

Chapter 6

Symmetries

6.1 Quantum dynamics

The state, or *ket*, vector $|\psi\rangle$ of a physical system completely characterizes the system at a given instant. The corresponding *bra* vector $\langle\psi|$ is the Hermitian conjugate of $|\psi\rangle$. Properly normalized states satisfy $\langle\psi|\psi\rangle = 1$.

Let $|\psi(t)\rangle$ denote the state of a system at time t . Given an initial state $|\psi(0)\rangle$, the goal of quantum dynamics is to predict $|\psi(t)\rangle$ for $t \neq 0$. The superposition principle of quantum mechanics implies that there is a linear operator $U(t)$, called the *time-evolution operator*, which maps any state at time zero into the corresponding state at time t ,

$$|\psi(t)\rangle = U(t) |\psi(0)\rangle. \quad (6.1.1)$$

Time evolution must map any properly normalized state at one time into a normalized state at another time. This implies that the time evolution operator is *unitary*,

$$U(t)^\dagger = U(t)^{-1}. \quad (6.1.2)$$

It is often convenient to consider a differential form of time evolution. The time derivative of any state must again (by the superposition principle) be given by some linear operator acting on the state. That linear operator, times $i\hbar$, is called the Hamiltonian, denoted H . In other words,

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = H |\psi(t)\rangle. \quad (6.1.3)$$

This is the (time dependent) Schrodinger equation. It is a linear first order differential equation, whose solution can be written immediately in terms of an exponential,

$$|\psi(t)\rangle = e^{-iHt/\hbar} |\psi(0)\rangle. \quad (6.1.4)$$

Comparing with the definition (6.1.1), one sees that this exponential of the Hamiltonian (times $-it/\hbar$) is precisely the time evolution operator,

$$U(t) = e^{-iHt/\hbar}. \quad (6.1.5)$$

The Hamiltonian must be Hermitian, $H^\dagger = H$, in order for $U(t)$ to be unitary.

We have just solved all quantum dynamics! Of course, evaluating this exponential of the Hamiltonian can be (and usually is) hard. A quantum system whose space of states is N -dimensional will have a Hamiltonian which is an $N \times N$ matrix. Most systems of interest will have a very large, or infinite, dimensional space of states.

6.2 Symmetries

A linear transformation T which maps an arbitrary state $|\psi\rangle$ into some different state $|\tilde{\psi}\rangle$ is called a *symmetry* if T is unitary, $T^\dagger = T^{-1}$, and T commutes with the time evolution operator,¹

$$T U(t) = U(t) T. \quad (6.2.1)$$

To understand this, consider some arbitrary initial state $|\psi(0)\rangle$, and imagine that you have worked out how this state evolves in time so that you know $|\psi(t)\rangle = U(t)|\psi(0)\rangle$. Applying the transformation T to the initial state $|\psi(0)\rangle$ will produce a different state $|\tilde{\psi}(0)\rangle = T|\psi(0)\rangle$. This transformed initial state will evolve in time into $|\tilde{\psi}(t)\rangle = U(t)|\tilde{\psi}(0)\rangle = U(t)T|\psi(0)\rangle$. But if condition (6.2.1) is satisfied, then one can interchange $U(t)$ and T and write this result as $|\tilde{\psi}(t)\rangle = T U(t)|\psi(0)\rangle = T|\psi(t)\rangle$. In other words, if T is a symmetry transformation, transforming and then time-evolving any state is the same as first time-evolving, and then applying the symmetry transformation. This is summarized by the diagram²

$$\begin{array}{ccc} |\psi(t)\rangle & \xrightarrow{T} & |\tilde{\psi}(t)\rangle \\ U(t) \uparrow & & \uparrow U(t) \\ |\psi(0)\rangle & \xrightarrow{T} & |\tilde{\psi}(0)\rangle \end{array} \quad (6.2.2)$$

showing that $|\tilde{\psi}(t)\rangle$ can be constructed from $|\psi(0)\rangle$ by following either path.

The condition (6.2.1) that the transformation T commute with the time evolution operator is equivalent to the condition that T commute with the Hamiltonian,

$$[T, H] \equiv T H - H T = 0. \quad (6.2.3)$$

Symmetries have many useful consequences. One class of applications follows directly from the basic definition embodied in the diagram (6.2.2) — if you understand how some state $|\psi\rangle$ evolves in time, you can immediately predict how the transformed state $|\tilde{\psi}\rangle$ will evolve. For example, we will be discussing a transformation known as charge conjugation which interchanges particles and antiparticles, turning a proton into an antiproton, a π^+ into a π^- , etc. Charge conjugation is a symmetry of strong and electromagnetic interactions. This symmetry implies that the rate at which a Δ^{++} baryon decays to a proton and π^+ is the same as the rate at which the $\bar{\Delta}^{--}$ antibaryon (the antiparticle of the Δ^{++}) decays to an antiproton and π^- . And it implies that the cross section for π^+ scattering on protons must be the same as the cross section for π^- mesons to scatter on antiprotons.

A second category of applications follows from the commutativity (6.2.3) of a symmetry transformation with the Hamiltonian. Recall, from linear algebra, that two matrices (or linear operators)

¹This definition applies to time-independent symmetry transformations. A more general formulation is required for Lorentz boosts and time-reversal transformations which have the effect of changing the meaning of time.

²Mathematicians call this a *commutative diagram*.

are simultaneously diagonalizable if and only if they commute. Consequently, if T is a symmetry then there exist states $\{|\psi_n\rangle\}$ which are simultaneous eigenstates of the Hamiltonian and of the transformation T ,

$$H|\psi_n\rangle = E_n|\psi_n\rangle, \quad (6.2.4a)$$

$$T|\psi_n\rangle = t_n|\psi_n\rangle. \quad (6.2.4b)$$

The eigenvalue E_n of the Hamiltonian is the energy of the state $|\psi_n\rangle$ — Hamiltonian eigenstates are called energy eigenstates or *stationary states*. The latter name reminds one that energy eigenstates have simple harmonic time dependence; the time-dependent Schrodinger equation (6.1.3) implies that the time evolution of an energy eigenstate is just $|\psi_n(t)\rangle = e^{-iE_n t/\hbar}|\psi_n(0)\rangle$. Hence, an eigenstate of the Hamiltonian is also an eigenstate of the time evolution operator $U(t)$, with eigenvalue $e^{-iE_n t/\hbar}$.

Because the Hamiltonian is a Hermitian operator, its eigenvalues E_n must be real. Because the symmetry transformation T is a unitary operator, its eigenvalues t_n must be phase factors, $t_n = e^{i\phi_n}$ for some real phase ϕ_n .³ The simultaneous diagonalizability of H and T implies that energy eigenstates can also be labeled by an additional number, t_n (or equivalently ϕ_n), which characterizes the effect of the symmetry transformation T on the state. Phrased differently, the eigenvalues of a symmetry transformation T define a quantum number which distinguishes different classes of eigenstates.

There are many examples of this. A particle moving in a (one dimensional) square well potential,

$$V(x) = \begin{cases} 0, & |x| < L/2; \\ \infty, & \text{otherwise,} \end{cases}$$

is an example of a theory in which a parity transformation, $x \rightarrow -x$, is a symmetry. Consequently, energy eigenstates in this potential can be labeled by their parity; their wavefunctions must either be even, $\psi_n(x) = \psi_n(-x)$, or odd, $\psi_n(x) = -\psi_n(-x)$, under $x \rightarrow -x$.

More generally, real particles (in infinite, empty space) can be labeled by their momentum and energy, as well as their angular momentum, electric charge, baryon number, and lepton number. As we will discuss below, these are all examples of quantum numbers which are associated with specific symmetries.

A third category of applications of symmetries involves time evolution of states which are eigenstates of some symmetry T but are not eigenstates of the Hamiltonian. Let $|\psi_{\text{in}}\rangle$ be some initial state which is an eigenstate of the symmetry T with eigenvalue t_{in} . Let $|\psi_{\text{out}}\rangle$ be some final state which is an eigenstate of the symmetry T with eigenvalue t_{out} . For example, think of $|\psi_{\text{in}}\rangle$ as the initial state of some scattering experiment involving two incoming particles of types a and b , while $|\psi_{\text{out}}\rangle$ is a final state describing outgoing particles of types c and d . Can the scattering process $a + b \rightarrow c + d$ occur? In other words, can the matrix element $\langle\psi_{\text{out}}|U(t)|\psi_{\text{in}}\rangle$, giving the amplitude for the initial state to evolve into the chosen final state, be non-zero? The answer is no — unless the symmetry eigenvalues of the initial and final states coincide. That is,⁴

$$\langle\psi_{\text{out}}|U(t)|\psi_{\text{in}}\rangle = 0 \quad \text{if} \quad t_{\text{in}} \neq t_{\text{out}}. \quad (6.2.5)$$

³To show this, multiply each side of the eigenvalue condition (6.2.4b) by its Hermitian conjugate to obtain $\langle\psi_n|T^\dagger T|\psi_n\rangle = t_n^* t_n \langle\psi_n|\psi_n\rangle$. The left hand side is just $\langle\psi_n|\psi_n\rangle$ since T is unitary, so this condition can only be satisfied if $|t_n| = 1$.

⁴To see this, use the fact that T is unitary to write $1 = T^\dagger T$. Inserting the identity operator does nothing, so $\langle\psi_{\text{out}}|U(t)|\psi_{\text{in}}\rangle = \langle\psi_{\text{out}}|T^\dagger T U(t)|\psi_{\text{in}}\rangle = \langle\psi_{\text{out}}|T^\dagger U(t)T|\psi_{\text{in}}\rangle$. The last step used the condition that T is a symmetry to interchange T and $U(t)$. By assumption, $|\psi_{\text{in}}\rangle$ is an eigenstate of T , $T|\psi_{\text{in}}\rangle = t_{\text{in}}|\psi_{\text{in}}\rangle$, and similarly $T|\psi_{\text{out}}\rangle = t_{\text{out}}|\psi_{\text{out}}\rangle$. Taking the Hermitian conjugate of this last relation gives $\langle\psi_{\text{in}}|T^\dagger = \langle\psi_{\text{in}}|t_{\text{in}}^*$. Use these eigenvalue relations for $|\psi_{\text{in}}\rangle$ and $\langle\psi_{\text{out}}|$ to simplify $\langle\psi_{\text{out}}|T^\dagger U(t)T|\psi_{\text{in}}\rangle$. The result is $\langle\psi_{\text{out}}|U(t)|\psi_{\text{in}}\rangle = t_{\text{out}}^* t_{\text{in}} \langle\psi_{\text{out}}|U(t)|\psi_{\text{in}}\rangle$. Note that exactly the

The key point here is that symmetries can be used to understand what types of final states can, or cannot, occur in many scattering experiments. As we will discuss below, conservation laws for energy, momentum, angular momentum, electric charge, and baryon and lepton number (and more) can all be viewed as particular cases of this general result.

A final type of application concerns sets of multiple symmetry transformations. Suppose transformations T_1 and T_2 are both symmetries, and hence both commute with the Hamiltonian. But suppose that T_1 and T_2 do not commute with each other. Then one cannot simultaneously diagonalize the Hamiltonian and *both* T_1 and T_2 , although one can find a basis in which H and, say, T_1 are diagonal. Let $|\psi_n\rangle$ be one of these basis states, so that $H|\psi_n\rangle = E_n|\psi_n\rangle$ and $T_1|\psi_n\rangle = t_{1,n}|\psi_n\rangle$. Applying the symmetry transformation T_2 to the state $|\psi_n\rangle$ will produce some state $|\tilde{\psi}_n\rangle$ which must also be an eigenstate of the Hamiltonian with exactly the same energy E_n .⁵ It may be a linearly independent state — $|\tilde{\psi}_n\rangle$ need not be proportional to $|\psi_n\rangle$. Consequently, the existence of symmetries which do not mutually commute can lead to degenerate energy levels, *i.e.*, multiple linearly independent states with exactly the same energy. Angular momentum eigenstates provide a familiar example of this. In any theory which is rotationally invariant, every energy eigenstate with non-zero angular momentum must be part of a degenerate multiplet. If the angular momentum is $j\hbar$, then the multiplet will contain $(2j + 1)$ states, since the projection of the angular momentum along a chosen quantization axis can take any of $2j + 1$ values, $\{-j, -j+1, \dots, j-1, j\}$, but the energy cannot depend on the value of this projection.

6.3 Continuous symmetries

Continuous symmetries are symmetries which depend (continuously!) on some parameter which controls the magnitude of the transformation. Examples include translations and rotations. Let $T(a)$ denote a continuous symmetry depending on the real parameter a . Assume (without loss of generality) that $a = 0$ corresponds to doing nothing, so that $T(0)$ equals the identity operator. One can always choose to define the parameterization so that $T(a/2)^2 = T(a)$, or more generally that $T(a/N)^N = T(a)$ for any N . This implies that $T(a)$ depends exponentially on the parameter a , so that one can write

$$T(a) = e^{iQa}, \quad (6.3.1)$$

for some operator Q , which is called the *generator* of the symmetry $T(a)$. In order for $T(a)$ to be unitary (as required), the generator Q must be Hermitian. Note that the relation between $T(a)$ and Q is completely analogous to the relation between the time evolution operator and the Hamiltonian; the Hamiltonian (divided by $-\hbar$) is the generator of time evolution.

The condition (6.2.3) that $T(a)$ commute with H implies that the generator Q of any continuous symmetry must also commute with the Hamiltonian,

$$[Q, H] = 0. \quad (6.3.2)$$

Once again, this implies that Q and H are simultaneously diagonalizable.

same matrix element appears on both sides. To satisfy this equation either $t_{\text{out}}^* t_{\text{in}}$ must equal 1, or else the matrix element $\langle\psi_{\text{out}}|U(t)|\psi_{\text{in}}\rangle$ must vanish. Because the eigenvalues of T are pure phases, $t_{\text{out}}^* = 1/t_{\text{out}}$. Hence the condition that $t_{\text{out}}^* t_{\text{in}} = 1$ is the same as the statement that t_{in} and t_{out} coincide.

⁵This follows from the given assumption that T_2 is a symmetry, so that it commutes with H . Consequently, $H|\tilde{\psi}_n\rangle = H(T_2|\psi_n\rangle) = T_2(H|\psi_n\rangle) = T_2(E_n|\psi_n\rangle) = E_n|\tilde{\psi}_n\rangle$.

Note that, given some continuous symmetry $T(a)$, one can extract the associated generator Q by performing a Taylor series expansion of $T(a)$ about $a = 0$. Keeping just the first non-trivial term gives $T(a) = 1 + iQa + \dots$, so that $Q = -i \frac{d}{da} T(a)|_{a=0}$. Alternatively, given any Hermitian operator Q which commutes with the Hamiltonian, one can construct a unitary symmetry transformation by exponentiating iQ (times an arbitrary real number), as in (6.3.1). So one can regard either the generator Q , or the finite transformation $T(a)$, as defining a continuous symmetry.

6.4 Spacetime symmetries

Spacetime symmetries are symmetries which reflect the underlying geometry of Minkowski space. Translations in both space and time, spatial rotations, and Lorentz boosts are all continuous spacetime symmetries. These are symmetries of the laws of physics, as currently understood. The additional discrete transformations of parity and time reversal are *not* exact symmetries, and will be discussed below as examples of approximate symmetries.

The total momentum operator \vec{P} (divided by \hbar) is the generator of spatial translations. Hence, the unitary operator $T_{\text{trans}}(\Delta\vec{x})$ which has the effect of performing a spatial translation through a displacement $\Delta\vec{x}$ is an exponential of momentum,⁶

$$T_{\text{trans}}(\Delta\vec{x}) = e^{i\vec{P}\cdot\Delta\vec{x}/\hbar}. \quad (6.4.1)$$

In any translationally invariant theory, the total momentum \vec{P} commutes with the Hamiltonian (and hence with the time evolution operator). Therefore, conservation of momentum is a direct consequence of spatial translation invariance.

The Hamiltonian H (divided by $-\hbar$) is the generator of time translations, and the associated unitary operator which has the effect of performing a time translation through an interval Δt is precisely the time evolution operator

$$U(\Delta t) = e^{-iH\Delta t/\hbar}. \quad (6.4.2)$$

The Hamiltonian commutes with itself, and therefore it satisfies the conditions defining the generator of a symmetry. Since the Hamiltonian is the operator which measures energy, this shows that conservation of energy is a direct consequence of time translation invariance.

A general spacetime translation with displacement $\Delta x = (\Delta x^0, \Delta\vec{x})$ is just a combination of a spatial translation through $\Delta\vec{x}$ and a time translation through $\Delta t = \Delta x^0/c$. The unitary operator which implements this spacetime translation is the product of $T_{\text{trans}}(\Delta\vec{x})$ and $U(\Delta t)$.⁷ Defining $P^0 \equiv H/c$ allows one to write this as a single exponential of a Minkowski space dot product,

$$T_{\text{trans}}(\Delta x) = T_{\text{trans}}(\Delta\vec{x}) \times U(\Delta x^0/c) = e^{iP^\mu \Delta x_\mu/\hbar}. \quad (6.4.3)$$

The total angular momentum \vec{J} (divided by \hbar) is the generator of rotations. The unitary operator which implements a rotation through an angle θ about an axis defined by a unit vector \hat{n} is an

⁶Recall that in single particle quantum mechanics in, for simplicity, one dimension, the coordinate representation of the momentum operator is $\frac{\hbar}{i} \frac{\partial}{\partial x}$. So acting with $\exp(iP\Delta x/\hbar) = 1 + iP\Delta x/\hbar - \frac{1}{2}(iP\Delta x/\hbar)^2 + \dots$ on an arbitrary state $|\Psi\rangle$ is the same as acting with $\exp(\Delta x \frac{\partial}{\partial x}) = 1 + \Delta x \frac{\partial}{\partial x} + \frac{1}{2}(\Delta x \frac{\partial}{\partial x})^2 + \dots$ on the wavefunction $\Psi(x)$. This produces $\Psi(x) + \Delta x \Psi'(x) + \frac{1}{2}(\Delta x)^2 \Psi''(x) + \dots$ which is a Taylor series expansion of the translated wavefunction $\Psi(x + \Delta x)$.

⁷Because $T_{\text{trans}}(\Delta\vec{x})$ commutes with $U(\Delta t)$ (or equivalently, because \vec{P} commutes with H), the order in which one performs this product doesn't matter.

exponential of the angular momentum projection along \hat{n} ,

$$T_{\text{rot}}(\theta, \hat{n}) = e^{i\theta \hat{n} \cdot \vec{J}/\hbar}. \quad (6.4.4)$$

The total angular momentum \vec{J} commutes with the Hamiltonian in any rotationally invariant theory. Hence, conservation of angular momentum is a direct consequence of spatial rotation invariance.

One can also define operators \vec{G} which are the generators of Lorentz boosts, so that the unitary operator which implements a boost along some direction \hat{n} can be written as an exponential,

$$T_{\text{boost}}(\eta, \hat{n}) = e^{i\eta \hat{n} \cdot \vec{G}/\hbar}. \quad (6.4.5)$$

The parameter η , is precisely the *rapidity*, introduced in section 2.6. Recall that the rapidity determines the boost velocity via $v/c = \tanh \eta$. In contrast to the situation with rotations and translations, the boost generators \vec{G} do not commute with H because Lorentz boosts change the meaning of time.⁸ Because of this, invariance under Lorentz boosts does not lead to any additional conserved quantities analogous to momentum or angular momentum.

6.5 Charge, lepton, and baryon number

The electric charge Q is an operator which, when acting on any state containing particles with individual charges $\{q_i\}$ ($i = 1, \dots, N$), measures the sum of all these charges,

$$Q|\Psi\rangle = q_{\text{tot}}|\Psi\rangle \quad (6.5.1)$$

with $q_{\text{tot}} \equiv \sum_{i=1}^N q_i$. So, as its name suggests, Q measures the total electric charge of any state. More precisely, each q_i should be understood as the charge of a particle in units of $|e|$. The electric charges of all known particles which can be produced in isolation are integer multiples of $|e|$; this is known as *charge quantization*. Hence the operator Q will always have integer eigenvalues.

Maxwell's equations are inconsistent if electric charge is not conserved. Therefore, Q must commute with the Hamiltonian (or with the time evolution operator), and hence Q is the generator of a continuous symmetry,

$$T_Q(\alpha) \equiv e^{i\alpha Q}. \quad (6.5.2)$$

Applying this transformation to any state multiplies the state by a phase proportional to its electric charge, $T_Q(\alpha)|\Psi\rangle = e^{i\alpha q_{\text{tot}}}|\Psi\rangle$.

In precisely the same fashion, one may regard baryon number B and lepton number L as quantum operators which measure the total baryon number or lepton number, respectively. And one may exponentiate either of these operators to form continuous symmetry transformations generated by B and L ,

$$T_B(\alpha) \equiv e^{i\alpha B}, \quad T_L(\alpha) \equiv e^{i\alpha L}. \quad (6.5.3)$$

⁸The boost generators \vec{G} depend explicitly on time, and the required condition that they must satisfy turns out to be $\frac{\partial}{\partial t} \vec{G} + i[H, \vec{G}] = 0$.

6.6 Approximate symmetries

There are many circumstances where it is useful to consider transformations which are almost, but not quite, symmetries of a theory. Consider, for example, a hydrogen atom in a weak background magnetic field. If the magnetic field were zero, then the Hamiltonian describing the dynamics of the atom would be rotationally invariant. As noted above, this implies that energy eigenstates with non-zero angular momentum must form degenerate energy levels. Turning on a magnetic field breaks three dimensional rotation invariance, since the Hamiltonian will now contain terms which depend on the direction of the background magnetic field. (More precisely, turning on a magnetic field reduces the symmetry from three dimensional rotation invariance down to one dimensional rotation invariance with respect to rotations about the direction of the magnetic field.) The presence of the magnetic field will perturb the energy levels of the atom, and lift the degeneracy of energy eigenstates with differing angular momentum projections along the direction of the field. But if the magnetic field is sufficiently weak, then the splitting induced by the field will be small (compared to the spacings between energy levels in the absence of the field). In this circumstance, it makes sense to regard the Hamiltonian of the system as the sum of a “big” rotationally invariant piece H_0 , which describes the atom in the absence of a magnetic field, plus a “small” perturbation ΔH which describes the interaction with the weak magnetic field,

$$H = H_0 + \Delta H. \quad (6.6.1)$$

One can systematically calculate properties of the atom as a Taylor expansion in the size of ΔH . The starting point involves ignoring ΔH altogether and understanding the properties of H_0 . And when studying the physics of H_0 , one can use full three dimensional rotation symmetry.

Exactly the same approach can be applied to particle physics to separate the effects of weak and electromagnetic interactions from those of strong interactions.

6.7 Flavor symmetries

Strong interactions, as described by quantum chromodynamics, preserve the net number of quarks of each flavor. Strong interactions can cause the creation or annihilation of quark-antiquark pairs of any given flavor, but this does not change the number of quarks minus antiquarks of each flavor. This is also true of electromagnetic interactions, but not weak interactions. Consequently, in a hypothetical world in which weak interactions are turned off, operators which measure the number of quarks minus antiquarks of each flavor,

$$N_u = (\# u \text{ quarks}) - (\# \bar{u} \text{ quarks}), \quad (6.7.1a)$$

$$N_d = (\# d \text{ quarks}) - (\# \bar{d} \text{ quarks}), \quad (6.7.1b)$$

$$N_s = (\# s \text{ quarks}) - (\# \bar{s} \text{ quarks}), \quad (6.7.1c)$$

$$N_c = (\# c \text{ quarks}) - (\# \bar{c} \text{ quarks}), \quad (6.7.1d)$$

$$N_b = (\# b \text{ quarks}) - (\# \bar{b} \text{ quarks}), \quad (6.7.1e)$$

$$N_t = (\# t \text{ quarks}) - (\# \bar{t} \text{ quarks}), \quad (6.7.1f)$$

all commute with the Hamiltonian. Therefore, all these operators may be regarded as generators of continuous symmetries. Note that baryon number equals the total number of quarks minus

antiquarks, divided by three,

$$B = \frac{1}{3} [(\# \text{ quarks}) - (\# \text{ antiquarks})] = \frac{1}{3} \sum_{f=u,d,s,c,b,t} N_f, \quad (6.7.2)$$

since baryons contain three quarks, while antibaryons contain three antiquarks. For historical reasons, it is conventional to refer to *strangeness* as the number of strange *anti*-quarks minus quarks,

$$S \equiv -N_s = (\# \bar{s} \text{ quarks}) - (\# s \text{ quarks}). \quad (6.7.3)$$

This definition assigns strangeness +1 to the K^+ meson (and this convention predates the development of QCD and the quark model of hadrons).

In a world without weak interactions, π mesons would be absolutely stable because there are no lighter strongly interacting hadrons which pions could decay into. Kaons (K mesons) would also be stable, even though they are over three times heavier than pions, because K mesons are the lightest hadron with nonzero strangeness (and strangeness is conserved by strong interactions). Similarly, the Ω^- baryon, containing three strange quarks, would be stable because there is no other combination of hadrons it might decay to which have lower energy, together with baryon number one and strangeness minus three.

Completely analogous arguments apply to hadrons containing heavier charm and bottom quarks. In the absence of weak interactions, there would be many additional stable hadrons containing net charm, or net “bottomness”.

6.8 Isospin

Figure 6.1 graphically displays the mass spectrum of light mesons and baryons. Looking at this figure, or the tables containing information about hadrons in section 5.2, many degeneracies or near-degeneracies are immediately apparent. For example, the masses of the π^+ and π^- mesons are the same, while the mass of the π^0 meson is only slightly different. The mass of the neutron is quite close to that of the proton. The masses of the Σ^+ , Σ^0 and Σ^- baryons are all different, but very close to each other. And likewise for many other “multiplets” of mesons and baryons. What explains all these near degeneracies?

Comparing the quark content of various hadrons (and referring to Tables 5.3–5.6 as needed), one sees that the near-degeneracies are all associated with substitutions of u for d quarks, or vice-versa. For example, the Σ^+ baryon has two up and one strange quark. Replacing one up quark by a down converts the Σ^+ into a Σ^0 , whose mass is larger than that of the Σ^+ by $3.3 \text{ MeV}/c^2$, which is less than 0.3% of the Σ^+ mass. Replacing the remaining up quark by a down converts the Σ^0 into a Σ^- , whose mass is an additional $4.8 \text{ MeV}/c^2$ larger.

The mass differences among the three Σ baryons, the three π mesons, between the neutron and proton, or within any of the other nearly degenerate multiples, must arise from some combination of two effects. First, the masses of up and down quarks are not quite the same. The mass of a down quark (*c.f.* Table 5.1) is a few MeV/c^2 larger than that of an up quark. This mass difference is tiny compared to the masses of hadrons, but it is comparable to the few MeV/c^2 mass splittings within the various near-degenerate multiplets.

Second, the interactions of up and down quarks are different. They have differing electric charges ($2/3$ for u , and $-1/3$ for d), which means that their electromagnetic interactions are not the same.



Figure 6.1: The mass spectrum of light mesons and baryons. Each column show mesons or baryons with a particular charge. Only meson states in the lightest $J = 0$ and $J = 1$ nonets, and baryons in the lightest $J = 1/2$ octet and $J = 3/2$ decuplet, are labeled. The unlabeled light grey lines show higher excited states. Many of these higher states are quite broad, with decay widths ranging from 50 MeV to several hundred MeV.

Their weak interactions also differ. But, as far as hadronic masses are concerned, the effects of weak and electromagnetic interactions are small perturbations on top of the dominant effects due to strong interactions, and strong interactions are flavor-blind. In a hypothetical world in which weak and electromagnetic interactions are absent, and in which up and down quarks have the same mass, these near-degeneracies would all become exact degeneracies.

This should remind you of angular momentum multiplets. In any rotationally invariant theory, every state with angular momentum $J\hbar$ is part of a multiplet containing $2J + 1$ degenerate states. A rotation transforms the different states in the multiplet into linear combinations of each other. The simplest non-trivial case is $J = \frac{1}{2}\hbar$, whose multiplets contain two (linearly independent) states conventionally chosen to have angular momentum up or down along some given axis, $|\uparrow\rangle$ and $|\downarrow\rangle$. The action of a rotation corresponds to a linear transformation,

$$\begin{pmatrix} |\uparrow'\rangle \\ |\downarrow'\rangle \end{pmatrix} = M \begin{pmatrix} |\uparrow\rangle \\ |\downarrow\rangle \end{pmatrix}. \quad (6.8.1)$$

For a rotation about an axis \hat{n} through an angle θ , the matrix M has the form

$$M = e^{i(\theta/2)\hat{n}\cdot\vec{\sigma}} = \left(\cos \frac{\theta}{2} + i\hat{n} \cdot \vec{\sigma} \sin \frac{\theta}{2} \right), \quad (6.8.2)$$

with $\vec{\sigma}$ denoting the Pauli matrices, $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, and $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. You can easily check that M is a unitary matrix with determinant equal to one. The space (or group) of such 2×2 matrices is called $SU(2)$.

When u and d quarks are degenerate (and weak and electromagnetic interactions are turned off), there is an analogous symmetry which transforms up and down quark states into new linear combinations of the two flavors,

$$\begin{pmatrix} |u'\rangle \\ |d'\rangle \end{pmatrix} = M \begin{pmatrix} |u\rangle \\ |d\rangle \end{pmatrix}, \quad (6.8.3)$$

where M is any 2×2 unitary matrix with determinant one. This symmetry is called *isospin* (or isotopic spin) rotation.

The mathematical structure of isospin rotations is identical to the structure of spatial rotations (although isospin has nothing to do with ordinary spatial rotations). There are three generators of isospin rotations, I_1 , I_2 and I_3 . Their commutation relations have the same form as the commutation relations of angular momentum operators (which are the generators of spatial rotations),

$$[I_a, I_b] = i\epsilon_{abc} I_c. \quad (6.8.4)$$

Total isospin is denoted by I , and can have either integer or half-integer values. An up quark has $I_3 = +1/2$, while a down quark has $I_3 = -1/2$. Hence an up quark behaves (with respect to isospin rotations) just like an up spin does (with respect to spatial rotations). This parallel is the origin of the names ‘up’ and ‘down’ for the two lightest quarks. For antiquarks, the assignments are reversed, a \bar{u} quark has $I_3 = -1/2$ while a \bar{d} has $I_3 = +1/2$.⁹

When you build states containing multiple up and down quarks (or antiquarks), the addition of isospin works just like adding angular momentum. For example, combining two isospin one-half

⁹It is $\begin{pmatrix} -|\bar{d}\rangle \\ |\bar{u}\rangle \end{pmatrix}$ which transforms in the same manner as $\begin{pmatrix} |u\rangle \\ |d\rangle \end{pmatrix}$, namely $\begin{pmatrix} -|\bar{d}'\rangle \\ |\bar{u}'\rangle \end{pmatrix} = M \begin{pmatrix} -|\bar{d}\rangle \\ |\bar{u}\rangle \end{pmatrix}$.

objects can yield either isospin 0 or isospin 1. An antisymmetric combination of u and d quarks,

$$(ud - du), \tag{6.8.5}$$

gives $I = 0$, while a symmetric combination gives isospin one. Hence, the three $I = 1$ flavor states of two quarks are

$$(uu), (ud + du), (dd), \tag{6.8.6}$$

with I_3 for these states equaling $+1$, 0 , and -1 , respectively. Similarly, when three u or d quarks are combined (as in a baryon), the resulting isospin can be either $1/2$ or $3/2$.

Looking back at the nearly degenerate set of particles shown in Figure 6.1, the π^+ , π^0 and π^- mesons form an $I = 1$ multiplet, whose masses would be exactly equal were it not for the perturbing effects of weak and electromagnetic interactions and the up and down quark mass difference. Similarly, the K^+ and K^0 mesons (whose quark contents are $u\bar{s}$ and $d\bar{s}$, respectively) form an isospin $1/2$ multiplet with strangeness one, while the K^- and \bar{K}^0 mesons (with quark content $s\bar{u}$ and $s\bar{d}$) form another $I = 1/2$ multiplet with $S = -1$. The three rho mesons form another $I = 1$ multiplet. Turning to the baryons, the two nucleons form an $I = 1/2$ multiplet (as do the Ξ baryons), while the three Σ baryons have $I = 1$ and the four Δ baryons have $I = 3/2$.

Conservation of isospin (by strong interactions) can also be used to explain a variety of more detailed hadronic properties, including the fraction of Δ^+ decays which yield $p\pi^0$ versus $n\pi^+$, or the fraction of different pion pairs produced by ρ decays. Isospin conservation can also be used to explain the absence of many unseen decay modes. For example, the $\Lambda(1690)$ is an excited state of the Λ baryon, with 1690 MeV rest energy and quark content uds . Roughly 25% of the time, a $\Lambda(1690)$ decays to a Λ plus two pions. But it doesn't decay to a Λ plus a single pion, despite that fact that more energy would be available for conversion into kinetic energy if only a single pion were produced. To understand why decays to a Λ plus two pions are favored, note that the Λ baryon, and its excited states like the $\Lambda(1690)$ have $I = 0$, while pions have $I = 1$. So the decay $\Lambda(1690) \rightarrow \Lambda + \pi$ would have $\Delta I = 1$ — an initial state with isospin zero and a final state of isospin one. But in the final state of the observed decay $\Lambda(1690) \rightarrow \Lambda + \pi + \pi$, the total isospin is the combination of two $I = 1$ pions plus the $I = 0$ Λ baryon. Adding two isospin one objects can yield isospin two, one, or zero. So if the final pions combine to form zero isospin, then isospin will be conserved in this decay.

Because isospin is only an approximate symmetry, predictions one can make using isospin invariance are not exact results in the real world. However, because the up and down quark mass difference is so small, and weak and electromagnetic interactions are *much* weaker than strong interactions, predictions which follow from isospin invariance are quite accurate — violations are typically at or below the 1% level.

6.9 Parity

A parity transformation, denoted P , has the effect of reversing all spatial coordinate axes. Therefore, a parity transformation acting on a state of a single particle located at some spatial position \vec{x} produces a state in which the particle is located at $-\vec{x}$. Fourier transforming to the momentum representation, one can equally well say that a parity transformation acting on a single particle state with momentum \vec{p} will produce a state with momentum $-\vec{p}$. Written symbolically, this suggests that if $|\vec{p}\rangle$ represents a state of some particle with momentum \vec{p} , then the parity transformed state should

be $P|\vec{p}\rangle = |-\vec{p}\rangle$. This is not quite right, however, as the unitary transformation P can also produce a change in the overall phase of the state. Therefore, in general one must write

$$P|\vec{p}\rangle = \eta_P |-\vec{p}\rangle, \quad (6.9.1)$$

where η_P is some phase factor which can depend on the type of particle under consideration. A parity transformation does not change the spin or angular momentum of a particle.¹⁰

Applying two parity transformations amounts to reversing the directions of all spatial coordinate axes, and then reversing them all over again. This has no net effect. Hence, as an operator, parity must square to the identity, $P^2 = 1$. This implies that the phase η_P appearing in Eq. (6.9.1) must square to one, $\eta_P^2 = 1$, so either $\eta_P = +1$ or $\eta_P = -1$. This sign is called the *intrinsic parity* of a particle. Some particles (such as protons and neutrons) have positive intrinsic parity, while others (such as pions and photons) have negative parity. One can show (from relativistic quantum mechanics) that for particles which are bosons, the intrinsic parities of antiparticles are the same as the corresponding particles, while for fermions, antiparticles have intrinsic parities which are opposite to the corresponding particle.

For multiparticle states, the form of the wavefunction describing the relative motion of the particles also affects the behavior of the state under a parity transformation. If two particles A and B , viewed in their mutual center-of-momentum frame, have orbital angular momentum ℓ , then an additional factor of $(-1)^\ell$ appears in the result of a parity transformation,¹¹

$$P|\Psi_\ell^{A+B}\rangle = \eta_P^A \eta_P^B (-1)^\ell |\Psi_\ell^{A+B}\rangle, \quad (6.9.2)$$

where η_P^A and η_P^B are the intrinsic parities of the individual particles.

Intrinsic parities can be assigned to particles in such a way that parity is a symmetry of strong and electromagnetic interactions. In particular, the light mesons in Tables 5.3 and 5.4 are all parity-odd (*i.e.*, they have negative intrinsic parity).¹² The photon is also parity-odd. The baryons listed in Tables 5.5 and 5.6 are all parity-even.

Parity is *not* a symmetry of weak interactions. This will be discussed further in the next chapter. So parity is an approximate symmetry, useful for understanding strong or electromagnetic processes, but is not a true symmetry of nature.

6.10 Charge conjugation

A charge conjugation transformation, denoted C , has the effect of interchanging particles and antiparticles, $C|A\rangle = |\bar{A}\rangle$. So, for example, charge conjugation turns a proton into an antiproton, an electron into a positron, and a π^+ into a π^- . For particles which are their own antiparticles (“self-conjugate” particles), such as the photon and π^0 , there can also be an overall phase factor,

$$C|A\rangle = \eta_C |A\rangle \quad (\text{self-conjugate particles}). \quad (6.10.1)$$

¹⁰Recall that $\vec{L} = \vec{r} \times \vec{p}$. Since a parity transformation reverses both \vec{r} and \vec{p} , the (orbital) angular momentum \vec{L} does not change. The intrinsic spin transforms in the same fashion as \vec{L} .

¹¹This factor comes from the behavior of spherical harmonics under the transformation $\vec{x} \rightarrow -\vec{x}$, namely $Y^{lm}(-\hat{x}) = (-1)^l Y^{lm}(\hat{x})$.

¹²All these mesons are *s*-wave quark-antiquark bound states, so they have no orbital angular momentum. Their negative parity reflects the opposite intrinsic parities of fermions and antifermions. Higher energy even-parity mesons do exist; these may be understood as bound states with non-zero orbital angular momentum.

These phases, which depend on particle type, can be defined in such a way that charge conjugation is a symmetry of strong and electromagnetic interactions. However, charge conjugation is not an invariance of weak interactions. So charge conjugation is only an approximate symmetry, like parity, but is very useful when considering strong or electromagnetic processes.

Charge conjugation has no effect on momenta or spins of particles, but the electric charge and other flavor quantum numbers (B , L , S , I_3) all have their signs changed by a charge conjugation transformation. Hence, only particles which are neutral (and whose strangeness, I_3 , baryon, and lepton numbers all vanish) can be self-conjugate.

The photon is charge-conjugation odd (*i.e.*, its phase $\eta_C = -1$). To understand why, consider a classical electromagnetic field produced by some charge or current density. A charge conjugation transformation would change the electrically charged particles which are the source of the electromagnetic field into their oppositely charged antiparticles. In other words, the charge and current densities appearing in Maxwell's equations would change sign. Since Maxwell's equations are linear, this implies that the electromagnetic field itself would change sign. The photon is a quantized excitation in the electromagnetic field. Its behavior under charge conjugation reflects the behavior of a classical EM field: it changes sign.

Since a single photon is charge-conjugation odd, a multi-photon state containing N photons is charge-conjugation even if N is even, and charge-conjugation odd if N is odd. The neutral pion (dominantly) decays to two photons, while π^0 decay to three photons has not been observed. Pion decay is an electromagnetic process, which for which charge conjugation is a symmetry. Hence, the neutral pion is charge-conjugation even.

As an example of the utility of charge conjugation symmetry, consider positronium. This is the name given to bound states of an electron and a positron. Since an electron and positron have opposite electric charges, they have an attractive Coulomb interaction, and consequently form Coulombic bound states — just like the electron and proton in a hydrogen atom. Relative to hydrogen, there are two noteworthy differences. First, because the positron mass equals the electron mass (instead of being much much heavier like a proton), spacings between energy levels in positronium are half the corresponding spacings in hydrogen. More importantly, positronium is not stable. Unlike a hydrogen atom, the electron and positron in positronium can (and eventually will) annihilate into photons.

Consider positronium in its $1s$ ground state. How many photons will be produced when it decays? Answering this requires a consideration of symmetries, not hard calculations. Energy and momentum conservation forbid decay into a single photon. To understand whether decay into two photons is possible, we need to specify the initial state more carefully. Since the electron and positron each have spin $1/2$, the total spin of positronium can be either $S = 0$ or $S = 1$. Since a $1s$ state has no orbital angular momentum, the total angular momentum is the same as the spin. The singlet ($S = 0$) state of positronium is known as *para-positronium*, while the triplet ($S = 1$) state is called *ortho-positronium*. Recall that when two spins are combined to form $S = 1$, the spin wavefunction [either $\uparrow\uparrow$, $(\uparrow\downarrow + \downarrow\uparrow)$, or $\downarrow\downarrow$, depending on the value of S_z] is symmetric under interchange of the two spins. But the singlet spin wavefunction, $(\uparrow\downarrow - \downarrow\uparrow)$, is antisymmetric under interchange of spins.

The Pauli principle implies that a state of two fermions must be antisymmetric under interchange of the two fermions. This means interchanging the types, spins, and positions simultaneously. In contrast, charge conjugation in positronium interchanges the types of the two fermions, without affecting their spins or positions. Interchanging the positions of the electron and positron flips their relative separation, $\vec{r} \rightarrow -\vec{r}$. Since a $1s$ state has a rotationally invariant spatial wavefunction, swapping the positions of the electron and positron does not change the value of the wavefunction. In

the $S = 1$ spin triplet (ortho-positronium), swapping the two spins also does not change the value of the wavefunction, since the spin wavefunction is symmetric. Hence, the action of charge conjugation in ortho-positronium is the same as completely interchanging the two particles (because the state is symmetric under interchange of positions and spins). And we know from the Pauli principle that a complete interchange of fermions must flip the sign of the state. Consequently, ortho-positronium must be charge-conjugation odd.

In contrast, in the $S = 0$ spin singlet (para-positronium), the action of charge conjugation differs from that of a complete interchange of the two fermions by an extra minus sign coming from the antisymmetry of the spin wavefunction. Therefore, para-positronium is charge-conjugation even.

We noted above that a multi-photon state is charge conjugation even or odd depending on whether the number of photons is even or odd. Hence, charge conjugation invariance (of electromagnetic interactions) implies that para-positronium must decay to an even number of photons, while ortho-positronium must decay to an odd number of photons. Every additional photon in the final state decreases the rate of decay (by at least one factor of the fine structure constant α). Therefore, singlet positronium should decay to two photons, while triplet positronium should decay, more slowly, to three photons. This is precisely what is observed. The lifetime of spin singlet positronium is 125 ps, while the lifetime of spin triplet positronium is 142 ns.

6.11 Time reversal and CPT

Time reversal, denoted T , is a transformation which has the effect of flipping the sign of time, $t \rightarrow -t$.¹³ So time reversal interchanges the past and the future. If some state $|\Psi_1\rangle$ evolves into state $|\Psi_2\rangle$ after a time interval Δt , then the time-reversed final state $T|\Psi_2\rangle$ will evolve into the time-reversed initial state $T|\Psi_1\rangle$ (after the same time interval Δt) — if time reversal is a symmetry of the dynamics.

As with C and P , time reversal is a symmetry of strong and electromagnetic interactions, but not of weak interactions. However, the product of charge conjugation, parity, and time reversal, or CPT , is a symmetry of all known interactions. In fact, one can prove that any Lorentz invariant theory (which satisfies causality) must be CPT invariant. This is one of the deepest results which follows from combining special relativity and Lorentz invariance, and essentially follows from analytic continuation applied to Lorentz transformations.¹⁴

¹³Because this transformation changes the meaning of time, it is not represented by a unitary operator which commutes with the Hamiltonian. In fact, unlike all other symmetries discussed so far, time reversal, in quantum mechanics, is not represented by a linear operator, but rather by an “anti-linear” operator. Such operators do not satisfy the defining relation of linear operators, $\mathcal{O}(c_1|\Psi_1\rangle + c_2|\Psi_2\rangle) = c_1(\mathcal{O}|\Psi_1\rangle) + c_2(\mathcal{O}|\Psi_2\rangle)$. Instead, for anti-linear operators, $\mathcal{O}(c_1|\Psi_1\rangle + c_2|\Psi_2\rangle) = c_1^*(\mathcal{O}|\Psi_1\rangle) + c_2^*(\mathcal{O}|\Psi_2\rangle)$.

¹⁴The Wikipedia entry on CPT symmetry has a nice sketch of the proof of the CPT theorem, together with a summary of its history.