the effective surface area. However, the fin itself represents a conduction resistance to heat transfer from the original surface. For this reason, there is no assurance that the heat transfer rate will be increased through the use of fins. An assessment of this matter may be made by evaluating the *fin effectiveness* ε_f. It is defined as the *ratio of the fin heat transfer rate to the heat transfer rate that would exist without the fin.* Therefore

$$\varepsilon_f = \frac{q_f}{hA_{c,b}\theta_b} \tag{3.86}$$

where $A_{c,b}$ is the fin cross-sectional area at the base. In any rational design the value of ε_f should be as large as possible, and in general, the use of fins may rarely be justified unless $\varepsilon_f \gtrsim 2$.

Subject to any one of the four tip conditions that have been considered, the effectiveness for a fin of uniform cross section may be obtained by dividing the appropriate expression for q_f in Table 3.4 by $hA_{c,b}\theta_b$. Although the installation of fins will alter the surface convection coefficient, this effect is commonly neglected. Hence, assuming the convection coefficient of the finned surface to be equivalent to that of the unfinned base, it follows that, for the infinite fin approximation (Case D), the result is

$$\varepsilon_f = \left(\frac{kP}{hA_c}\right)^{1/2} \tag{3.87}$$

Several important trends may be inferred from this result. Obviously, fin effectiveness is enhanced by the choice of a material of high thermal conductivity. Aluminum alloys and copper come to mind. However, although copper is superior from the standpoint of thermal conductivity, aluminum alloys are the more common choice because of additional benefits related to lower cost and weight. Fin effectiveness is also enhanced by increasing the ratio of the perimeter to the cross-sectional area. For this reason, the use of *thin,* but closely spaced fins, is preferred, with the proviso that the fin gap not be reduced to a value for which flow between the fins is severely impeded, thereby reducing the convection coefficient.

Equation 3.87 also suggests that the use of fins can be better justified under conditions for which the convection coefficient h is small. Hence from Table 1.1 it is evident that the need for fins is stronger when the fluid is a gas rather than a liquid and when the surface heat transfer is by *free* convection. If fins are to be used on a surface separating a gas and a liquid, they are generally placed on the gas side, which is the side of lower convection coefficient. A common example is the tubing in an automobile radiator. Fins are applied to the outer tube surface, over which there is flow of ambient air (small h), and not to the inner surface, through which there is flow of water (large h). Note that, if $\varepsilon_f > 2$ is used as a criterion to justify the implementation of fins, Equation 3.87 yields the requirement that $(kP/hA_c) > 4$.

Equation 3.87 provides an upper limit to ε_f, which is reached as L approaches infinity. However, it is certainly not necessary to use very long fins to achieve near maximum heat transfer enhancement. As seen in Example 3.9, 99% of the maximum possible fin heat transfer rate is achieved for $mL = 2.65$. Hence, it would make no sense to extend the fins beyond $L = 2.65/m$.

Fin performance may also be quantified in terms of a thermal resistance. Treating the difference between the base and fluid temperatures as the driving potential, a *fin resistance* may be defined as

$$R_{t,f} = \frac{\theta_b}{q_f} \tag{3.88}$$

This result is extremely useful, particularly when representing a finned surface by a thermal circuit. Note that, according to the fin tip condition, an appropriate expression for q_f may be obtained from Table 3.4.

Dividing Equation 3.88 into the expression for the thermal resistance due to convection at the exposed base,

$$R_{t,b} = \frac{1}{hA_{c,b}} \tag{3.89}$$

and substituting from Equation 3.86, it follows that

$$\varepsilon_f = \frac{R_{t,b}}{R_{t,f}} \tag{3.90}$$

Hence the fin effectiveness may be interpreted as a ratio of thermal resistances, and to increase ε_f it is necessary to reduce the conduction/convection resistance of the fin. If the fin is to enhance heat transfer, its resistance must not exceed that of the exposed base.

Another measure of fin thermal performance is provided by the *fin efficiency* η_f. The maximum driving potential for convection is the temperature difference between the base ($x = 0$) and the fluid, $\theta_b = T_b - T_\infty$. Hence the maximum rate at which a fin could dissipate energy is the rate that would exist *if* the entire fin surface were at the base temperature. However, since any fin is characterized by a finite conduction resistance, a temperature gradient must exist along the fin and the preceding condition is an idealization. A logical definition of fin efficiency is therefore

$$\eta_f \equiv \frac{q_f}{q_{\max}} = \frac{q_f}{hA_f\theta_b} \tag{3.91}$$

where A_f is the surface area of the fin. For a straight fin of uniform cross section and an adiabatic tip, Equations 3.81 and 3.91 yield

$$\eta_f = \frac{M \tanh mL}{hPL\theta_b} = \frac{\tanh mL}{mL} \tag{3.92}$$

Referring to Table B.1, this result tells us that η_f approaches its maximum and minimum values of 1 and 0, respectively, as L approaches 0 and ∞.

In lieu of the somewhat cumbersome expression for heat transfer from a straight rectangular fin with an active tip, Equation 3.77, it has been shown that approximate, yet accurate, predictions may be obtained by using the adiabatic tip result, Equation 3.81, with a corrected fin length of the form $L_c = L + (t/2)$ for a rectangular fin and $L_c = L + (D/4)$ for a pin fin [14]. The correction is based on assuming equivalence between heat transfer from the actual fin with tip convection and heat transfer from a longer, hypothetical fin with an adiabatic tip. Hence, with tip convection, the fin heat rate may be approximated as

$$q_f = M \tanh mL_c \tag{3.93}$$

and the corresponding efficiency as

$$\eta_f = \frac{\tanh mL_c}{mL_c} \tag{3.94}$$

Errors associated with the approximation are negligible if (ht/k) or $(hD/2k) \lesssim 0.0625$ [15].

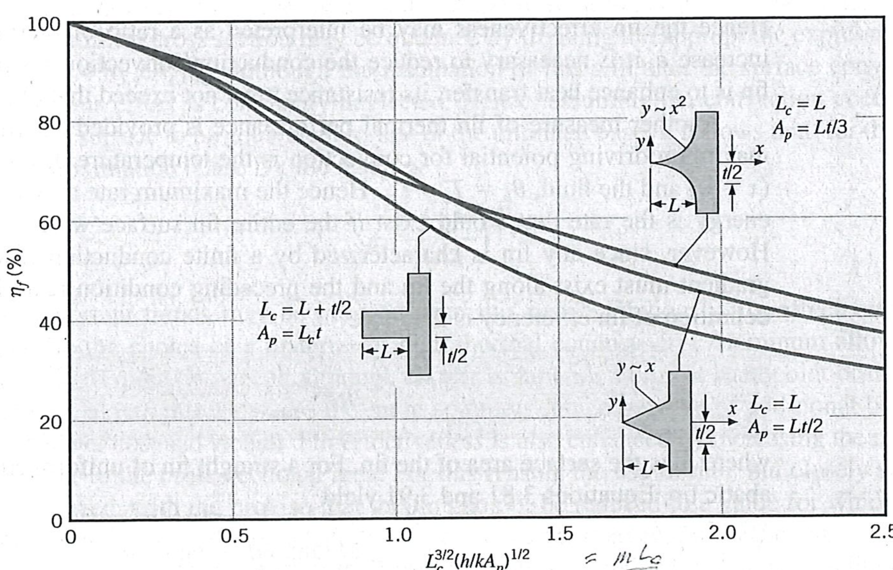
If the width of a rectangular fin is much larger than its thickness, $w \gg t$, the perimeter may be approximated as $P = 2w$, and

$$mL_c = \left(\frac{hP}{kA_c}\right)^{1/2} L_c = \left(\frac{2h}{kt}\right)^{1/2} L_c$$

Multiplying numerator and denominator by $L_c^{1/2}$ and introducing a corrected fin profile area, $A_p = L_c t$, it follows that

$$mL_c = \left(\frac{2h}{kA_p}\right)^{1/2} L_c^{3/2} \tag{3.95}$$

Hence, as shown in Figures 3.19 and 3.20, the efficiency of a rectangular fin with tip convection may be represented as a function of $L_c^{3/2}(h/kA_p)^{1/2}$.



$= \frac{mL_c}{\sqrt{2}}$

FIGURE 3.19 Efficiency of straight fins (rectangular, triangular, and parabolic profiles).

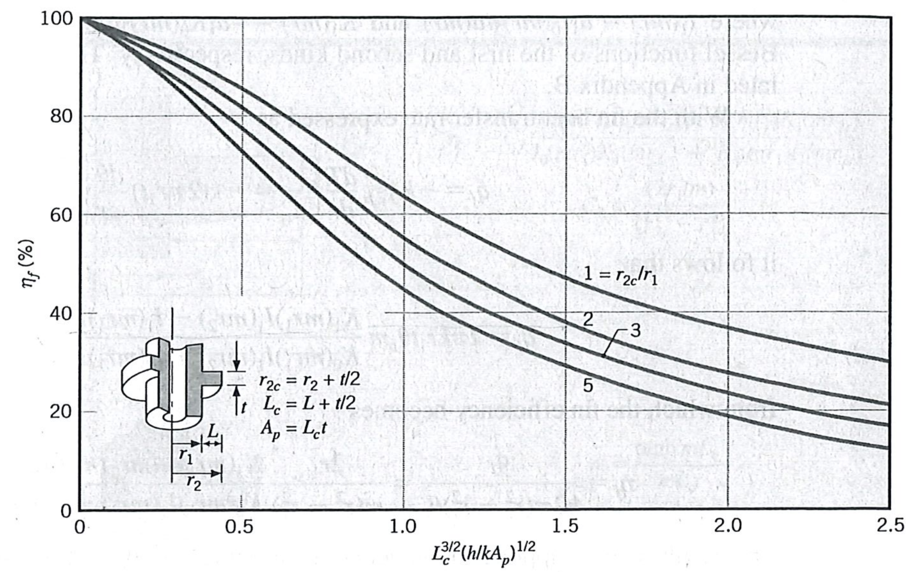


FIGURE 3.20 Efficiency of annular fins of rectangular profile.

3.6.4 Fins of Nonuniform Cross-Sectional Area

Analysis of fin thermal behavior becomes more complex if the fin is of nonuniform cross section. For such cases the second term of Equation 3.66 must be retained, and the solutions are no longer in the form of simple exponential or hyperbolic functions. As a special case, consider the annular fin shown in the inset of Figure 3.20. Although the fin thickness is uniform (t is independent of r), the cross-sectional area, $A_c = 2\pi r t$, varies with r. Replacing x by r in Equation 3.66 and expressing the surface area as $A_s = 2\pi(r^2 - r_1^2)$, the general form of the fin equation reduces to

$$\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr} - \frac{2h}{kt}(T - T_\infty) = 0$$

or, with $m^2 \equiv 2h/kt$ and $\theta \equiv T - T_\infty$,

$$\frac{d^2\theta}{dr^2} + \frac{1}{r}\frac{d\theta}{dr} - m^2\theta = 0$$

The foregoing expression is a *modified Bessel equation* of order zero, and its general solution is of the form

$$\theta(r) = C_1 I_0(mr) + C_2 K_0(mr)$$

where I_0 and K_0 are modified, zero-order Bessel functions of the first and second kinds, respectively. If the temperature at the base of the fin is prescribed, $\theta(r_1) = \theta_b$, and an adiabatic tip is presumed, $d\theta/dr|_{r_2} = 0$, C_1 and C_2 may be evaluated to yield a temperature distribution of the form

$$\frac{\theta}{\theta_b} = \frac{I_0(mr)K_1(mr_2) + K_0(mr)I_1(mr_2)}{I_0(mr_1)K_1(mr_2) + K_0(mr_1)I_1(mr_2)}$$

where $I_1(mr) = d[I_0(mr)]/d(mr)$ and $K_1(mr) = -d[K_0(mr)]/d(mr)$ are modified, first-order Bessel functions of the first and second kinds, respectively. The Bessel functions are tabulated in Appendix B.

With the fin heat transfer rate expressed as

$$q_f = -kA_{c,b}\left.\frac{dT}{dr}\right|_{r=r_1} = -k(2\pi r_1 t)\left.\frac{d\theta}{dr}\right|_{r=r_1}$$

it follows that

$$q_f = 2\pi k r_1 t\theta_b m \frac{K_1(mr_1)I_1(mr_2) - I_1(mr_1)K_1(mr_2)}{K_0(mr_1)I_1(mr_2) + I_0(mr_1)K_1(mr_2)}$$

from which the fin efficiency becomes

$$\eta_f = \frac{q_f}{h2\pi(r_2^2 - r_1^2)\theta_b} = \frac{2r_1}{m(r_2^2 - r_1^2)}\frac{K_1(mr_1)I_1(mr_2) - I_1(mr_1)K_1(mr_2)}{K_0(mr_1)I_1(mr_2) + I_0(mr_1)K_1(mr_2)} \qquad (3.96)$$

This result may be applied for an active (convecting) tip, if the tip radius r_2 is replaced by a corrected radius of the form $r_{2c} = r_2 + (t/2)$. Results are represented graphically in Figure 3.20.

Knowledge of the thermal efficiency of a fin may be used to evaluate the fin resistance, where, from Equations 3.88 and 3.91, it follows that

$$R_{t,f} = \frac{1}{hA_f\eta_f} \qquad (3.97)$$

Expressions for the efficiency and surface area of several common fin geometries are summarized in Table 3.5. Although results for the fins of uniform thickness or diameter

TABLE 3.5 Efficiency of common fin shapes

Straight Fins

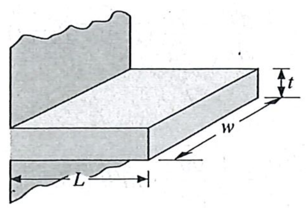
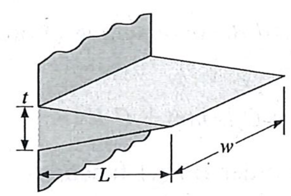
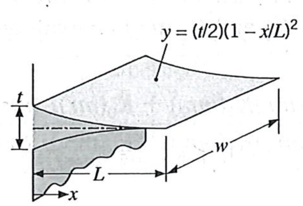
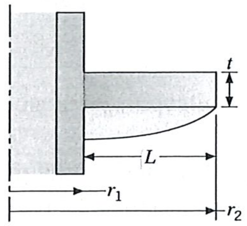
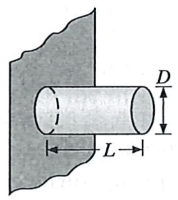
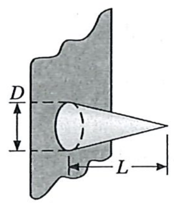
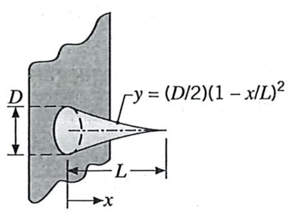
Rectangular[a] $A_f = 2wL_c$ $L_c = L + (t/2)$ $A_p = tL$		$\eta_f = \dfrac{\tanh mL_c}{mL_c}$	(3.94)
Triangular[a] $A_f = 2w[L^2 + (t/2)^2]^{1/2}$ $A_p = (t/2)L$		$\eta_f = \dfrac{1}{mL}\dfrac{I_1(2mL)}{I_0(2mL)}$	(3.98)
Parabolic[a] $A_f = w[C_1L +$ $(L^2/t)\ln(t/L + C_1)]$ $C_1 = [1 + (t/L)^2]^{1/2}$ $A_p = (t/3)L$		$\eta_f = \dfrac{2}{[4(mL)^2 + 1]^{1/2} + 1}$	(3.99)

TABLE 3.5 *Continued*

Description	Figure	Fin Efficiency	Eq.
Annular Fin			
Rectangular[a] $A_f = 2\pi\,(r_{2c}^2 - r_1^2)$ $r_{2c} = r_2 + (t/2)$ $V = \pi\,(r_2^2 - r_1^2)t$		$\eta_f = C_2 \dfrac{K_1(mr_1)I_1(mr_{2c}) - I_1(mr_1)K_1(mr_{2c})}{I_0(mr_1)K_1(mr_{2c}) + K_0(mr_1)I_1(mr_{2c})}$ $C_2 = \dfrac{(2r_1/m)}{(r_{2c}^2 - r_1^2)}$	(3.96)
Pin Fins			
Rectangular[b] $A_f = \pi D L_c$ $L_c = L + (D/4)$ $V = (\pi D^2/4)L$		$\eta_f = \dfrac{\tanh mL_c}{mL_c}$	(3.100)
Triangular[b] $A_f = \dfrac{\pi D}{2}[L^2 + (D/2)^2]^{1/2}$ $V = (\pi/12)D^2L$		$\eta_f = \dfrac{2}{mL}\dfrac{I_2(2mL)}{I_1(2mL)}$	(3.101)
Parabolic[b] $A_f = \dfrac{\pi L^3}{8D}\{C_3C_4 - \dfrac{L}{2D}\ln[(2DC_4/L) + C_3]\}$ $C_3 = 1 + 2(D/L)^2$ $C_4 = [1 + (D/L)^2]^{1/2}$ $V = (\pi/20)D^2 L$		$\eta_f = \dfrac{2}{[4/9(mL)^2 + 1]^{1/2} + 1}$	(3.102)

[a] $m = (2h/kt)^{1/2}$.
[b] $m = (4h/kD)^{1/2}$.

were obtained by assuming an adiabatic tip, the effects of convection may be treated by using a corrected length (Equations 3.94 and 3.100) or radius (Equation 3.96). The triangular and parabolic fins are of nonuniform thickness that reduces to zero at the fin tip.

Expressions for the profile area, A_p, or the volume, V, of a fin are also provided in Table 3.5. The volume of a straight fin is simply the product of its width and profile area, $V = wA_p$.

Fin design is often motivated by a desire to minimize the fin material and/or related manufacturing costs required to achieve a prescribed cooling effectiveness. Hence, a straight *triangular* fin is attractive because, for equivalent heat transfer, it requires much less volume (fin material) than a rectangular profile. In this regard, heat dissipation per unit volume, $(q/V)_f$, is largest for a *parabolic* profile. However, since $(q/V)_f$ for the parabolic profile is only slightly larger than that for a triangular profile, its use can rarely be justified in view of its larger manufacturing costs. The *annular* fin of rectangular profile is commonly used to enhance heat transfer to or from circular tubes.

3.6.5 Overall Surface Efficiency

In contrast to the fin efficiency η_f, which characterizes the performance of a single fin, the *overall surface efficiency* η_o characterizes an *array* of fins and the base surface to which they are attached. Representative arrays are shown in Figure 3.21, where S designates the fin pitch. In each case the overall efficiency is defined as

$$\eta_o = \frac{q_t}{q_{\max}} = \frac{q_t}{hA_t\theta_b} \tag{3.103}$$

where q_t is the total heat rate from the surface area A_t associated with both the fins and the exposed portion of the base (often termed the *prime* surface). If there are N fins in the array, each of surface area A_f, and the area of the prime surface is designated as A_b, the total surface area is

$$A_t = NA_f + A_b \tag{3.104}$$

The maximum possible heat rate would result if the entire fin surface, as well as the exposed base, were maintained at T_b.

The total rate of heat transfer by convection from the fins and the prime (unfinned) surface may be expressed as

$$q_t = N\eta_f hA_f\theta_b + hA_b\theta_b \tag{3.105}$$

where the convection coefficient h is assumed to be equivalent for the finned and prime surfaces and η_f is the efficiency of a single fin. Hence

$$q_t = h[N\eta_f A_f + (A_t - NA_f)]\theta_b = hA_t\left[1 - \frac{NA_f}{A_t}(1 - \eta_f)\right]\theta_b \tag{3.106}$$

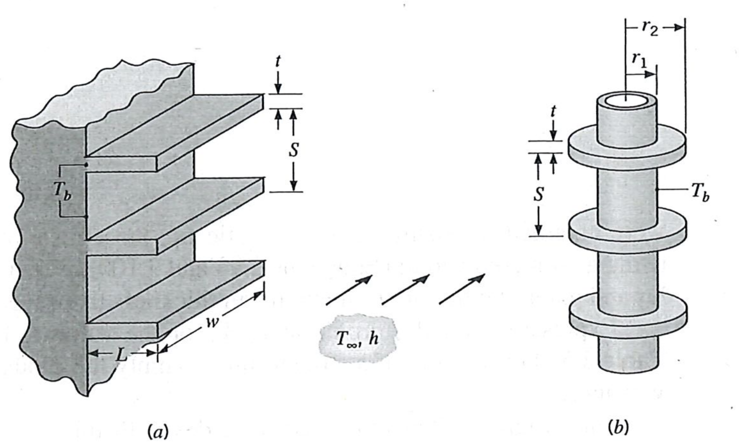


FIGURE 3.21 Representative fin arrays. (*a*) Rectangular fins. (*b*) Annular fins.

Substituting Equation (3.106) into (3.103), it follows that

$$\eta_o = 1 - \frac{NA_f}{A_t}(1 - \eta_f) \tag{3.107}$$

From knowledge of η_o, Equation 3.103 may be used to calculate the total heat rate for a fin array.

Recalling the definition of the fin thermal resistance, Equation 3.88, Equation 3.103 may be used to infer an expression for the thermal resistance of a fin array. That is,

$$R_{t,o} = \frac{\theta_b}{q_t} = \frac{1}{\eta_o h A_t} \tag{3.108}$$

where $R_{t,o}$ is an effective resistance that accounts for parallel heat flow paths by conduction/convection in the fins and by convection from the prime surface. Figure 3.22 illustrates the thermal circuits corresponding to the parallel paths and their representation in terms of an effective resistance.

If fins are machined as an integral part of the wall from which they extend (Figure 3.22*a*), there is no contact resistance at their base. However, more commonly, fins are manufactured separately and are attached to the wall by a metallurgical or adhesive joint. Alternatively, the attachment may involve a *press fit,* for which the fins are forced into slots machined on the wall material. In such cases (Figure 3.22*b*), there is a thermal contact resistance $R_{t,c}$, which

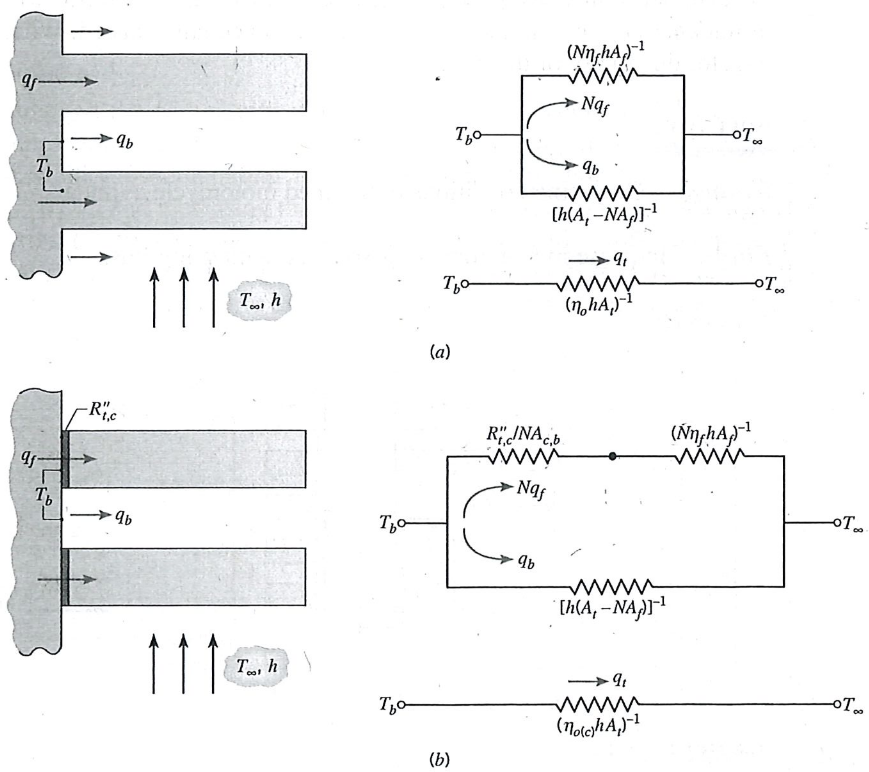


FIGURE 3.22 Fin array and thermal circuit. (*a*) Fins that are integral with the base. (*b*) Fins that are attached to the base.

may adversely influence overall thermal performance. An effective circuit resistance may again be obtained, where, with the contact resistance,

$$R_{t,o(c)} = \frac{\theta_b}{q_t} = \frac{1}{\eta_{o(c)} h A_t} \tag{3.109}$$

It is readily shown that the corresponding overall surface efficiency is

$$\eta_{o(c)} = 1 - \frac{NA_f}{A_t}\left(1 - \frac{\eta_f}{C_1}\right) \tag{3.110a}$$

where

$$C_1 = 1 + \eta_f h A_f (R''_{t,c}/A_{c,b}) \tag{3.110b}$$

In manufacturing, care must be taken to render $R_{t,c} \ll R_{t,f}$.

EXAMPLE 3.10

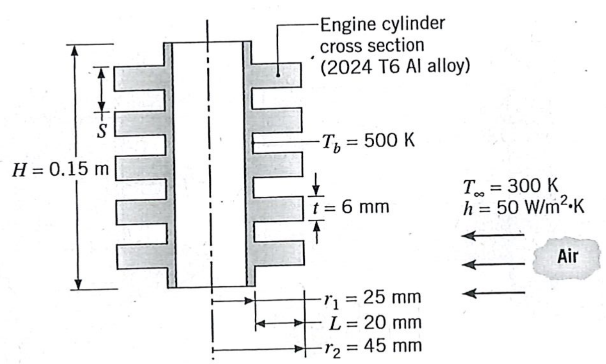
The engine cylinder of a motorcycle is constructed of 2024-T6 aluminum alloy and is of height $H = 0.15$ m and outside diameter $D = 50$ mm. Under typical operating conditions the outer surface of the cylinder is at a temperature of 500 K and is exposed to ambient air at 300 K, with a convection coefficient of 50 W/m$^2\cdot$K. Annular fins are integrally cast with the cylinder to increase heat transfer to the surroundings. Consider five such fins, which are of thickness $t = 6$ mm, length $L = 20$ mm, and equally spaced. What is the increase in heat transfer due to use of the fins?

SOLUTION

Known: Operating conditions of a finned motorcycle cylinder.

Find: Increase in heat transfer associated with using fins.

Schematic:



Assumptions:

1. Steady-state conditions.
2. One-dimensional radial conduction in fins.
3. Constant properties.