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A critical review on multiaxial fatigue assessments of metals

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Multiaxial fatigue analysis is categorized into five viewpoints, i.e. empirical formulas and modifications of the Coffin–Manson equation, application of stress or strain invariants, use of the space averages of stress or strain, critical plane approaches, and use of energy which has accumulated on the materials. Garud¹ (J. Test. Evaluations 1981, **9**, 165) reviewed the results of multiaxial fatigue researches proposed up to 1980 in chronological order and evaluated them. In this paper multiaxial fatigue researches which have been suggested after 1980 were classified into the five viewpoints and some problems which existed in those studies were examined. During these periods major progresses in multiaxial fatigue analysis are the consideration of anisotropy of materials and the suggestion of the energy method using Mohr's circles. Additionally, existing equations or parameters were modified to consider the mean stress, loading path, etc. There have also been some trials for the generalization of the existing theories.

(Keywords: fatigue; multiaxial fatigue; critical plane; energy)

The safety and durability of structures has become more important than before because the sudden failure of complex systems such as nuclear power plants, automobiles, aircraft and pressure vessels may cause many injuries, much financial loss and even environmental damage. Since many of these parts are subjected to repeated multiaxial loadings, fatigue evaluation becomes one of the major considerations in the design of structures. In general, applied loads are often complex, that is, the corresponding principal stresses are non-proportional, or whose directions change during a cycle of such loadings. Under such loadings, it is very difficult to define the fatigue behaviour of materials and structures. According to Lee,² different investigators reported different conclusions on complex (non-proportional, out-of-phase) multiaxial loadings.

Multiaxial fatigue assessments have been carried out with the help of an appropriate rule or criterion that reduces the complex multiaxial loadings to an equivalent uniaxial loading. Several researchers^{1–6} reviewed various theories which had been suggested up to the 1970s. From their evaluations, multiaxial fatigue assessments are affected by environments and material characteristics.

The field of multiaxial fatigue theories can be classified into five viewpoints, i.e. empirical formulas and modifications of the Coffin–Manson equation; application of stress or strain invariants; use of the space averages of stress or strain; critical plane approaches;

and use of energy which has accumulated on the materials under consideration. In this paper, multiaxial fatigue theories or criteria that have been suggested after 1980 are classified according to the five viewpoints chronologically and the validity region and limitations of those multiaxial fatigue theories are critically evaluated.

EMPIRICAL FORMULAS AND MODIFICATIONS OF THE COFFIN–MANSON EQUATION

As stated in Garud,¹ yield theories suggested by von Mises and Tresca are preferred in multiaxial fatigue analysis. Although the results of their applications are nonconservative, the yield theories have been used due to convenience. In order to solve such nonconservativeness of yield theories, there are some trials to analyse the multiaxial fatigue behaviour by using the additional factors which reflect the variation of fatigue properties due to multiaxial loading and environments, and some researches to simplify the criteria by using the stress amplitude directly.

High-cycle fatigue

Lee² proposed an equivalent stress criterion for the complex multiaxial fatigue of out-of-phase bending and torsion and compared it with other equivalent stress criteria by using discriminating specimens. Lee^{2,3} modified the ellipse quadrant of Gough⁷ to incorporate the phase difference between loadings and suggested the following Equation (1) and validated it with the experimental data of several materials.

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$$\sigma_{eq} = \sigma_a [1 + (bK/2t)^\alpha]^{1/\alpha}, \alpha = 2(1 + \beta \sin \phi), \quad (1)$$

where K is $2\tau_a/\sigma_a$, τ_a and σ_a are torsional stress and bending stress amplitude, b and t are bending and torsional fatigue strength for a given fatigue life, respectively, β is a material constant, and ϕ is the phase difference between bending and torsion. Lee⁸ also modified Equation (1) to consider the bending mean stress and verified the following equation with multiaxial bending and torsion tests of SM45C structural steel.

$$\sigma_{eq} = \sigma_a [1 + (bK/2t)^\alpha]^{1/\alpha} / (1 - (\sigma_m/\sigma_u)^n), \quad (2)$$

where σ_u is the tensile strength of material, σ_m is a bending mean stress, and n is an empirical constant between 1 and 2.

Lee and Chiang⁹ combined the equations of Findley *et al*¹⁰ and Lee³ into Equation (3) for the prediction of multiaxial fatigue life. However, as t/b has a value from 0.5 to 0.8^{3,11} (in general, 0.577), Equation (3) is similar to the Equation (1) suggested by Lee.³

$$\left(\frac{\sigma_a}{b}\right)^{b(1+\gamma \sin \phi)/t} + \left(\frac{\tau_a}{t}\right)^{2(1+\gamma \sin \phi)} = 1 \quad (3)$$

where γ is a material constant for consideration of material hardening under out-of-phase loading and ϕ is the phase difference between bending and torsion. Lee and Chiang⁹ modified Equation (3) into the following Equation (4) to consider the effects of shear mean stress, which were neglected by other researchers because the effects were so small.

$$\left(\frac{\sigma_a}{b}\right)^{b(1+\gamma \sin \phi)/t} + \left(\frac{\tau_a}{t}\right)^{2(1+\gamma \sin \phi)} = (1 - \left(\frac{\sigma_m}{\sigma'_f}\right)^{n1}) (1 - \left(\frac{\tau_m}{\tau'_f}\right)^{n2}), \quad (4)$$

where τ'_f is the shear fatigue strength coefficient, σ_m and τ_m are bending and shear mean stress, respectively, and $n1$ and $n2$ are empirical constants. However Equation (4) has severe scatter and needs supplements of multiaxial fatigue tests. Moreover the determination of $n1$ and $n2$ in Equation (4) is ambiguous.

Froustey and Lasserre¹² and Lasserre and Froustey¹³ modified some criteria, which was proposed by Sines,¹⁴ Crossland,¹⁵ Galtier and Sequestre,¹⁶ and Dang Van *et al*,^{17,18} for the case of in-phase bending and torsional loading. Then they concluded from the endurance test results that Dang Van's criterion was very optimistic.

Low-cycle fatigue

Mowbray¹⁹ suggested the equation considering the effect of hydrostatic stress under biaxial fatigue loading. By using von Mises' equivalent stress or strain, he proposed the functions $f(\cdot)$ and $g(\cdot)$ in terms of the hydrostatic stress ratio to explain the variation of two fatigue properties, σ'_f and ϵ'_f , which depend on multiaxial loading and hydrostatic stresses. He then constructed the Coffin-Manson type equation for prediction of fatigue life as follows:

$$\frac{\Delta \epsilon_1}{2} = \frac{\sigma'_f}{E} f(\lambda_\sigma, \nu) (2N_f)^b + \left(\frac{3}{3-A}\right) \epsilon'_f g(\lambda_\sigma, A) (2N_f)^c, \quad (5)$$

where

$$f(\lambda_\sigma, \nu) = (1 - \nu \lambda_\sigma) / \sqrt{(1 - \lambda_\sigma + \lambda_\sigma^2)}, \quad (6)$$

$$g(\lambda_\sigma, A) = (2 - \lambda_\sigma) [3 \sqrt{(1 - \lambda_\sigma + \lambda_\sigma^2)} - A(1 + \lambda_\sigma)] / (6(1 - \lambda_\sigma + \lambda_\sigma^2)), \quad (7)$$

and A is an empirical constant, λ is Poisson's ratio, and σ_σ is the hydrostatic stress ratio. However, $\Delta \epsilon_1/2$ in the left hand side of Equation (5) is not a proper parameter in multiaxial fatigue analysis because of its nonconservativeness. Moreover, severe scatter exists in the verification with experimental results. So it is necessary to choose a correct parameter with caution. Using the relationship of von Mises' equivalent stress and strain, the change of fatigue properties by hydrostatic terms is seen to be explained. But Equation (5) is not testified under out-of-phase multiaxial fatigue and it overlooks the interaction between out-of-phase stress components.

Multiaxial fatigue loadings can change the fatigue properties. Especially fatigue ductility decreases in high temperatures. Zamrik *et al*²⁰ thought that the transition region in strain-fatigue life curve represented the fatigue strength coefficient and fatigue ductility coefficient of material and then, in order to incorporate the changes of fatigue properties under multiaxial loading, introduced the Z -parameter from uniaxial and pure torsional strain-fatigue life curves as follows:

$$Z = \frac{3}{2(1 + \nu_e)} \frac{\tau'_f / G}{\sigma'_f / E} \frac{\epsilon'_f}{\gamma'_f}, \quad (8)$$

where ν_e is Poisson's ratio in the elastic region, τ'_f and γ'_f are shear fatigue strength and ductility coefficient, respectively. Parameter Z united the two curves of uniaxial and pure torsional strain-life into one. So, it is thought to be useful in multiaxial fatigue as well as TF (triaxiality factor) and MF (multiaxiality factor) which were suggested by Davis and Connelly²¹ and Manson and Halford,²² respectively, but experimental verification is needed.

Kalluri and Bonacuse²³ further investigated the multiaxial fatigue behaviour of Haynes 188 super alloy at 760°C. They constructed the following Equation (9) with MF to incorporate the decrease of fatigue ductility and fatigue strength under multiaxial fatigue:

$$\Delta \epsilon_{eq} = \left(\frac{\sigma'_f / E}{MF^{b/c}} \right) (2N_f)^b + \frac{\epsilon'_f}{MF} (2N_f)^c, \quad (9)$$

where

$$MF = \begin{cases} 1/(2 - TF) & \text{for } TF \leq 1 \\ TF & \text{for } TF \geq 1 \end{cases} \quad (10)$$

$$TF = \frac{\sqrt{2}(\sigma_1 + \sigma_2 + \sigma_3)}{[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]^{1/2}}, \quad (11)$$

and $\Delta \epsilon_{eq}$ is von Mises' equivalent strain, σ_1 , σ_2 , and σ_3 are the principal stresses. Kalluri and Bonacuse²³ compared three parameters, i.e. von Mises' equivalent strain; Fatemi and Socie's²⁴ strain parameter on the critical plane; and SWT-parameter²⁵ and then concluded that the von Mises' equivalent strain was a suitable parameter for multiaxial fatigue at high temperatures. It seems, from their comparison only, that the von Mises' equivalent parameter predicts the multiaxial

fatigue life and better results may be had if the fatigue properties are modified with the consideration of the environment and loading conditions. However, in the case of out-of-phase multiaxial fatigue, *TF* and *MF* change during one cycle of loading, so that this approach is confined to in-phase multiaxial fatigue.

USE OF STRESS (OR STRAIN) INVARIANTS

Sines and Ohgi²⁶ described fatigue strength with the stress invariants of Equation (12) and modified it to include the mean stress effect, as Equation (13)

$$(J_2')_a^{1/2} \geq A \quad (12)$$

$$(J_2')_a^{1/2} \geq A - \alpha_1(J_1)_m - \alpha_2(J_2')_m^n, \quad (13)$$

where

$$J_1 = \sigma_1 + \sigma_2 + \sigma_3, \quad (14)$$

$$J_2' = \frac{1}{6} \{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\},$$

and n and A are material constants, $\alpha_2(J_2')_m^n$ is a term to reflect the nonlinear effect in the case that higher mean stress is applied, and α_1 and α_2 are empirical constants. Equations (12) and (13) state that when the square root of the second invariant of the stress deviator of the stress amplitude exceeds a certain constant characteristic of the material, failure occurs. However, no experimental validation was provided for multiaxial fatigue. Besides, stress invariants are the average value of applied stresses, so that nonconservative analysis may be invoked.

Hashin²⁷ generalized the multiaxial fatigue failure criterion as follows:

$$F(\sigma_{m(ij)}, \sigma_{a(ij)}, \phi_{ij}, 2N_f) = 1, \quad (15)$$

where (ij) denotes the stress component, σ_m is the magnitude of mean stress, σ_a is the amplitude of alternating stress, $2N_f$ is load reversals, and ϕ_{ij} is the phase difference between the (ij) stress component and a reference stress component. Hashin proposed that Equation (15) could be expressed as a polynomial with coefficients which were determined from fatigue tests, and also showed that Equation (15) was the locus of all cyclic stress states (with the same frequency and the same load ratio) which produced failure after $2N_f$ reversals. That is, when $2N_f$ varies, Equation (15) becomes a parametric family of surfaces with parameter $2N_f$ in stress space. But the use of Equation (15) is limited to the cases of the constant load-ratio multiaxial fatigue loading only. Therefore the verification with various loading state should be followed.

In the case of rotating octahedral planes, the shear stress amplitudes in the plane of the maximum oscillating octahedral shear stress can be considered as the decisive factor determining the material behaviour. Dietmann *et al*²⁸ insisted that, in order to determine the resulting fatigue strength of the material under complex loading, the time-dependence of the stress wave form, the frequency, and the phase-difference between the stress components as well as the various mean and alternating stress components and the number of cycles must be considered. Then Dietmann *et al*²⁸ proposed the modified octahedral shear stress theory from the tests of cyclic axial loading and cyclic internal pressure with trapezoidal, triangular and sinusoidal loading shape.

Chaudonneret²⁹ presented a multiaxial fatigue damage model, derived from Chaboche's uniaxial formulation^{30,31} of damage, for the cases that uniaxial cyclic loading and multiaxial cyclic loading with or without mean stress are applied, but the formulation uses the cyclic motion only. The definition of quantities such as the octahedral shear stress amplitude or the mean hydrostatic pressure seems difficult to adapt to the case of fully complex loading.

Ballard *et al*³² thought that fatigue rupture will not occur, if and only if the response of the grains, most favourably oriented and subjected to the microscopic fluctuations of the periodic loading, is elastic shake-down. Ballard *et al*³² reconstructed the four criteria suggested by Dang Van *et al*,^{17,18} Papadopoulos,³³ Crossland¹⁵ and Deperrois³⁴ with a microscopic viewpoint.

Munday and Mitchell,³⁵ considered that slip occurs on the maximum shear stress plane, tried to analyse the multiaxial fatigue behaviour and modified the Gough's ellipse,⁷ as Equation (16), with maximum distortion energy and stress gradient.

$$\left(\frac{\sigma_{xa}}{\bar{S}_s}\right)^2 + 3\left(\frac{\tau_{xya}}{\bar{S}_{se}}\right)^2 \geq 1, \quad (16)$$

where σ_{xa} and τ_{xya} represent alternating normal stress and alternating shearing stress, respectively, and $\bar{S}_s$ and $\bar{S}_{se}$ are the pure reversed bending and torsion fatigue limits, respectively. In their work all consideration of mean stress components has been avoided, but the gradients associated with mean stress components seemed to influence the fatigue test results. However, a better understanding of gradient effects is needed to permit a full interpretation of the fatigue data, and also Galtier and Seguret¹⁶ and Lemaître and Chaboche,³⁶ have tried to analyse the multiaxial fatigue behaviour with the strain and stress invariants.

USE OF SPACE AVERAGES OF STRESS OR STRAIN

Papadopoulos,³³ through the complicated calculation of parameters, suggested a generalized failure criterion as follows:

$$\sqrt{\langle \tau \rangle^2} + \alpha \left(\max_t \langle \sigma_n \rangle \right) \leq \beta, \quad (17)$$

where $\langle \bullet \rangle$ means the average value of argument, α and β are material constants, τ and σ_n are the shear stress and the normal stress acting on critical plane by Papadopoulos' definition, respectively. Papadopoulos³⁷ also constructed the microscopic version of Equation (17). Equation (17) can be changed into the criterion suggested by Gough⁷ in the case of multiaxial bending and torsional loading. Shear stress, τ , in Equation (17) includes the term which considers the effect of phase difference at the early step in the calculation. However, during the induction of Equation (17), the term to explain the effects of phase difference disappears. So, Papadopoulos concluded that the phase difference, especially the phase difference of shear stress, had no effect on multiaxial fatigue behaviour, but Papadopoulos' conclusion can be applied to hard metals only and contradicts many results of the multiaxial tests.^{1-3,38-40} Consequently, it is desirable to

modify the Papadopoulos' equation to accommodate the effects of out-of-phase multiaxial loadings.

In terms of low-cycle fatigue, Sonsino and Grubisic^{41,42} investigated the fatigue behaviour of cyclically softening and hardening steels under multiaxial loading. From the tests it was observed that the decrease of fatigue life under out-of-phase strains was caused by the changing direction of principal strains resulting in an interaction of the deformations in all directions on the surface. Then Sonsino and Grubisic proposed that the interaction could be taken into account by the arithmetic mean value of shear amplitudes acting in all interference planes of the surface and showed a fairly good coincidence with the comparison of test results. However, since Sonsino and Grubisic's proposal is confined to sinusoidal loading, other complicated loading shapes, such as trapezoidal loading, impact, etc. can cause some trouble in multiaxial fatigue analysis due to the averaged value of the parameter.⁴³⁻⁴⁵

Additionally, the works of Zenner *et al.*⁴⁶ Sonsino and Grubisic,⁴² and Liu⁴⁷⁻⁴⁹ have been introduced to explain the fatigue behaviour with the space averages of stress or strain.

USE OF THE CRITICAL PLANE

Fatigue analysis using the concept of critical plane is very effective because the critical plane concept is based upon the fracture mode or the initiation mechanism of cracks. In the critical plane concept, after determining the maximum shear strain (or stress) plane, many researchers define the parameter as the combination of the maximum shear strain (or stress) and normal strain (or stress) on that plane to explain the multiaxial fatigue behaviour.^{1,3,4,10,50}

Strain terms are used in the region of LCF and stress terms in the HCF region with the critical plane in multiaxial fatigue analysis. Brown and Miller⁴ tried to analyse the multiaxial fatigue in the LCF region by using the state of strain on the plane where maximum shear strain occurred, while Findley *et al.*¹⁰ and Stulen and Commings⁵¹ used the HCF region with stress terms. In the HCF region the planes on which maximum shear strain and stress occurs are coincident, but they are different in the LCF region because of the nonlinear stress-strain relation in the LCF region. Therefore the parameter must be distinguished according to the fatigue region (LCF or HCF) and the magnitudes of the applied strains or stresses.

Application of most of the stress- or strain-based criteria proposed up to 1980 is limited to the constant amplitude monofrequency type of combined loading and smooth surface condition.^{1,3} Recent investigations after 1980 have been generalized to solve that problem. But the crack-stage distinction, such as initiation and propagation (or the definition of failure), is often lacking.

Low-cycle fatigue

Reviews by Garud¹ and Brown and Miller^{4,5} revealed that many multiaxial fatigue theories were limited to multiaxial loading of simple wave shapes (such as, sine wave). Several generalizations of the theories using the critical plane concepts and the energy concepts have been constantly tried. Recent modifications of multiaxial fatigue parameters include the effects of loading shape,

loading path, material anisotropy and crack initiation types on the basis of the critical plane concept.

In examination of the results using strain-state on the critical plane, Lohr and Ellison⁵² indicated that Brown and Miller's parameters,⁴ $\gamma_{\max}$ and ϵ_n , had some problems. They found that there was discontinuity according to the variation of strain ratio in the Brown and Miller's Γ -plane⁴ and an initial crack grew along the thickness direction. So Lohr and Ellison⁵² modified the parameter as follows:

$$\gamma^* = \begin{cases} \gamma_{\max} & \text{for } -\nu \leq \lambda_\epsilon \leq 1 \\ \gamma_{\max}/2 & \text{for } \lambda_\epsilon = -1 \end{cases}, \quad (18)$$

where λ_ϵ is the ratio of shear strain to axial strain and ν is Poisson's ratio. Lohr and Ellison⁵² also induced the following failure criterion from the several experimental results of other researchers:

$$\gamma^* + 0.2\epsilon_n^* = C, \quad (19)$$

where γ^* is the maximum shear strain that occurs on the 45°-slanted plane with respect to the surface along the thickness, ϵ_n^* is the normal strain on the γ^* -plane on which γ^* occurred, and C is a material constant. But Lohr and Ellison's parameter⁵² is confined to in-phase multiaxial loading because the γ^* varies according to the strain ratio in out-of-phase loadings. As the maximum shear strain plane is different from the 45°-slanted plane with respect to the surface along the thickness, verification should be given. The coefficient of ϵ_n^* was introduced to explain the secondary effect of the normal strain on the initiation and initial growth of cracks. Since the coefficient of ϵ_n^* in Equation (19) is an empirical constant, it is not to be fixed as 0.2.

Socie and Shield⁵³ performed the experiments to examine the effect of mean stress under multiaxial fatigue loading with tubular specimens of Inconel 718. They showed in their experiments that the multiaxial fatigue life could be predicted well by using a linear combination of Brown and Miller's parameters,⁴ $\gamma_{\max}$ and ϵ_n to explain the mean stress effect as follows:

$$\gamma_{\max,pl} + \epsilon_{n,pl} + \sigma_{no}/E = \gamma_f'(2N_f)^{co}, \quad (20)$$

where $\gamma_{\max,pl}$ and $\epsilon_{n,pl}$ are the values of Brown and Miller's parameters in the plastic region, respectively, σ_{no} is the component of mean stress applied on the $\gamma_{\max}$ -plane, and co and γ_f' are shear fatigue ductility exponent and coefficient, respectively. From their results, tensile mean stress applied on the maximum shear strain plane increases the frequency and distribution of microcracks and promotes the initiation of cracks.

Subsequently, Socie *et al.*⁵⁴ proposed an equation for predicting fatigue life similar to that of Socie and Shield.⁵³ They predicted nearly the same results with Socie and Shield⁵³ by using the linear combination of Lohr and Ellison's parameter⁵² and the term which explains the effect of mean stress. Two researches of Socie and Shield⁵³ and Socie *et al.*⁵⁴ each using different parameters showed the same results, because there was little difference in fatigue life due to the relaxation of mean stress with relatively large strain loading condition and the value of the two parameters were the same, due to the same shape of specimen. But Socie and Shield⁵³ and Socie *et al.*⁵⁴ did not consider the case of out-of-phase loading. Hence, it is advisable

to use the equations suggested by Socie and Shield⁵³ and Socie *et al*⁵⁴ in simple (proportional, in-phase) multiaxial loadings.

Jordan *et al*⁵⁵ performed experiments with various loading shapes and frequencies and showed that life prediction with amplitudes of applied strains only could underestimate the multiaxial fatigue damage. Jordan *et al*, under the test conditions of in-phase multiaxial strain loading and impact tension–torsion loading, computed RMS values of variable amplitude loading and modified the equivalent strain of Kandil *et al*⁵⁶ as follows:

$$\Delta\gamma = \Delta\gamma_{\max}(1 + S\zeta^n)^{1/n}, \quad \zeta = 4\sqrt{\omega/\omega'}\epsilon_{np}/\Delta\gamma_{\max}, \quad (21)$$

where ω and ω' are frequencies of each loading, n and S are empirical constants, ϵ_{np} is a peak value of ϵ_n on the $\gamma_{\max}$ -plane. By comparing with the plastic work of Garud,⁵⁷ Jordan *et al*⁵⁵ showed that the accuracy of the life prediction by plastic work reduced as the life increased. However the equivalent strain equation of Jordan *et al*⁵⁵ cannot consider the effects of out-of-phase multiaxial loading and mean stress.

Jacquelin *et al*⁵⁸ performed fatigue tests with 316 stainless steel and Inconel 718 under static and alternating torsion. Under the same strain condition the crack initiation modes in two materials are different from each other, i.e. cracks perpendicular to the axis of the specimen were occurred in 316 stainless steel and, in Inconel 718, cracks were detected in the region between 45° and 90° with respect to the axis of specimen. These differences are due to the characteristics of the materials according to the microstructural difference. They defined 30 μm crack length as the failure condition for determining fatigue life in their experiments. However, a 30 μm crack length may not be adequate as a fatigue failure condition due to overestimation. It is unknown whether the method suggested by Jacquelin *et al*⁵⁸ is generally applicable to multiaxial fatigue because the parameters for life prediction are different with respect to the characteristics of materials.⁵⁹

Wu and Yang⁴³ examined the effect of loading path in the multiaxial fatigue tests. The fatigue characteristics of the material were defined by investigating the initial crack growth mode on 304 stainless steel under axial strain and shear strain along a rectangular path. They calculated the effect of normal strain on the crack surface for one cycle with the effective average strain theory of Taira⁴⁴ as follows:

$$\Delta\epsilon_{eq} = 2\frac{\omega}{\pi} \int_0^{\pi/\omega} |\epsilon_{eq}|^{1/\alpha} dt, \quad (22)$$

where ϵ_{eq} is von Mises' equivalent strain and α is the exponent of the Coffin–Manson equation. From comparison with their experiments, it was found that the plastic work theory suggested by Garud⁵⁷ could not analyse the effects of loading paths. They thought that the effect by normal strain on the microcrack growth was larger than that by shear strain on the crack initiation and initial crack growth, so they defined normal strain as the primary factor for the analysis of multiaxial fatigue.

To consider loading paths, loading shapes, and cracking modes, Socie⁵⁹ performed multiaxial fatigue tests with two materials (304 Stainless steel and Inconel

718) which had different cracking modes under various loading paths. According to the Socie⁵⁹ experiments, 90° out-of-phase multiaxial loading paths among several loading paths showed severe cyclic hardening in the two materials. Besides, the 304 stainless steel showed a tensile cracking mode, while Inconel 718 revealed a shear cracking mode. This phenomenon is thought to be the characteristic of the materials. Socie suggested following Equation (23) for the material which reveals a shear cracking mode.

$$\gamma_{\max} + \epsilon_n + \sigma_{no}/E = \gamma_f'(2N_f)^{co} + \frac{\tau_f'}{G}(2N_f)^{bo}, \quad (23)$$

where co and bo are shear fatigue ductility exponent and strength exponent, respectively, and τ_f' and γ_f' are the shear fatigue strength and ductility coefficient, respectively, $\gamma_{\max}$ and ϵ_n are Brown and Miller's parameters,⁴ and σ_{no} is the component of mean stress on the maximum shear strain plane. However, for the tensile cracking mode, Socie recommended the following Equation (24):

$$\sigma_1^{\max}\Delta\epsilon_1/2 = \sigma_f'\epsilon_f'(2N_f)^{b+c} + \frac{\sigma_f'^2}{E}(2N_f)^{2b}, \quad (24)$$

where $\sigma_1^{\max}$ is the maximum principal stress for one cycle of loading and $\Delta\epsilon_1$ is the principal strain range.

Fatemi and Socie²⁴ performed multiaxial fatigue tests with Inconel 718 and proposed a modified equation which was originated from Brown and Miller's parameter⁴ as follows:

$$\begin{aligned} \gamma_{\max}(1 + n\frac{\sigma_n^{\max}}{\sigma_y}) &= (1 + \nu_e)\frac{\sigma_f'}{E}(2N_f)^b \\ &+ \frac{n}{2}(1 + \nu_e)\frac{\sigma_f'^2}{E\sigma_y}(2N_f)^{2b} \\ &+ (1 + \nu_p)\epsilon_f'(2N_f)^c \\ &+ \frac{n}{2}(1 + \nu_p)\frac{\epsilon_f'\sigma_f'}{\sigma_y}(2N_f)^{b+c}, \end{aligned} \quad (25)$$

where n is an empirical constant, ν_e and ν_p are poisson's ratio in the elastic and plastic region, respectively. From the experimental results,^{24,59} it was found that 90° out-of-phase multiaxial fatigue loading brought about additional cyclic hardening. This additional hardening results from the rotation of the principal axes. Equation (25) shows some consistency with the experiments, but it does not explain the additional hardening. On the other hand, the equation by Fatemi and Socie²⁴ can accommodate the effect of mean stress in normal strain term linearly and coincides with the test results.

Nitta *et al*³⁹ executed multiaxial fatigue tests at a high temperature with 304 Stainless steel and compared them with several parameters and analysed fractography. After comparing the test results with the von Mises' equivalent strain, Brown and Miller's parameter, and the energy method, they suggested that Brown and Miller's parameter was a better predictor.

Doquet and Pineau⁴⁰ performed multiaxial fatigue tests with mild steels of BCC structure instead of FCC structured material and compared some parameters with the test results. In their experiments, it was shown that the BCC structured mild steel had intergranular cracks mostly under out-of-phase loading and did not have remarkable crack initiation modes under in-phase load-

ing. They also showed that Brown and Miller's parameter⁴ was consistent with their experimental results.

Macha⁶⁰ suggested fatigue criteria by means of the strain-state on the critical plane under multiaxial random fatigue loading. Macha defined the critical plane by average of direction cosines, which expressed the maximum shear strain plane with respect to time during an arbitrary period. He then proposed the following formula:

$$\begin{aligned} \max_t \{b\epsilon_{\eta s}(t) + k\epsilon_{\eta}(t)\} = \\ \max_t \{b\hat{\beta}_{\eta i}\hat{\beta}_{sj}\epsilon_{ij}(t) + k\hat{\alpha}_{\eta i}\hat{\alpha}_{\eta j}\epsilon_{ij}(t)\} = f, \end{aligned} \quad (26)$$

where $\epsilon_{\eta s}(t) = \hat{\beta}_{\eta i}\hat{\beta}_{sj}\epsilon_{ij}(t)$, $\epsilon_{\eta}(t) = \hat{\alpha}_{\eta i}\hat{\alpha}_{\eta j}\epsilon_{ij}(t)$, $\hat{\alpha}$ and $\hat{\beta}$ are average values of direction cosines, f is a material constant, and b and k are empirical constants. Equation (16), however, lacks in the consideration of the effects of mean stress and load sequences under random loading. Additionally different results can be induced by a different selection of the arbitrary period which determines the critical plane.

Lefebvre *et al.*⁶¹ concentrated on the crack initiation mode according to the phase difference. They examined the variations of shear strain and normal strain on the maximum shear strain plane for one cycle and calculated the principal strains in the three regions where cracks could occur. Then, they related the intensities of normal and shear strain with fatigue life. Lefebvre *et al.* examined the fatigue behaviour with Equation (27) and reconstructed the Γ -plane of Brown and Miller⁴.

$$I(\gamma_{\max}/2) + f\{I(\epsilon_n)\} = C, \quad I(\bullet) = \int_{\omega t_1}^{\omega t_2} \bullet d(\omega t) \quad (27)$$

where $I(\cdot)$ denotes the intensity of the argument, ω is the frequency of loading, and t_1 and t_2 represent the beginning and the end of any particular zone. But the crack initiation modes, case A and case B,⁴ are determined not by phase difference, but by the magnitudes of loading and the shapes of the specimens. The case A crack starts and grows along the surface, but the case B crack penetrates into the surface. Their prediction deviates from the experimental results and the discontinuities are shown in some regions of phase difference on their reconstructed Γ -plane.

Wang and Brown⁶² investigated the effects of loading path on multiaxial fatigue life with four materials and proposed a method applicable to variable amplitude loading. They determined the magnitude of normal strain in the excursion region of maximum shear strain as shown in Figure 1, when the loads with different

loading shapes and frequencies were applied on materials. From Figure 1 the effective normal strain can be defined as

$$\epsilon_{n, \text{eff}} = \max_{t_A \leq t \leq t_B} (\epsilon_n(t)) - \min_{t_A \leq t \leq t_B} (\epsilon_n(t)) \quad (28)$$

and according to the crack initiation modes, the effective shear strain range as a multiaxial fatigue parameter will be given as follows:

$$\Delta\hat{\gamma} = \begin{cases} \Delta\gamma_{\max} + k\epsilon_{n, \text{eff}} & \text{for case A crack} \\ \Delta\gamma_{\max} & \text{for case B crack}^4 \end{cases} \quad (29)$$

One reversal in Figure 1 denotes the period between the two turning points of maximum shear strain which are assumed to correspond to the reversing of the shear deformation process, which will be reflected in flow and decohesion at the crack tip.

Their results are persuasive and coincident with the experimental results under variable multiaxial loading, but they do not consider the mean stress effect. Since there is not the term for considering the mean stress effect in the calculation of normal strain, it is helpful to modify fatigue properties using similar factors shown in the right hand side terms of the life prediction Equation (5). Then the scatterband in the full region of fatigue life can be decreased by such modifications.

In general, multiaxial fatigue theories are based on the assumption of isotropic materials. But there exists anisotropy by manufacturing (initial forming, machining or post-processing) and anisotropy becomes one of the factors which changes the fatigue behaviour of components according to the loading directions. There had been some trials to consider the anisotropy in multiaxial fatigue theory before 1980. However, such attempts were not general and also many difficulties arose in implementation of the theory.¹

Lin and Nayeab-Hashemi⁶³ accomplished the multiaxial fatigue tests with aluminium for the analysis of anisotropic effect in multiaxial fatigue. They modified Brown and Miller's suggestion by using the anisotropic plasticity of Shih and Lee⁶⁴ as follows:

$$A_i(\gamma_{\max} + B_i\epsilon_n) = f(N_f), \quad (30)$$

where A_i and B_i are coefficients for consideration of material anisotropy. Although their tests were confined to the LCF region, their suggestion could be used in both of LCF and HCF regions because the effects of anisotropy are so small in the HCF region. The parameters suggested by other researchers showed some scatter in the comparison of the experimental results with the same materials which had different material orientations.^{1,63} The predictions by Equation (30) are coincident with the test results, but the research of Lin and Nayeab-Hashemi is performed under in-phase multiaxial fatigue so that it is difficult to determine A_i and B_i in the case of out-of-phase multiaxial fatigue.

Shatil *et al.*⁶⁵ examined the differences between anisotropy and isotropy which occurred due to different heat treatments in their biaxial fatigue tests. From the comparison of three parameters for multiaxial fatigue analysis with experimental results of EN15R steel, they found that the difference in fatigue life stemmed from the post-processing, the microstructure of material, and multiaxiality of loadings.

From two-level tests under tension and torsion at

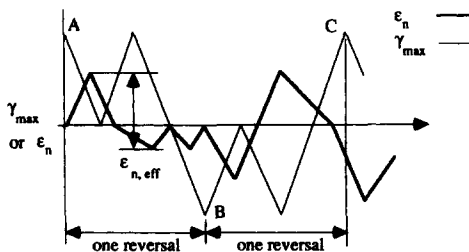


Figure 1 Determination of $\epsilon_{n, \text{eff}}$ ⁶² where A, B, and C are turning points of maximum shear strain

equivalent strain ranges Robillard and Cailletaud⁶⁶ observed that the anisotropy of the material damage was shown along with the crack aspects, and that the life accumulation was not symmetrical. They proposed that damage on each facet inside the material was defined by an incremental law, and failure happened when, on a given facet, the damage reached a critical value. Thus Robillard and Cailletaud constructed the directionally-defined damage model, which involved a damage variable in each space direction where the asymmetry of damage accumulation on the facets appears naturally due to their respective orientation. However, the use of this model is actually limited to the number of cycles to failure and the main crack direction prediction in tension and torsion.

Chu⁶⁷ constituted the closed form for six damage criteria, i.e. the normal strain amplitude; the Smith–Watson–Topper parameter; the shear strain amplitude; the Brown and Miller damage parameter; the Findley type damage parameter; and a biaxial Smith–Watson–Topper type parameter,⁶⁸ that had been proposed in the literature. And he determined the most critical plane and the largest damage parameter from the transformation of strains and stresses onto a critical plane and from a generalized Mroz⁶⁹ model. However, Chu's damage model is confined to in-phase multiaxial fatigue loading, so that some modifications are needed with experimental verifications.

High-cycle fatigue

To consider the mean stress effects, McDiarmid⁷⁰ modified his suggestion⁷¹ for the calculation of fatigue strength. With the test results of Gough,⁷ he⁷⁰ proposed the equivalent shear stress amplitude as follows

$$\tau_a = t - \frac{(t - b/2)}{(b/2)^{1.5}} \sigma_a^{1.5} - C\sigma_m - 0.081\tau_m, \quad (31)$$

where, b and t are bending and shear fatigue strength, σ_a , σ_m and τ_m denote the amplitude of alternating stress and the magnitude of normal and shear mean stress on the maximum shear stress plane, respectively, and C is an empirical constant. McDiarmid⁷⁰ suggested that the influence of shear mean stress on the critical plane was so little that the term for shear mean stress, $0.081\tau_m$, in Equation (31) could be neglected, but the removal of shear mean stress needs further verification.

McDiarmid⁷² suggested that SCF (stress concentration factor) for multiaxial fatigue of a notched member should be determined according to the dominant loading between bending and torsion, and constructed a new equation for fatigue strength by using the results of McDiarmid⁷¹ as follows

$$\tau_a = \frac{1}{K_t} \left[t - \frac{(t - b/2)}{(b/2)^{1.5}} (K_t \sigma_a)^{1.5} - \frac{b/2}{(\sigma_u/2)^2} \sigma_m \right], \quad (32)$$

where K_t is the theoretical stress concentration factor and σ_u is the tensile strength. However, since a large deformation occurs locally in notched region repeatedly, it is not appropriate, but conservative to use K_t by the dominant loading between loadings instead of K_f (fatigue notch factor) in the calculation of fatigue strength. Moreover Equation (32) needs verifications for out-of-phase multiaxial fatigue.

Flavenot and Skalli⁵⁰ suggested the concepts of critical depth for the multiaxial fatigue behaviour of

notched components. The critical depth is related to the microstructure of material. They considered the effects of the stress gradient around the notch by defining the shear stress and hydrostatic stress at the critical depth, not on the surface. They verified the validity of critical depth from the comparison with Sines,¹⁴ Crossland,¹⁵ and Dang Van *et al.*^{17,18} but they did not apply their concept on out-of-phase multiaxial loading. Besides, Flavenot and Skalli⁵⁰ analysed the fatigue strength of engineering parts with residual stresses and then showed that Dang Van's equation^{17,18} was a good life predictor. However, they excluded the case of out-of-phase and did not reflect the relaxation of residual stresses under repeated loading.

Macha⁷³ constituted a generalized failure criterion similar to Equation (26) for multiaxial random fatigue loading in stress terms.

$$\begin{aligned} & \max_t \{B\tau_{\eta s}(t) + K\sigma_{\eta}(t)\} \\ & = \max_t \{B\hat{\beta}_{\eta i}\hat{\beta}_{sj}\sigma_{ij}(t) + K\hat{\alpha}_{\eta i}\hat{\alpha}_{\eta j}\sigma_{ij}(t)\} = F, \end{aligned} \quad (33)$$

where $\tau_{\eta s}(t) = \hat{\beta}_{\eta i}\hat{\beta}_{sj}\sigma_{ij}(t)$, $\sigma_{\eta}(t) = \hat{\alpha}_{\eta i}\hat{\alpha}_{\eta j}\sigma_{ij}(t)$, B and K are empirical constants, and F is a material constant. Macha compared Equation (33) with the theories of McDiarmid,⁷⁰ Mataka⁷⁴ and Stanfield⁷⁵ and proved the validity of Equation (33). However, Equation (33) has the same limitations as in Equation (26) of Macha.⁶⁰

McDiarmid³⁸ proposed a more generalized failure criterion including the crack initiation modes from the synthesis of researches^{70–73} as follows:

$$\frac{\tau_a}{t_{A,B}} + \frac{\sigma_{n,max}}{2\sigma_u} = 1 \quad (34)$$

$$0.5t \leq \tau_a \leq t, 0 \leq \sigma_{n,max} \leq \sigma_u,$$

where τ_a and $\sigma_{n,max}$ are shear stress and maximum normal stress on the maximum shear stress plane and $t_{A,B}$ is shear fatigue strength for each cases of crack initiation modes A and B.⁴ Experimental verification of the criterion was provided, but Equation (34) also has limitations of the range of loading conditions as $0.5t \leq \tau_a \leq t$ and $0 \leq \sigma_{n,max} \leq \sigma_u$ and does not explain the mean stress effect. The scatter becomes prominent as τ_a goes away from the shear fatigue strength and $\sigma_{n,max}/2\sigma_u$ is greater in comparison with the experimental results.

Besides, critical plane approaches have been dealt in the works of Grubisic and Sonsino,⁷⁶ Dang Van and Papadopoulos,⁷⁷ Robert *et al.*,⁷⁸ Lin,⁷⁹ Papadopoulos and Panoskaltsis⁸⁰ and Chen and Keer.⁸¹

USE OF ENERGY CONCEPTS

All aforementioned stress- or strain-based criteria are lacking in consideration of the multiaxial stress–strain response of the material that is a crucial part of the fatigue process. The fatigue process is generally believed to involve cyclic plastic deformations which are dependent on the stress–strain path. Thus, the stress- or strain-based criteria cannot reflect the path dependence of the fatigue process sufficiently. Some researches which consider the multiaxial stress–strain response were introduced. However, most of these approaches appear to be difficult to implement.¹ The

energy concept includes the explicit consideration of the multiaxial stress-strain response and its promise was shown for complex multiaxial loading by Garud.⁵⁷

There are not any remarkable developments in multiaxial fatigue analysis using energy concept after the researches of Garud⁵⁷ and Leis.⁸² Some researchers^{45,55,63} suggested the validity of Garud's plastic work theory.⁵⁷ However, they did not explain the effects of mean stress and scatter in the HCF region; and also the study on the plasticity needs to include the effects of loading path.

The damage accumulated in material by fatigue loading can be related with energy due to loading. Ellyin *et al.*⁸³⁻⁸⁶ tried to express the multiaxial fatigue behaviour with the total strain energy density. Ellyin and Golos⁸³ proposed that the durability of components should be characterized with the quantity of energy which material could contain, and, using the master curve of material, they suggested total strain energy density, ΔW_t , as follows:

$$\Delta W_t = \Delta W_e + \Delta W_p \quad (35)$$

where

$$\Delta W_e = \frac{1+\nu}{3E} (\sigma^{\max})^2 + \frac{1-2\nu}{6E} (\sigma_{kk}^{\max})^2, \quad (36)$$

$$\Delta W_p = \frac{2(1-n')(2K')^{-1/n'}}{1+n'} (\Delta \bar{\sigma})^{(1+n')/n'}, \quad (37)$$

and $\bar{\sigma}$ is von Mises' equivalent stress, n' and K' are cyclic hardening exponent and cyclic strength coefficient, respectively. Ellyin and Golos⁸³ proved that Equations (35)–(37) could predict fatigue life better in the LCF and HCF region under uniaxial fatigue loading and in-phase multiaxial fatigue loading, but there were some discrepancies according to the ratio of shear strain to axial strain in multiaxial fatigue tests. This fact means that Equations (35)–(37) do not hold any terms to reflect the strain ratio. Besides, this method cannot explain the additional cyclic hardening or softening of the material due to the interaction between loadings under out-of-phase multiaxial fatigue loading.

Ellyin *et al.*⁸⁴ in 1991, using MCF (multiaxial constraint factor), performed in-phase and out-of-phase multiaxial fatigue test on ASTM A516Gr. 70 material and supplemented the result of Ellyin and Golos.⁸³ They could correlate the data of different strain ratios and specimen shapes by utilizing MCF, $\bar{\rho}$, as follows:

$$\bar{\rho} = (1 + \bar{\nu}) \frac{\epsilon_1}{\gamma_{\max}}, \quad (38)$$

where

$$\bar{\nu} = \frac{\nu_p(1 - \nu_e)(\epsilon_a + \epsilon_t) + (\nu_e - \nu_p)(\epsilon_{a,e} + \epsilon_{t,e})}{(1 - \nu_e)(\epsilon_a + \epsilon_t) + (\nu_e - \nu_p)(\epsilon_{a,e} + \epsilon_{t,e})}, \quad \epsilon_a \neq \epsilon_t, \quad (39)$$

and ϵ_1 is principal strain, $\gamma_{\max}$ is maximum shear strain, ϵ_a and ϵ_t are applied axial and shear strain, ν_e and ν_p denote Poisson's ratio in the elastic and plastic region, respectively. The combination of ΔW_t and MCF, as found in the following equation, provides a fairly good correlation between in-phase multiaxial and uniaxial fatigue test:

$$\frac{\Delta W_t - 0.1}{\bar{\rho}} = A(2N_f)^c, \quad (40)$$

where A and c are material constants. However, the database for the out-of-phase tests is rather limited and more experimental results are needed in this approach.

On the other hand, Ellyin and Kusawaski⁸⁵ added a new factor to the ΔW_t suggested by Ellyin *et al.*⁸⁴ to reflect the mean stress effect and material anisotropy. They showed the usefulness of their suggestion with experimental validation, but the problems of out-of-phase remain unsolved. Additionally the effect of anisotropy needs validation.

Ellyin and Xia,⁸⁶ considering the out-of-phase multiaxial loading, calculating the MCF with Equation (41) and explained the decrease in fatigue life due to the phase difference.

$$\bar{\rho} = \frac{(1 - \bar{\nu}^2)(1 + \bar{\nu}\lambda_e \cos \phi)}{(1 + \bar{\nu}\lambda_e \cos \phi)^2 + (\bar{\nu}\lambda_e \sin \phi)^2}, \quad (41)$$

where, λ_e is the strain ratio of shear strain to axial strain, ϕ is the phase difference, and $\bar{\nu}$ is shown in Equation (39). While the maximum shear strain plane is different according to the strain ratio and the loading type, Ellyin and Xia⁸⁶ calculated the maximum shear strain on the 45°-slanted plane with respect to surface along the thickness.

Liu⁸⁷ calculated the VSE (virtual strain energy) by use of the crack initiation modes (cases A and B⁴) and two of Mohr's circles, which are shown in Figure 2. In the calculation of VSE, Liu divided the work into the following three equations, as in Equation (42) according to the crack initiation modes and verified the usefulness of VSE from the comparison with another experimental results:

$$\Delta W_I = \{2(OA) \cdot 2(O'A')\}_{\max}, \text{ for tensile dominant,}$$

$$\Delta W_{II,A} = 2(OB) \cdot 2(O'B'), \text{ for case A crack,} \quad (42)$$

$$\Delta W_{II,B} = 2(OC) \cdot 2(O'C'), \text{ for case B crack,}$$

where OA , $O'A'$, OB , $O'B'$, OC , and $O'C'$ are defined in Figure 2c. It is found that Liu's VSE predicts fatigue life better, regardless of temperature, materials, and load ratio. It is expected that the development of Mohr's circles for out-of-phase multiaxial fatigue loading and mean stresses makes Liu's VSE more available.

CONCLUDING REMARKS

Many multiaxial fatigue theories developed after 1980 were examined from five viewpoints, i.e. empirical formulas and modifications of the Coffin–Manson equation, application of stress or strain invariants, use of the space averages of stress or strain, critical plane approaches, and use of energy which accumulates on the materials. It was found that there have been some trials to generalize the previously suggested theories by considering the effects of mean stress and loading path, rather than the suggestion of new theories. However, multiaxial fatigue analyses including material anisotropy are made and the energy method using Mohr's circles is newly proposed. The following are summary of these investigations:

- (1) It is helpful to utilize the von Mises' equivalent parameter on some materials and under some environments. However, the utilization of that parameter is not general. So, by using the additional factors to reflect the variation of fatigue properties

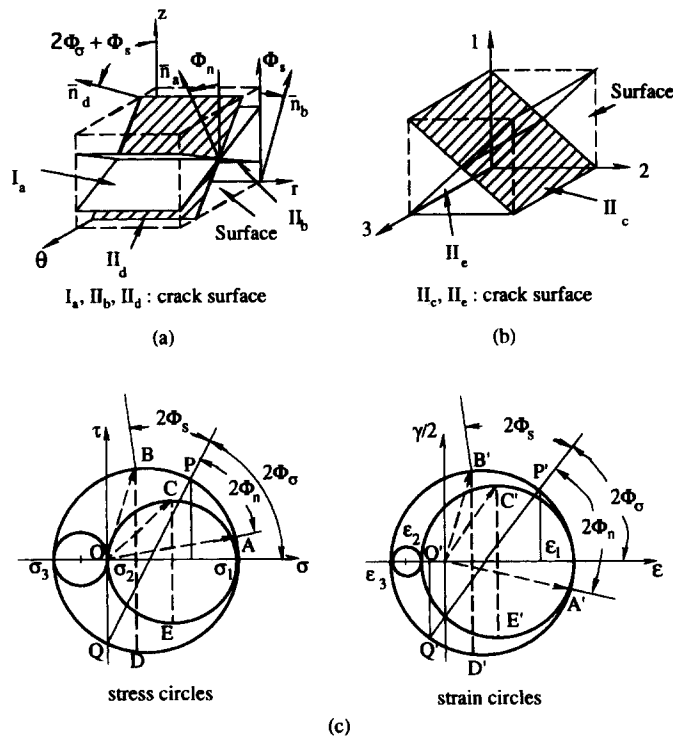


Figure 2 Crack initiation modes of (a) tensile dominant or case A, (b) case B, and (c) two Mohr's circles⁸⁷

due to multiaxial loading and environments, it is favourable to predict the fatigue life and explain the fatigue behaviour.¹⁹⁻²¹

- (2) Several investigators^{3,8,9,26,27} suggested relatively simple parameters based on loading conditions only to analyse the multiaxial fatigue behaviour without complicated calculation of parameters. However, the equations suggested by Sines and Ohgi²⁶ and Hashin²⁷ need verification under out-of-phase loadings or another factors to predict the fatigue behaviour more accurately.
- (3) If the average or RMS value of each strain parameter during one cycle and the frequency of a specific loading are used in multiaxial fatigue theories, the effects of loading shape can be analysed better.^{43,53,59} Crack initiation modes seem to be related with the characteristics of the materials, the strain ratio, and the shape of specimen. Therefore considerations of the crack initiation modes^{53,57} can improve the predictions of the multiaxial fatigue behaviour.
- (4) The effect of material anisotropy on the fatigue behaviour can be observed better if material orientation produced by post-processing is computed by means of the plasticity. But the applications are limited to in-phase and large strain multiaxial conditions.⁶²
- (5) Some researchers have defined the failure conditions for determining fatigue life as 1 mm crack length and 10% load drop, which is vague under multiaxial fatigue loading. The fatigue life is different according to the cracking mode and the definition of failure conditions. Therefore suitable conditions are necessary.
- (6) The generalized fatigue strength equation, which was suggested by McDiarmid from his series of researches, does not explain the effects of mean

stress, out-of-phase multiaxial loading, and stress concentration sufficiently. Despite those problems there are better predictions in some ranges of fatigue loading.³⁸

- (7) After the Garud's study, Ellyin *et al*⁸³⁻⁸⁶ suggested the energy method for multiaxial fatigue analysis. But the equation proposed by Ellyin *et al*, as in the case of Garud's, cannot incorporate the effect of loading path and the interaction between out-of-phase strain or stress components.⁸³⁻⁸⁶ Generalization of the total strain energy density needs experimental verification on various materials and loading shapes.
- (8) If one constructs Mohr's circle for out-of-phase multiaxial fatigue loading or mean stress, Liu's VSE (virtual strain energy) method can be more valuable.

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