Appendix A

Basic mathematics

Elementary functions

The *exponential function* e^x (also denoted $\exp x$) is the unique solution to the differential equation $\frac{df}{dx} = f$ with initial condition $f(0) = 1$. Exponential identities include:

$$\tfrac{d}{dx} e^x = e^x \,, \quad e^{-x} = 1/e^x \,, \quad e^x \, e^y = e^{x+y} \,, \quad (e^x)^y = e^{xy} \,. \tag{A.1}$$

If $z = x + iy$ is a complex number, then $e^z = e^x \, e^{iy}$.

The *natural logarithm* $\ln x$ is the inverse function of the exponential, and satisfies $\ln(e^x) = x$. Logarithm identities include:

$$\tfrac{d}{dx} \ln x = 1/x \,, \quad \ln(xy) = \ln x + \ln y \,, \quad \ln(x/y) = \ln x - \ln y \,, \quad \ln(x^y) = y \ln x \,, \quad x^y = e^{y \ln x} \,. \tag{A.2}$$

For complex arguments the logarithm is multi-valued, if $z = re^{i\theta}$ then $\ln z = \ln r + i(\theta + 2\pi n)$ for any integer n. When not specified, $\ln z$ generally refers to the *principal value* for which $-\pi < \mathrm{Im}(\ln z) \le \pi$.

The *trigonometric* functions $\sin x$ and $\cos x$ are linearly independent solutions of the differential equation $\frac{d^2 f}{dx^2} = -f$. The *sine* function, $\sin x$, is the solution with initial conditions $f(0) = 0$ and $f'(0) = 1$, while the *cosine* function, $\cos x$, is the solution with initial conditions $f(0) = 1$ and $f'(0) = 0$. The function $\sin x$ is an odd function of x, while $\cos x$ is an even function. Both $\sin x$ and $\cos x$ are periodic functions of their argument with period 2π. Basic identities satisfied by trigonometric functions include

$$e^{ix} = \cos x + i \sin x \,, \qquad 1 = \cos^2 x + \sin^2 x \,, \tag{A.3}$$

along with:

$$\begin{aligned}
\tfrac{d}{dx} \sin x &= \cos x \,, & \cos x &= \tfrac{1}{2}(e^{ix} + e^{-ix}) \,, && \text{(A.4)} \\
\tfrac{d}{dx} \cos x &= -\sin x \,, & \sin x &= \tfrac{1}{2i}(e^{ix} - e^{-ix}) \,, && \text{(A.5)} \\
\sin(x+y) &= \sin x \cos y + \cos x \sin y \,, & \sin 2x &= 2 \sin x \cos x \,, && \text{(A.6)} \\
\cos(x+y) &= \cos x \cos y - \sin x \sin y \,, & \cos 2x &= \cos^2 x - \sin^2 x = 1 - 2 \sin^2 x \,, && \text{(A.7)} \\
\sin(x + n\pi) &= (-1)^n \sin x \,, & \sin(x + \tfrac{\pi}{2}) &= \cos x \,, && \text{(A.8)} \\
\cos(x + n\pi) &= (-1)^n \cos x \,, & \cos(x + \tfrac{\pi}{2}) &= -\sin x \,. && \text{(A.9)}
\end{aligned}$$

(In the periodicity relations (A.8) and (A.9), n must be an integer.) The auxiliary trigonometric functions *tangent*, *secant*, and *cosecant* are defined by

$$\tan x \equiv \sin x / \cos x\,, \quad \sec x \equiv 1/\cos x\,, \quad \csc x \equiv 1/\sin x\,, \tag{A.10}$$

respectively.

The *hyperbolic* functions $\sinh x$ and $\cosh x$ are linearly independent solutions to the differential equation $\frac{d^2 f}{dx^2} = f$. The *hyperbolic sine* function, $\sinh x$, is the solution with initial conditions $f(0) = 0$ and $f'(0) = 1$, while the *hyperbolic cosine* function, $\cosh x$, is the solution with initial conditions $f(0) = 1$ and $f'(0) = 0$. The function $\sinh x$ is an odd function of x, while $\cosh x$ is an even function. Basic identities satisfied by hyperbolic functions include

$$e^x = \cosh x + \sinh x\,, \qquad \cosh^2 x - \sinh^2 x = 1\,, \tag{A.11}$$

along with:

$$\frac{d}{dx}\sinh x = \cosh x\,, \qquad \cosh x = \tfrac{1}{2}(e^x + e^{-x}) = \cos(ix)\,, \tag{A.12}$$

$$\frac{d}{dx}\cosh x = \sinh x\,, \qquad \sinh x = \tfrac{1}{2}(e^x - e^{-x}) = -i\sin(ix)\,. \tag{A.13}$$

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y\,, \quad \sinh 2x = 2\sinh x \cosh x\,, \tag{A.14}$$

$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y\,, \quad \cosh 2x = \cosh^2 x + \sinh^2 x = 1 + 2\sinh^2 x\,. \tag{A.15}$$

The *hyperbolic tangent* $\tanh x \equiv \sinh x / \cosh x$. For real values of x, $\tanh x$ runs from -1 to $+1$ as x runs from $-\infty$ to ∞.

Series expansions

Binomial
$$(1+x)^a = \sum_{k=0}^{\infty} \binom{a}{k} x^k \quad = 1 + a\,x + \tfrac{1}{2}a(a-1)\,x^2 + \cdots\,, \tag{A.16}$$

Logarithmic
$$\ln(1+x) = \sum_{k=1}^{\infty} (-1)^k \frac{x^k}{k} \quad = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \cdots\,, \tag{A.17}$$

Exponential
$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \quad = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{3!}x^3 + \cdots\,, \tag{A.18}$$

Trigonometric
$$\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} \quad = 1 - \tfrac{1}{2}x^2 + \tfrac{1}{4!}x^4 - \cdots\,, \tag{A.19}$$

$$\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \quad = x - \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 - \cdots\,, \tag{A.20}$$

Hyperbolic
$$\cosh x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} \quad = 1 + \tfrac{1}{2}x^2 + \tfrac{1}{4!}x^4 + \cdots\,, \tag{A.21}$$

$$\sinh x = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!} \quad = x + \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 + \cdots\,. \tag{A.22}$$

In these series expansions, x may be real or complex. The binomial and logarithmic series converge for $|x| < 1$, while the exponential, trigonometric and hyperbolic series converge for all x. If $|x| \ll 1$, then retaining only the first few terms in these series provides good approximations to the given functions, as successive terms in the series rapidly decrease. (In the binomial series (A.16), if the exponent a is a positive integer, then the expansion terminates after the term with $k = a$.)

Linear algebra

An $N \times N$ matrix M represents a *linear transformation* which may be applied to any N-component vector v. If M_{ij} denotes the row i, column j component of the matrix M, and v_j is the j'th component of the vector v, then the linear transformation $u = Mv$ may be written explicitly in components as $u_i = M_{ij}\, v_j$ with an implied sum on the index j (with indices i and j running from 1 to N). One often writes $M = \|M_{ij}\|$ to indicate that M is the matrix constructed from the components M_{ij}, and similarly $v = \{v_j\}$. If A and B are both $N \times N$ matrices, then the matrix product $C = AB$ is equivalent to the component form $C_{ik} = A_{ij}\, B_{jk}$.

The *identity matrix* $I \equiv \|\delta_{ij}\|$ has components equal to the *Kronecker delta symbol* defined by $\delta_{ij} \equiv \begin{cases} 1 & i = j; \\ 0 & i \neq j, \end{cases}$ and represents the linear transformation which leaves every vector unchanged. The *inverse* of an $N \times N$ matrix M is denoted M^{-1} and, if it exists, satisfies both $M^{-1}M = I$ and $MM^{-1} = I$. The inverse M^{-1} exists provided the determinant of the matrix, denoted by $\det M$ or $|M|$, is non-zero. The linear equation $Mx = y$ has a unique solution given by $x = M^{-1}y$ provided $\det M \neq 0$. If $\det M = 0$ one says that the matrix M is *singular*. (A linear equation with a singular matrix may have zero solutions, or infinitely many solutions, depending on whether the vector y lies in the range of the matrix.)

Given some square matrix M, an *eigenvalue* λ and corresponding *eigenvector* v solve the *eigenvalue equation* $Mv = \lambda v$. The set of all eigenvalues equal the roots of the *characteristic equation* $\det(M - \lambda\, I) = 0$, which is an N'th order polynomial in λ.

The *transpose*, *complex conjugate*, and *Hermitian conjugate* of a matrix M are denoted by M^T, M^*, and $M^\dagger$, respectively, with

$$(M^T)_{ij} \equiv M_{ji}\,, \quad (M^*)_{ij} \equiv (M_{ij})^*\,, \quad (M^\dagger)_{ij} \equiv (M_{ji})^*\,. \tag{A.23}$$

A *symmetric* matrix is equal to its transpose, $M = M^T$. An *antisymmetric* matrix equals minus its transpose, $M = -M^T$. An *Hermitian* matrix is equal to its Hermitian conjugate, $M = M^\dagger$.

An *orthogonal* matrix O is a matrix whose inverse equals its transpose, so that $OO^T = 1$. A *unitary* matrix U is a matrix whose inverse equals its Hermitian conjugate, so that $UU^\dagger = 1$. A real symmetric matrix $M = M^T$ can be *diagonalized* by a real orthogonal transformation. In other words, there exists a real orthogonal matrix O such that $M = O\lambda\, O^T$ with λ a real diagonal matrix. The diagonal elements $\{\lambda_i\}$ are the eigenvalues of M, and the columns of O are the corresponding mutually orthogonal eigenvectors. Similarly, a complex Hermitian matrix $M = M^\dagger$ can be *diagonalized* by a unitary transformation. In other words, there exists a unitary matrix U such that $M = U\lambda\, U^T$ with λ a diagonal matrix of real eigenvalues.

Matrix multiplication is non-commutative, meaning that $AB \neq BA$ for arbitrary matrices A and B. In other words, the *commutator* $[A, B] \equiv AB - BA$ is generally non-zero (but vanishes in special cases where the product is independent of order). Two Hermitian matrices A and B are *simultaneously diagonalizable* if and only if their commutator vanishes. If the condition $[A, B] = 0$ holds, then there exists a single unitary matrix U such that $A = U\lambda^A\, U^\dagger$ and $B = U\lambda^B\, U^\dagger$ with λ^A and λ^B both real and diagonal. Equivalently, each column of U is an eigenvector of both A and B, with eigenvalues for each matrix given by the corresponding diagonal elements of λ^A and λ^B.

Vector spaces

The above relations involving $N \times N$ matrices and N component vectors generalize in a natural fashion to arbitrary *vector spaces*. This name refers to any collection of objects (such as N component column vectors, geometric vectors, polynomials, or various classes of functions) in which it makes sense to add or subtract any two elements (or "vectors") of the space, or multiply any element by an overall constant. Vector spaces can be real or complex, depending of whether it makes sense to multiply elements only be real numbers, or by arbitrary complex numbers. And vector spaces can be finite or infinite dimensional. Given some vector space, a *basis* for the space is a set of linearly independent elements, $\{\hat{e}_a\}$, (with $a = 1, 2, \cdots$), such that any vector x in the space can be written as a linear combination of the basis elements,

$$x = \sum_a \hat{e}_a \, x_a \,, \tag{A.24}$$

for some set of coefficients $\{x_a\}$. If the vector space is N dimensional, with N finite, then a basis for the space will contain N basis vectors (so the index a labeling basis elements runs from 1 to N). If the vector space is infinite dimensional, then so is any set of basis elements. In this case, the index a labeling basis elements runs from 1 to ∞.[1] (Or, in some circumstances, the natural label is a real number, in which case the sum (A.24) is replaced by an integral over this label.)

An *inner product* (or *dot product*) is some function which takes two vectors as arguments and returns a single number — a real number for real vector spaces, a complex number for complex vector spaces. The inner product of vectors x and y is commonly denoted as $\langle x, y \rangle$, (x, y), or $x \cdot y$. Regardless of which notation is used, an inner product must satisfy:

Symmetry	$\langle x, y \rangle = \langle y, x \rangle^* \,,$	(A.25)
Linearity	$\langle x, \alpha y \rangle = \alpha \langle x, y \rangle \quad \text{and} \quad \langle x, y{+}z \rangle = \langle x, y \rangle + \langle x, z \rangle \,,$	(A.26)
Positivity	$\langle x, x \rangle \ge 0 \,,$	(A.27)
Non-degeneracy	$\langle x, x \rangle = 0 \text{ implies } x = 0 \,.$	(A.28)

Two vectors are *orthogonal* if their inner product vanishes. A basis is *orthonormal* if basis elements are mutually orthogonal, and every basis element is normalized so that its inner product with itself is unity. More succinctly, a basis is orthonormal if $\langle \hat{e}_a, \hat{e}_b \rangle = \delta_{ab}$.

A function T which acts on elements of a vector space and returns some element in the same vector space is called a *linear operator* if it satisfies the linearity conditions

$$T(\alpha \, x) = \alpha \, T(x) \,, \qquad T(x{+}y) = T(x) + T(y) \,. \tag{A.29}$$

Here α is an arbitrary real number for real vector spaces, or arbitrary complex number for complex spaces. A linear operator T is *Hermitian* if $\langle x, Ty \rangle = \langle Tx, y \rangle$ for all vectors x and y.

Given an orthonormal basis, determining the expansion coefficients $\{x_a\}$ of an arbitrary vector x is easy: they are simply given by the inner product of x with each basis element, $x_a = \langle \hat{e}_a, x \rangle$. In other

[1] In an infinite dimensional vector space, one may rightly ask whether the infinite sum (A.24) will converge for all vectors, or only for some vectors. A more formal mathematics class would carefully address this question, but for our purposes the claim that the sum will always make sense in physically sensible situations will have to suffice.

words, $x = \sum_a \hat{e}_a \langle \hat{e}_a, x \rangle$ for any vector x. This may be written without reference to any specific vector x as the *completeness relation*

$$I = \sum_a P_a \,, \tag{A.30}$$

where P_a is a *projection operator* onto vectors proportional to $\hat{e}_a$, and I is the identity operator which leaves all vectors invariant. (Explicitly, $P_a \, x \equiv \hat{e}_a \langle \hat{e}_a, x \rangle$ for any vector x.)

The above structure regarding abstract vector spaces, linear operators, and inner products is a natural generalization of N-component vectors, $N \times N$ matrices, and the usual definition of dot product. Definitions of eigenvectors and eigenvalues, and the above results on diagonalizability generalize directly from $N \times N$ matrices to arbitrary linear operators.