Chapter 1

Special relativity

1.1 Galilean relativity

Newton's laws of motion,

$$\frac{d\vec{p}}{dt} = \vec{F}\,, \qquad \frac{d\vec{x}}{dt} = \frac{\vec{p}}{m}\,, \tag{1.1.1}$$

retain the same form if one substitutes

$$\vec{x} \to \vec{x}\,' + \vec{u}\,t\,, \qquad \vec{p} \to \vec{p}\,' + m\,\vec{u}\,, \tag{1.1.2}$$

for any velocity $\vec{u}$ which is constant (independent of time). In other words, equations (1.1.1) and (1.1.2) imply that

$$\frac{d\vec{p}\,'}{dt} = \vec{F}\,, \qquad \frac{d\vec{x}\,'}{dt} = \frac{\vec{p}\,'}{m}\,. \tag{1.1.3}$$

This shows that changing coordinates to those of a moving (inertial) reference frame does not affect the form of Newton's equations. In other words, there is no preferred inertial frame in which Newton's equations are valid; if they hold in one frame, then they hold in all inertial frames. This is referred to as *Galilean* relativity. It is an example of an *invariance*, a change in the description of a system (in this case, a change in the coordinate system) which preserves the form of the equations of motion. An intrinsic aspect of Galilean relativity is the assumption that time has the same meaning in all inertial frames, so t represents time as measured by any good (and synchronized) clock, regardless of whether that clock is moving.

Consider a particle, or wave, which moves with some velocity $\vec{v}$ when viewed in the unprimed frame, so that the position of the particle (or crest of the wave) is given by $\vec{x}(t) = \vec{x}_0 + \vec{v}\,t$. In the primed frame, using (1.1.2), the location of the same particle or wave-crest is given by $\vec{x}\,'(t) = \vec{x}_0 + (\vec{v} - \vec{u})\,t$. Hence, when viewed in the primed frame, the velocity of the particle or wave is given by

$$\vec{v}\,' = \vec{v} - \vec{u}\,. \tag{1.1.4}$$

This *shift* in velocities upon transformation to a moving frame is completely in accord with everyday experience. For example, as illustrated in Figure 1.1 , if a person standing on the ground sees a

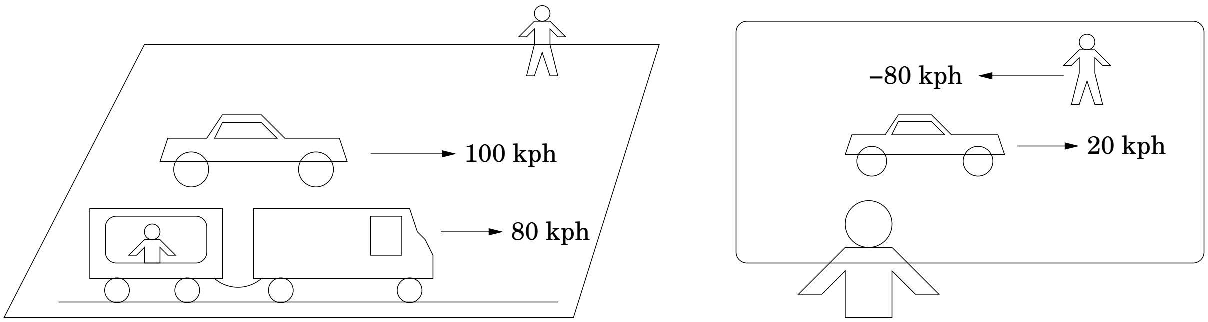

Figure 1.1: A moving train and car, as seen from the ground (left), and from the train (right).

car moving at 100 kph (kilometers per hour) parallel to a train moving at 80 kph, then a person sitting in the train will see that car moving with a relative velocity of 20 kph = (100 − 80) kph, while the person on the ground recedes from view at a velocity of −80 kph. Similarly, a sound wave propagating at the speed of sound v_s (in a medium), as seen by an observer at rest with respect to the medium, will be seen (or heard) as propagating with speed $v' = v_s - u$ by an observer moving in the same direction as the sound wave with speed u (with respect to the medium). Consequently, the frequency $f' = v'/\lambda$ heard by the moving observer (*i.e.*, the number of wave fronts passing the observer per unit time) will differ from the frequency $f = v_s/\lambda$ heard by the stationary observer,

$$f' = \frac{v_s - u}{\lambda} = f\left(1 - \frac{u}{v_s}\right) . \tag{1.1.5}$$

This is the familiar *Doppler shift* for the case of a moving observer and stationary source with respect to the medium. Recall that the medium plays an important role here. If, with respect to the medium, it is the observer who is stationary while the source moves away from the observer at speed u, then the result for the Doppler shift becomes

$$f' = f\Big/\left(1 + \frac{u}{v_s}\right) . \tag{1.1.6}$$

The two results coincide to first order in u/v_s (*i.e.*, (1.1.6) approaches (1.1.5) for $|u/v_s| \ll 1$), but as u approaches v_s (so that $u/v_s \to 1$) the two expressions are very different. This reflects the fact that for sound, there is a physically distinguished *special* reference frame, the rest frame of the medium through which the sound propagates.

1.2 Constancy of c

When applied to light (electromagnetic radiation), the Galilean relativity velocity transformation (1.1.4) predicts that observers moving at different speeds will measure different propagation velocities for light coming from a given source (perhaps a distant star). This conclusion is *wrong*. Many experiments, including the famous Michelson-Morley experiment, have looked for, and failed to find, any variation in the speed of light as a function of the velocity of the observer (or source). It has been unequivocally demonstrated that (1.1.4) does not apply to light. Moreover, unlike sound, light requires no medium in which to propagate.

Newton's laws, and the associated Galilean relativity relations (1.1.2) and (1.1.4), provide extremely accurate descriptions for the dynamics of particles and waves which move slowly compared to the speed of light,[1]

$$c = 2.99\,792\,458 \times 10^8 \text{ m/s}\,. \tag{1.2.1}$$

But Newtonian dynamics does not correctly describe the behavior of light or (as we will see) any other particle or wave moving at speeds which are not small compared to c. Our goal is to find a formulation of dynamics which does not have this limitation.

We will provisionally adopt two postulates:

Postulate 1 *The speed of light (in a vacuum) is the same in all inertial reference frames.*

Postulate 2 *There is no preferred reference frame: the laws of physics take the same form in all inertial reference frames.*

We will see that these postulates lead to a fundamentally different view of space and time, as well as to many predictions which have been experimentally tested — successfully.

1.3 Clocks and rulers

A clock is some construct which produces regular "ticks" that may be counted to quantify the passing of time. An ideal clock is one whose period is perfectly regular and reproducible. Real clocks must be based on some physical phenomenon which is nearly periodic — as close to periodic as possible. Examples include pendula, vibrating crystals, and sundials. All of these have limitations. The period of a pendulum depends on its length and the acceleration of Earth's gravity. Changes in temperature will change the length of a pendulum. Moreover, the Earth is not totally rigid: tides, seismic noise, and even changes in weather produce (small) changes in the gravitational acceleration at a given point on the Earth's surface. The frequency (or period) of vibration of a crystal is affected by changes in temperature and changes in mass due to adsorption of impurities on its surface. In addition to practical problems (weather), the length of days as measured by a sundial changes with the season and, on much longer time scales, changes due to slowing of the Earth's rotation caused by tidal friction.

An idealized clock, which is particularly simple to analyze, is shown in Fig. 1.2. A short pulse of light repeatedly bounces back and forth (in a vacuum) between two parallel mirrors. Each time the light pulse reflects off one of the mirrors constitutes a "tick" of this clock.[2] If L is the distance between the mirrors, then the period (round-trip light travel time) of this clock is $\Delta t = 2L/c$.

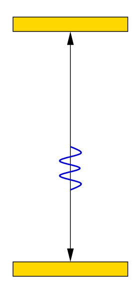

Figure 1.2: An idealized clock in which a pulse of light repeatedly bounces between two mirrors.

[1] This value is exact — because the meter is defined by this value for c and the international standard for time.

[2] To actually build such a clock, one would make one of the mirrors partially reflecting so that a tiny part of each light pulse is transmitted and measured by a photo-detector. These practical aspects are inessential for our purposes.

Now consider this same clock as seen by an observer moving to the left (perpendicular to the direction of the bouncing light) at velocity $-u$. In the observer's frame, the clock moves to the right at velocity u, as shown in Fig. 1.3. Let $\Delta t'$ be the period of the clock as viewed in this frame, so that the pulse of light travels from the lower mirror to the upper mirror and back to the lower mirror in time $\Delta t'$. The upper reflection takes place halfway through this interval, when the upper mirror has moved a distance $u\,\Delta t'/2$ to the right, and the light returns to the lower mirror after it has moved a distance $u\,\Delta t'$. Hence the light must follow the oblique path shown in the figure. The distance the light travels in one period is twice the hypotenuse, $D = 2\sqrt{L^2 + (u\,\Delta t'/2)^2} = \sqrt{4L^2 + (u\,\Delta t')^2}$. Now use the first postulate: the speed of light in this frame is c, exactly the same as in the original frame. This means that the distance D and the period $\Delta t'$ must be related via $D = c\,\Delta t'$. Combining these two expressions gives $c\,\Delta t' = \sqrt{4L^2 + (u\,\Delta t')^2}$ and solving for $\Delta t'$ yields $\Delta t' = 2L/\sqrt{c^2 - u^2}$. Inserting $2L = c\,\Delta t$ and simplifying produces

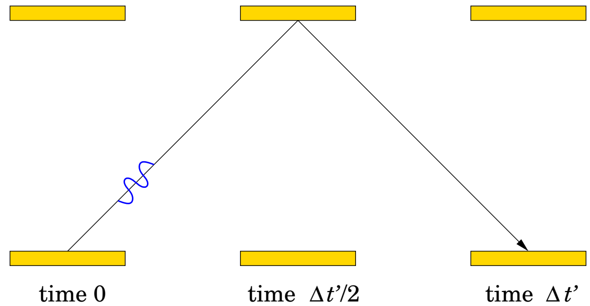

Figure 1.3: Three snapshots of the same clock viewed from a moving frame.

$$\Delta t' = \frac{\Delta t}{\sqrt{1-(u/c)^2}}\,. \tag{1.3.1}$$

This is a remarkable result. It shows that the period of a clock, when viewed in a frame in which the clock is moving, is different, and longer, than the period of the clock as viewed in its rest frame. This phenomena is known as *time dilation*. It is an inescapable consequence of the constancy of the speed of light. Although we have analyzed a particularly simple model of a clock to deduce the existence of time dilation, the result is equally valid for *any* good clock.[3] In other words, moving clocks run slower than when at rest, by a factor of

$$\gamma \equiv \frac{1}{\sqrt{1-(u/c)^2}}\,, \tag{1.3.2}$$

where u is the speed with which the clock is moving. Note that γ is greater than one for any non-zero speed u which is less (in magnitude) than c.

In the above discussion, we examined the case where the axis of our idealized clock was perpendicular to the direction of motion. What if the axis of the clock is parallel to the direction of motion? This situation is shown in Fig. 1.4 . Analyzing this case is also instructive.

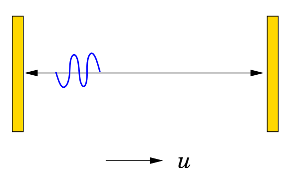

Figure 1.4: Our idealized clock, now rotated so that its axis is parallel to the direction of motion.

The round-trip light travel time (or period) must again be $\Delta t' = \gamma\,\Delta t$, because time dilation applies to *any* clock.[4] Let

[3] After all, if some other good clock remains synchronized with our idealized clock when viewed in their common rest frame, then postulate 2 implies that the same synchronization between the two clocks must also be present when the two clocks are viewed in a moving frame.

[4] To expand on this, imagine constructing two identical copies of our idealized clock. In their common rest frame, orient the axis of one clock perpendicular to the axis of the other clock. Since these two ideal clocks remain synchronized when viewed in their rest frame, by postulate 2 they must also remain synchronized when viewed from a moving frame whose velocity is parallel to one clock and perpendicular to the other.

L' be the distance between the mirrors, as viewed in the primed frame. The mirrors are moving to the right at velocity u, as shown in the figure. Suppose the light reflects off the right-hand mirror at time $\delta t'$ after leaving the left-hand mirror. During this time the right-hand mirror will have moved a distance $u\,\delta t'$ and therefore the distance light travels on this leg is $L' + u\,\delta t'$, longer than L' due to the motion of the mirror. Since $\Delta t'$ is the round-trip time, the light travel time for the return leg must be $\Delta t' - \delta t'$. On the way back, the light travel distance is $L' - u\,(\Delta t' - \delta t')$, since the motion of the left-hand mirror is decreasing the distance the light must travel.

Now use Postulate 1. For the first leg, the light travel distance $L' + u\,\delta t'$ must equal $c\,\delta t'$, since the speed of light in any (inertial) frame is c. Hence $\delta t' = L'/(c - u)$. And for the second leg, equating the distance $L' - u\,(\Delta t' - \delta t')$ with $c\,(\Delta t' - \delta t')$ implies that $\Delta t' - \delta t' = L'/(c + u)$. Substituting in $\delta t'$ gives

$$\Delta t' = \frac{L'}{c+u} + \frac{L'}{c-u} = \frac{2c\,L'}{c^2 - u^2} = \gamma^2\,(2L'/c)\,. \tag{1.3.3}$$

But we already know that $\Delta t' = \gamma\,\Delta t = \gamma\,(2L/c)$. The only way these two results for $\Delta t'$ can be consistent is if the distance L' between the mirrors, as seen in the frame in which the clock is moving parallel to its axis, is smaller than L by a factor of γ,

$$L' = \frac{L}{\gamma} = L\,\sqrt{1 - (u/c)^2}\,. \tag{1.3.4}$$

This phenomena is known as *Lorentz contraction*. We have deduced it by using an ideal clock to convert a measurement of distance (the separation between mirrors) into a measurement of time. But the same result must apply to the measurement of any length which is parallel to the direction of motion. In other words, a ruler whose length is L, as measured in its rest frame, will have a length of $L' = L/\gamma$ when viewed in a frame in which the ruler is moving with a velocity parallel to itself (*i.e.*, parallel to the long axis of the ruler).

1.4 Observational tests

As we have seen, both time dilation and Lorentz contraction are direct, logical consequences of the *frame-independence* of the speed of light. Therefore every experimental test of the frame independence of c is a test of the existence of both time dilation and Lorentz contraction. Nevertheless, it is interesting to ask how these effects can be observed directly.

One place where time dilation has a “real world” impact is in the functioning of the global positioning system (GPS). Time dilation, due to the orbital motion of GPS satellites, slows the atomic clocks carried in these satellites by about 7 microseconds per day. This is easily measurable, and is a huge effect compared to the tens of nanosecond (per day) timing accuracy which can be achieved using GPS signals.[5]

A different observable phenomena where time dilation plays a key role involves muons produced in cosmic ray showers. When a high energy cosmic ray (usually a proton or atomic nucleus) strikes

[5]However, this is only part of the story regarding relative clock rates in GPS satellites. The difference in gravitational potential between the satellites' orbits and the Earth's surface also produces a change in clock rates due to a general relativistic effect known as *gravitational redshift*. This effect goes in the opposite direction (speeding orbiting clocks relative to Earth-bound ones) and is larger in magnitude, 45 microseconds per day. So GPS clocks actually run faster than clocks on the ground by $45 - 7 = 38$ microseconds per day.

an air molecule in the upper reaches of the atmosphere (typically above 20 km), this can create a particle shower containing many elementary particles of various types (which we will be discussing later) including electrons, positrons, pions, and muons. Muons are unstable particles; their lifetime τ is 2.2 microseconds. Moving at almost the speed of light, a high energy muon will travel a distance of about $c\,\tau \approx (3\times 10^8\,\mathrm{m/s})\times(2\times 10^{-6}\,\mathrm{s}) = 600\,\mathrm{m}$ in time τ. This is small compared to the height of the atmosphere, and yet muons produced in showers originating in the upper atmosphere are easily observed on the ground. How can this be, if muons decay after merely a couple of microseconds?

The resolution of this apparent paradox is time dilation. Two microseconds is the lifetime of a muon *in its rest frame*. One may view a muon, or a bunch of muons moving together, as a type of clock. If there are N_0 muons initially, then after some time t (as measured in the rest frame of the muons), on average all but $N_1 = N_0\,e^{-t/\tau}$ muons will have decayed. Turning this around, if all but some fraction N_1/N_0 of muons decay after some interval of time, then the length of this interval equals $\tau\,\ln(N_0/N_1)$ — as measured in the muons' rest frame. But as we have seen above, a moving clock, any moving clock, runs slow by a factor of γ. Therefore, fast moving muons decay more slowly than do muons at rest. This means that muons produced in the upper atmosphere at a height H (typically tens of kilometers) will have a substantial probability of reaching the ground before decaying provided they are moving fast enough so that $\gamma\,c\,\tau > H$.

Muons produced in the upper atmosphere and reaching the earth before decaying also illustrate Lorentz contraction — if one considers what's happening from the muon's perspective. Imagine riding along with a muon produced in an atmospheric shower. Or, as one says more formally, consider the co-moving reference frame of the muon. In this frame, the muon is at rest but the Earth is flying toward the muon at nearly the speed of light. The muon will decay, on average, in two microseconds. But the thickness of the atmosphere, in this frame, is reduced by Lorentz contraction. Therefore, the surface of the Earth will reach the muon before it (typically) decays if $(H/\gamma)/c < \tau$. This is the same condition obtained above by considering physics in the frame of an observer on the ground. This example nicely illustrates the second relativity postulate: because the laws of physics are frame independent, one may use whatever frame is most convenient to analyze some particular phenomena. In this example, whether one regards time dilation or Lorentz contraction as being responsible for allowing muons produced in the upper atmosphere to reach the ground depends on the frame one chooses to use, but not the fact that high energy muons *can* reach the ground from the upper atmosphere.

1.5 Superluminal motion?

The time dilation (1.3.1) and Lorentz contraction (1.3.4) equations make no sense (*i.e.*, are no longer real) if $u > c$. As we will discuss further in Chapter 3, a basic feature of special relativity is that nothing (no signal, no particle, no information) can travel faster than the speed of light c. Consequently, there was considerable excitement in the autumn of 2011 when the OPERA neutrino experiment at the Gran Sasso laboratory in Italy reported that neutrinos (which are thought to have a very small but nonzero mass) appeared to travel to the detector from CERN in Geneva, Switzerland at a speed that exceeded c. (See, *e.g.*, this Science Daily story.) Such a measurement requires precise synchronization between clocks in Geneva and in Gran Sasso. A precision better than 50 nanoseconds is needed; this is achievable, with difficulty, using the GPS system.

The OPERA result fundamentally conflicts with special relativity. Either our postulates, or the experimental measurement, must be in error. All indications are that it is the measurement which was in error. In early 2012, the OPERA team reported that their original measurement may have suffered from a synchronization error caused by a loose connection in the cable which relayed GPS signals to the experiment's clocks. Moreover, in April 2012 the independent ICARUS experiment, also at Gran Sasso, reported their measurement of neutrino speed, which was found to be consistent with the speed of light.

1.6 Example problems

1.6.1 Earthbound clocks

Two ideal clocks, initially synchronized, sit on the surface of the Earth, one at the south pole and the other on the equator. (Ignore any difference in elevation between the two sites.)

(a) Q: Will these two clocks remain synchronized?

A: No, because the Earth rotates. As viewed in the frame in which the Earth's center of mass is at rest, which is also the frame in which the south pole is (essentially) at rest, the clock on the equator is moving, and hence will exhibit time dilation.

(b) Q: For the equatorial clock, how much does γ differ from 1?

A: The Earth's circumference is about 25,000 miles or 40,000 km. The rotation period of 1 day $= 24 \times 60 \times 60$ seconds $=$ 86,400 s. So the rotation speed of a point on the equator is $v \approx 40/86.4$ km/s $=$ 463 m/s. Dividing by c gives $v/c \approx 1.5 \times 10^{-6}$, or $(v/c)^2 \approx 2.4 \times 10^{-12}$.

To evaluate $\gamma - 1$, one must be careful not to lose all numerical accuracy. Blindly using (1.3.2), one must evaluate the denominator keeping more than 12 significant digits, or else the effect of the $(v/c)^2$ term will be completely lost. Since v/c is tiny, a better approach is to use the first two terms in the binomial expansion $1/\sqrt{1-x} = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \cdots$. Hence, $\gamma - 1 \approx \frac{1}{2}(v/c)^2 \approx 1.2 \times 10^{-12}$.

(c) Q: After one year, what is the time difference between the two clocks? Is this measurable (using current technology)?

A: Relative to the polar clock, the equatorial clock will have lost $(\gamma - 1) \times 86,400$ seconds, which is just over 100 ns. This is measurable. The best atomic clocks have long term frequency stability of a few parts in 10^{16}, much better than the part in 10^{12} effect of time dilation due to Earth's rotation.

1.7 Further resources

GPS and Relativity, R. Pogge

Relativity in the Global Positioning System, N. Ashby

Michelson-Morley experiment, Wikipedia

Global Positioning System, Wikipedia

Cosmic ray, Wikipedia

OPERA experiment, Wikipedia

ICARUS experiment, Wikipedia