Chapter 2

Minkowski spacetime

2.1 Events

An *event* is some occurrence which takes place at some instant in time at some particular point in space. Your birth was an event. JFK's assassination was an event. Each downbeat of a butterfly's wingtip is an event. Every collision between air molecules is an event. Snap your fingers right now — that was an event. The set of all possible events is called *spacetime*. A point particle, or any stable object of negligible size, will follow some trajectory through spacetime which is called the *worldline* of the object. The set of all spacetime trajectories of the points comprising an extended object will fill some region of spacetime which is called the *worldvolume* of the object.

2.2 Reference frames

To label points in space, it is convenient to introduce spatial coordinates so that every point is uniquely associated with some triplet of numbers (x^1, x^2, x^3). Similarly, to label events in spacetime, it is convenient to introduce spacetime coordinates so that every event is uniquely associated with a set of *four* numbers. The resulting spacetime coordinate system is called a *reference frame*. Particularly convenient are *inertial reference frames*, in which coordinates have the form (t, x^1, x^2, x^3) (where the superscripts here are coordinate labels, *not* powers). The set of events in which x^1, x^2, and x^3 have arbitrary *fixed* (real) values while t ranges from $-\infty$ to $+\infty$ represent the worldline of a particle, or hypothetical observer, which is subject to no external forces and is at rest in this particular reference frame. This is illustrated in Figure 2.1. In general, the *rest frame* of an inertial observer (or object) means the inertial frame in which the specified observer (or object) is at rest.

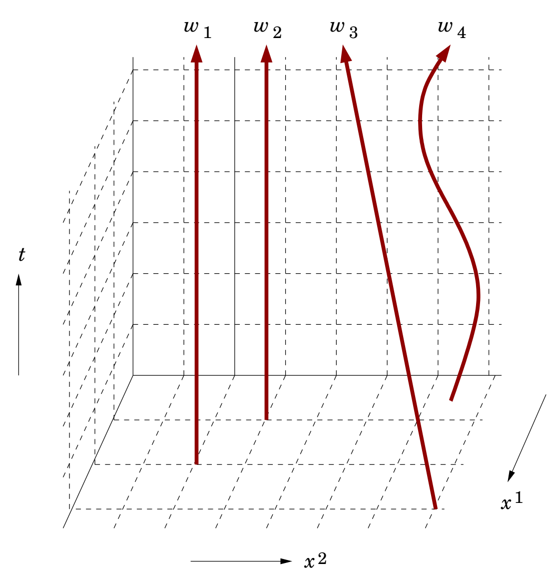

Figure 2.1: An inertial reference frame. Worldlines w_1 and w_2 represent observers at rest in this reference frame, w_3 is the spacetime trajectory of an inertial observer who is moving in this frame, and w_4 is the spacetime trajectory of a non-inertial object whose velocity and acceleration fluctuates.

As Figure 2.2 tries to suggest, one may view an inertial reference frame as being defined by an infinite set of inertial observers, one sitting at every point in space, all of whom carry synchronized (ideal) clocks and all of whom are at rest with respect to each other.[1] You can imagine every observer carrying a notebook and recording the time, according to his clock, of events of interest.

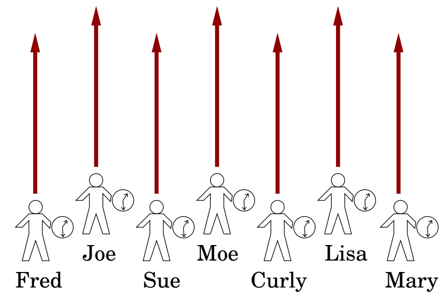

Figure 2.2: A family of inertial observers, all with synchronized watches and mutually at rest, defines an inertial reference frame.

As an example, consider the statement a moving rod "has length L". Suppose that the worldline of the left end of the rod intersects the worldline of some observer A at the event labeled A^* whose time, according to observer A's clock, is t_1. The worldline of the right end of the rod intersects the worldline of observer B at the event labeled B^* whose time (according to B) is also t_1, and then intersects the worldline of observer C at event C^* at the later time t_2 (according to C). The interior of the rod sweeps out a flat two-dimensional surface in spacetime — the shaded "ribbon" bounded by the endpoint worldlines shown in Figure 2.3.

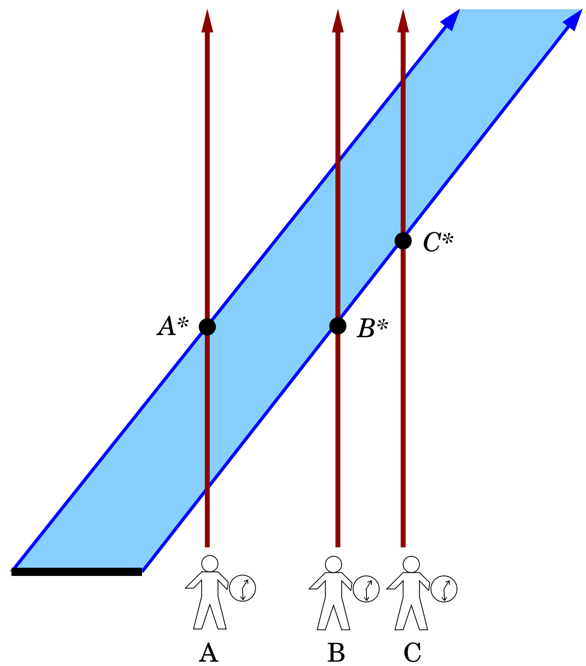

Figure 2.3: A moving rod which passes by three inertial observers who are at rest.

The *surface of simultaneity* of event A^*, in the reference frame in which observer A is at rest, is the set of all events whose time coordinates in this frame coincide with the time of event A^*. So event B^* is on the surface of simultaneity of event A^* (in Fig. 2.3, B^* is displaced precisely horizontally from A^*), while event C^* is not. The length of the rod, in this reference frame, is the spatial distance between events A^* and B^*, marking the endpoints of the rods on a surface of simultaneity. This is the same as the distance between observers A and B, who are mutually at rest. As usual, it is convenient to choose Cartesian spatial coordinates, so that if observers A and B have spatial coordinates (x_A^1, x_A^2, x_A^3) and (x_B^1, x_B^2, x_B^3), then their relative spatial separation is given by

$$d_{AB} = \left[(x_B^1 - x_A^1)^2 + (x_B^2 - x_A^2)^2 + (x_B^3 - x_A^3)^2\right]^{1/2} . \tag{2.2.1}$$

One should stop and ask how the observers defining an inertial reference frame could, in principle, test whether their clocks are synchronized, and whether they are all mutually at rest. The simplest approach is to use the propagation of light. Suppose observer A flashes a light, momentarily, while observer B holds a mirror which will reflect light coming from observer A back to its source. If the

[1] Achieving this synchronization can be a challenge in practice — as evident from the discussion of the OPERA experiment in section 1.5.

light is emitted at time t_A, according to A's clock, it will be reflected at time t_B, according to B's clock, and the reflected pulse will then be detected by A at some time $t_A + \Delta t$. If A and B's clocks are synchronized, then the time t_B at which B records the reflection must equal $t_A + \frac{1}{2}\Delta t$. Any deviation from this indicates that the clocks are not synchronized. If this experiment is repeated, then any drift in the value of Δt indicates that the two observers are not mutually at rest (or that their clocks are failing to remain synchronized).

2.3 Lightcones

Before proceeding further, it will be helpful to introduce a useful convention for spacetime coordinates. When one does dimensional analysis, it is customary to regard time and space as having different dimensions. If we define the spacetime coordinates of an event as the time and spatial coordinates in a chosen inertial frame, (t, x^1, x^2, x^3), then the differing dimensions of the time and space coordinates will be a nuisance. Because the value of the speed of light, c, is universal — independent of reference frame — we can use it as a simple *conversion factor* which relates units of time to units of distance. Namely, we define the length

$$x^0 \equiv c\,t\,, \tag{2.3.1}$$

which is the distance light can travel in time t. Henceforth, we will use x^0 in place of the time t as the first entry in the spacetime coordinates of an event, (x^0, x^1, x^2, x^3).

Now consider a flash of light which is emitted from the event with coordinates $x^0 = x^1 = x^2 = x^3 = 0$ — *i.e.*, the spacetime origin in this coordinate system. The light will propagate outward in a spherical shell whose radius at time t equals ct, which is x^0. Therefore, the set of events which form the entire history of this light flash are those events for which $\left[(x^1)^2 + (x^2)^2 + (x^3)^2\right]^{1/2} = x^0$. This set of events form a cone, illustrated in Figure 2.4. The intersection of this cone with the x^0–x^1 plane is the two half-lines at $\pm 45°$, for which $x^0 = \pm x^1$ and $x^0 > 0$. These 45° lines describe the path of light which is emitted from the origin traveling in the $\pm x^1$ direction.

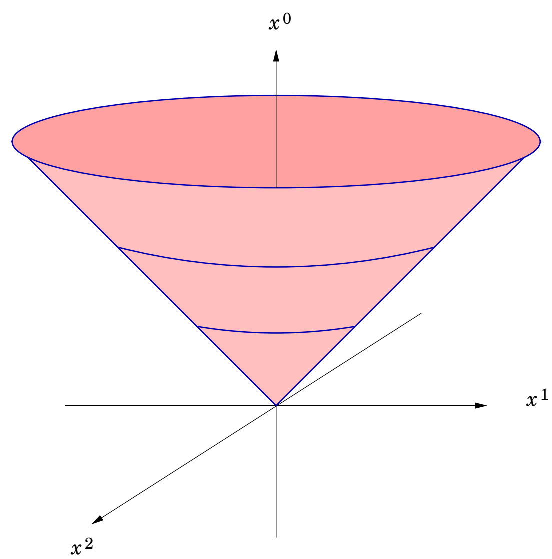

Figure 2.4: The "lightcone" of a flash of light emitted from the origin.

2.4 Simultaneity

Next, consider the reference frames of two different inertial observers, A and B, who are not at rest with respect to each other. As viewed in A's reference frame, suppose that observer B is moving with speed v in the x^1 direction, so that B's position satisfies

$$x^1 = vt = (v/c)\,x^0 \qquad \text{(in frame A)}\,.$$

Figure 2.5 depicts this situation graphically. (We have chosen the origin of time to be when A and B are at the same point.) In reference frame A, the worldline of observer A is the vertical axis, since this corresponds to all events with $x^1 = x^2 = x^3 = 0$ and x^0 arbitrary. The worldline of observer B (in reference frame A) is a tilted line with a slope of c/v, since this corresponds to all events with $x^0 = (c/v)\, x^1$ (and vanishing x^2 and x^3). Note that as $v \to 0$ the slope diverges and the line becomes vertical, coinciding with the worldline of A. As $v \to c$, the slope approaches one and the worldline of B approaches a 45° line lying on the lightcone.

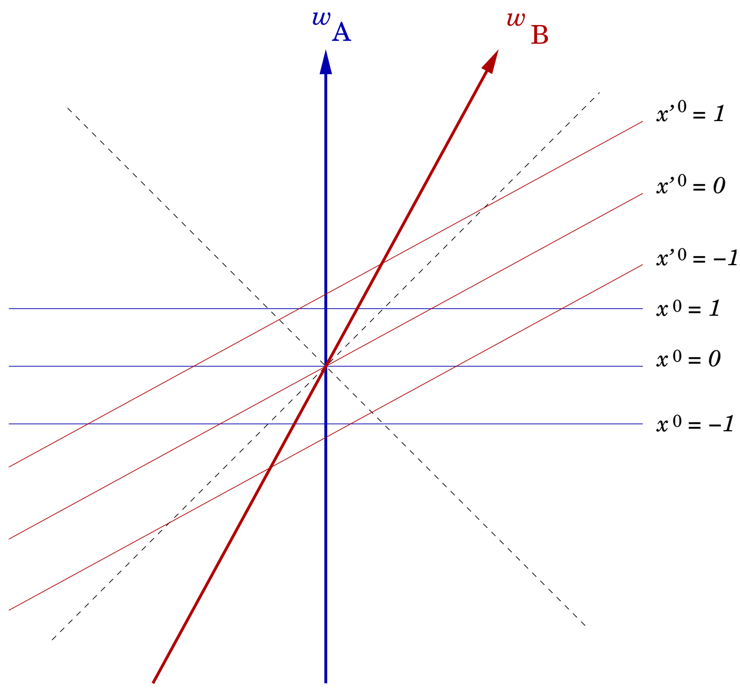

Figure 2.5: Worldline of two observers, and corresponding surfaces of simultaneity. The dashed lines show the lightcone of the origin.

Surfaces of simultaneity for observer A correspond to horizontal planes in this diagram, because such planes represent events with a common value of time (or x^0) according to A's clock. But what are surfaces of simultaneity for observer B? In other words, what set of events share a common value of time according to B's clock? These turn out to be tilted planes with slope v/c (not c/v), and are shown in the figure as thin red lines labeled $x'^0 = -1$, 0, or 1.

One way to see that this must be the case is to note that the 45° worldline of a light ray traveling from the origin in the $+x^1$ direction (the dashed line with unit slope in the figure) bisects the angle between observer A's worldline (the x^0 axis in the figure) and his surface of simultaneity defined by $x^0 = 0$. Exactly the same statement must be true for observer B — she will also describe the path of the light as bisecting the angle between her worldline and her surface of simultaneity (the red $x'^0 = 0$ line) which contains the origin. This is an application of our second postulate (physics looks the same in all inertial reference frames). Hence, when plotted in A's reference frame, observer B's worldline and surfaces of simultaneity must have *complementary* slopes (c/v versus v/c) so that they form *equal* angles with the lightcone at 45°.

The essential point, which is our most important result so far, is that the concept of simultaneity is observer dependent. Events which one observer views as occurring simultaneously (but at different locations) will not be simultaneous when viewed by a different observer moving at a non-zero relative velocity.

Because this is a key point, it may be helpful to go through the logic leading to this conclusion in a more explicit fashion. To do so, consider the experiment depicted in Figure 2.6 . Two flashes of light (shown as black lines in the figure) are emitted at events R and S and meet at event T. In observer B's frame, shown on the left, the emission events are *simultaneous* and displaced by some distance L'. The reception event T is necessarily equi-distant between R and S. Lines $w_{\rm B}$, $w_{\rm B'}$, and $w_{\rm B''}$ show the worldlines of observers who are at *rest* in this frame and who witness events R, T, and S, respectively. (In other words, $w_{\rm B}$ is the worldline of observer B, sitting at the origin in this frame, $w_{\rm B''}$ is the worldline of an observer sitting at rest a distance L' away, and $w_{\rm B'}$ is the worldline of an

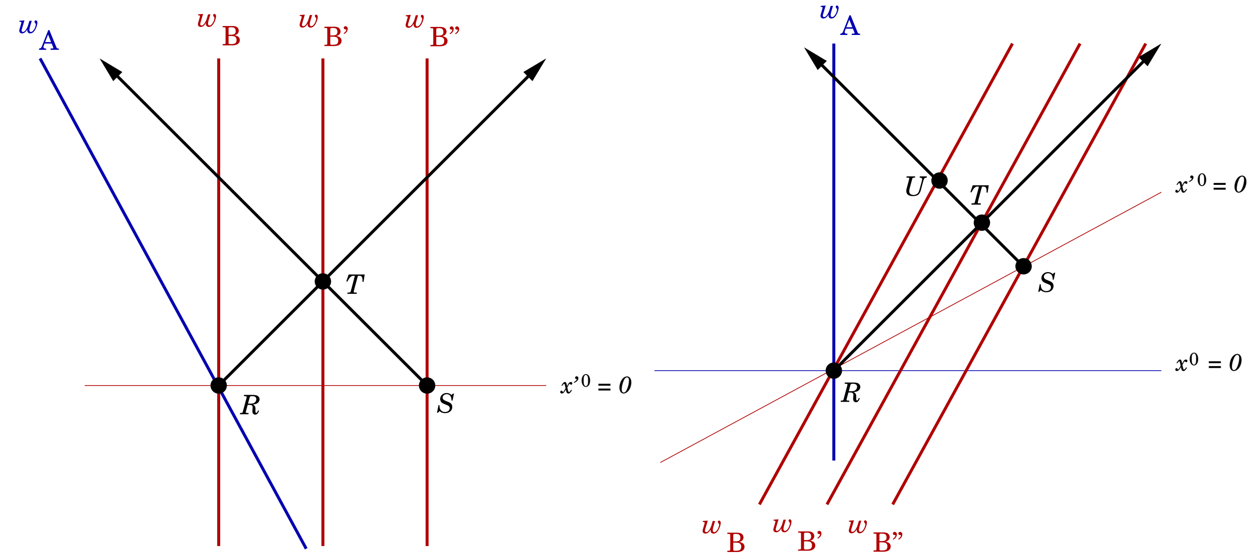

Figure 2.6: Two flashes of light emitted at events R and S which meet at event T, as described in two different frames.

observer at rest halfway in between B and B$''$.)

In observer A's frame, shown in the right panel of Figure. 2.6 , the worldlines of observers at rest in frame B are now tilted lines with slope c/v. But the paths of the light rays (propagating within the plane shown) lie at $\pm 45°$, because the speed of light is universal. The emission event S, which lies on B's surface of simultaneity, is the intersection between the leftward propagating light ray and the worldline $w_{B''}$ of an observer who is at rest in B's frame and twice as far from the origin as the worldline, $w_{B'}$, which contains the reception event T. Since events R and S are simultaneous, as seen in frame B, (and the distance L' in this construction is arbitrary) the frame B surface of simultaneity containing events R and S must, in frame A, appear as a straight line connecting these events. From the geometry of the figure, one can see that the triangles RTU and RTS are similar, and hence the angle between the simultaneity line RS and the the $45°$ lightcone is the same as the angle between the worldline w_B and the lightcone. This implies that the slope of the simultaneity line is the inverse of the slope of worldline w_B, as asserted above. (As an exercise, determine where event U lies in the left panel, and show that in this panel triangles RTU and RTS are again similar.)

2.5 Lorentz transformations

Just as many problems in ordinary spatial geometry are easier when one introduces coordinates and uses analytic geometry, spacetime geometry problems of the type just discussed are also simpler if one introduces and uses analytic formulas relating coordinates in different reference frames. These relations are referred to as *Lorentz transformations.*

Using the two frames discussed above, let (x^0, x^1, x^2, x^3) denote spacetime coordinates in the inertial reference frame of observer A, and let (x'^0, x'^1, x'^2, x'^3) denote spacetime coordinates in the inertial reference frame of observer B, who is moving in the x^1 direction with velocity v relative to observer A. How are these coordinates related?

Assume, for simplicity, that the spacetime origins of both frames coincide. Then there must be some linear transformation which relates coordinates in the two frames,

$$\begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \Lambda \begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix}, \tag{2.5.1}$$

where Λ is some 4×4 matrix. Since the transformation Λ describes the effect of switching to a moving frame, it is referred to as a *Lorentz boost*, or simply a 'boost'.

If the spatial coordinates of frame B are not rotated with respect to the axes of frame A, so that observer B describes observer A as moving in the $-x'^1$ direction with velocity $-v$, then Lorentz contraction will only affect lengths in the 1-direction, leaving the 2 and 3 directions unaffected. Therefore, we should have

$$x^2 = x'^2, \quad x^3 = x'^3 \qquad \text{(for boost along } x^1\text{)}, \tag{2.5.2}$$

implying that the boost matrix Λ has the block diagonal form

$$\Lambda = \begin{pmatrix} M & N & 0 & 0 \\ P & Q & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.5.3}$$

with an identity matrix in the lower 2×2 block, and some non-trivial 2×2 matrix in the upper block which we need to determine.

Now the coordinates of events on the worldline of observer B, in frame B coordinates, satisfy $x'^1 = x'^2 = x'^3 = 0$ since observer B is sitting at the spatial origin of his coordinate system. Specializing to this worldline, the transformation (2.5.1) gives

$$x^0 = M\, x'^0, \quad x^1 = P\, x'^0, \tag{2.5.4}$$

implying that $x^1 = (P/M)\, x^0$. But we already know that this worldline, in frame A coordinates, should satisfy $x^1 = (v/c)\, x^0$ since observer B moves with velocity v in the 1-direction relative to observer A. Therefore, we must have $P/M = v/c$. We also know that from observer A's perspective, clocks at rest in frame B run slower than clocks at rest in frame A by a factor of $\gamma = 1/\sqrt{1-(v/c)^2}$. In other words,

$$\gamma = \frac{\Delta t_A}{\Delta t_B} = \frac{dx^0}{dx'^0} = M\,. \tag{2.5.5}$$

Combining this with the required value of P/M implies that $P = \gamma\,(v/c)$. This determines the first column of the Lorentz boost matrix (2.5.3).

To fix the second column, consider the events comprising the x'^1 axis in frame B, or those events with $x'^0 = x'^2 = x'^3 = 0$ and x'^1 arbitrary. These events lie on the surface of simultaneity of the spacetime origin in frame B. Above, we learned that this surface, as viewed in reference frame A, is the tilted plane with slope v/c, whose events satisfy $x^0 = (v/c)\, x^1$. But applied to the x'^1 axis in frame B, the transformation (2.5.1) gives

$$x^0 = N\, x'^1, \qquad x^1 = Q\, x'^1, \tag{2.5.6}$$

or $x^0 = (N/Q)\, x^1$. Therefore, we must have $N/Q = v/c$. Finally, we can use the fact that events on the path of a light ray emitted from the spacetime origin and moving in the 1-direction must satisfy both $x'^1 = x'^0$ and $x^1 = x^0$, since observers in both frames will agree that the light moves with speed c. But if $x'^1 = x'^0$, then the transformation (2.5.1) gives $x^0 = (M+N)\, x'^0$, and $x^1 = (P+Q)\, x'^0$. Therefore, we must have $M+N = P+Q$. Inserting $M = \gamma$, $P = (v/c)\,\gamma$, $N = (v/c)\, Q$ and solving for Q yields $Q = \gamma$. Putting it all together, we have

$$\Lambda = \begin{pmatrix} \gamma & \gamma\,(v/c) & 0 & 0 \\ \gamma\,(v/c) & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.5.7}$$

for a boost along the 1-direction with velocity v. With this matrix, multiplying out the transformation (2.5.1) gives

$$x^0 = \gamma\left(x'^0 + \tfrac{v}{c}\, x'^1\right), \qquad x^2 = x'^2\,, \tag{2.5.8a}$$

$$x^1 = \gamma\left(\tfrac{v}{c}\, x'^0 + x'^1\right), \qquad x^3 = x'^3\,. \tag{2.5.8b}$$

With a little more work, one may show that the general Lorentz transformation matrix for a boost with speed v in an arbitrary direction specified by a unit vector $\hat{n} = (n_x, n_y, n_z)$ is given by

$$\Lambda = \begin{pmatrix} \gamma & \gamma\,(v/c)\, n_x & \gamma\,(v/c)\, n_y & \gamma\,(v/c)\, n_z \\ \gamma\,(v/c)\, n_x & 1+(\gamma{-}1)\, n_x^2 & (\gamma{-}1)\, n_x n_y & (\gamma{-}1)\, n_x n_z \\ \gamma\,(v/c)\, n_y & (\gamma{-}1)\, n_x n_y & 1+(\gamma{-}1)\, n_y^2 & (\gamma{-}1)\, n_y n_z \\ \gamma\,(v/c)\, n_z & (\gamma{-}1)\, n_x n_z & (\gamma{-}1)\, n_y n_z & 1+(\gamma{-}1)\, n_z^2 \end{pmatrix}. \tag{2.5.9}$$

Finally, it is always possible for two inertial reference frames to differ by a spatial rotation, in addition to a boost. The coordinate transformation corresponding to a spatial rotation may also be written in the form (2.5.1), but with a transformation matrix which has the block-diagonal form

$$\Lambda = \begin{pmatrix} 1 & \\ & R \end{pmatrix} \qquad \text{(spatial rotation)}\,, \tag{2.5.10}$$

where R is some 3×3 rotation matrix (an orthogonal matrix with determinant one). In other words, for such transformations the time coordinates are not affected, $x^0 = x'^0$, while the spatial coordinates are transformed by the rotation matrix R. The most general Lorentz transformation is a product of a rotation of the form (2.5.10) and a boost of the form (2.5.9),

$$\Lambda = \Lambda_{\text{boost}} \times \Lambda_{\text{rotation}}\,. \tag{2.5.11}$$

2.6 Rapidity

The mixing of time and space components of a four-vector generated by the Lorentz transformation matrix (2.5.7) may seem reminiscent of the mixing of spatial components of a vector undergoing a rotation. A closer connection is apparent if one introduces the "rapidity" η, which is monotonically related to v/c via

$$\tanh\eta = v/c\,. \tag{2.6.1}$$

The rapidity η ranges from $-\infty$ to $+\infty$ as v/c varies between -1 and $+1$. A short exercise (using the hyperbolic identity $1-\tanh^2 z = 1/\cosh^2 z$) shows that

$$\cosh\eta = \gamma\,, \qquad \sinh\eta = \gamma\,(v/c)\,, \tag{2.6.2}$$

so the non-trivial upper 2×2 block of the Lorentz transformation matrix (2.5.7) takes the form $\left(\begin{smallmatrix}\cosh\eta & \sinh\eta\\ \sinh\eta & \cosh\eta\end{smallmatrix}\right)$, with hyperbolic functions replacing the usual trigonometric functions appearing in a rotation. Characterizing a boost by its rapidity (instead of v/c) is often convenient; rapidity is commonly used when analyzing data from particle colliders such as the Large Hadron Collider (LHC) near Geneva.

2.7 Spacetime vectors

In ordinary three-dimensional (Euclidean) space, if one designates some point O as the spatial origin then one may associate every other point X with a vector which extends from O to X. One can, and should, regard vectors as geometric objects, independent of any specific coordinate system. However, it is very often convenient to introduce a set of basis vectors $\{\hat e_1, \hat e_2, \hat e_3\}$ (normally chosen to point along coordinate axes), and then express arbitrary vectors as linear combinations of the chosen basis vectors,

$$\vec v = \sum_{i=1}^{3} \hat e_i\, v^i\,. \tag{2.7.1}$$

The components $\{v^i\}$ of the vector depend on the choice of basis vectors, but the geometric vector $\vec v$ itself does not.

In exactly the same fashion, once some event O in spacetime is designated as the spacetime origin, one may associate every other event X with a spacetime vector which extends from O to X. Spacetime vectors (also called "4-vectors") are geometric objects, whose meaning is independent of any specific reference frame. However, once one chooses a reference frame, one may introduce an associated set of spacetime basis vectors, $\{\hat e_0, \hat e_1, \hat e_2, \hat e_3\}$, which point along the corresponding coordinate axes. And, as in any vector space, one may then express an arbitrary spacetime vector v as a linear combination of these basis vectors,

$$v = \sum_{\mu=0}^{3} \hat e_\mu\, v^\mu\,. \tag{2.7.2}$$

We will use Greek letters (most commonly α and β, or μ and ν) to represent spacetime indices which run from 0 to 3. And sometimes we will use Latin letters i, j, k to represent spatial indices which run from 1 to 3. We will often use an *implied summation* convention in which the sum sign is omitted, but is implied by the presence of repeated indices:

$$\hat e_\mu\, v^\mu \equiv \sum_{\mu=0}^{3} \hat e_\mu\, v^\mu\,. \tag{2.7.3}$$

We will generally *not* put vector signs over spacetime vectors, instead relying on the context to make clear whether some object is a 4-vector. But we will put vector signs over three-dimensional spatial vectors, to distinguish them from spacetime vectors.

The spacetime coordinates of an event are the components of the spacetime vector x associated with this event in the chosen reference frame,

$$x = \hat{e}_\mu \, x^\mu \equiv \hat{e}_0 \, x^0 + \hat{e}_1 \, x^1 + \hat{e}_2 \, x^2 + \hat{e}_3 \, x^3 \, . \tag{2.7.4}$$

A different reference frame will have basis vectors which are linear combinations of the basis vectors in the original frame. Consider a 'primed' frame whose coordinates $\{x'^\mu\}$ are related to the coordinates $\{x^\nu\}$ of the original frame via a Lorentz transformation (2.5.1). It is convenient to write the components of the transformation matrix as $\Lambda^\mu{}_\nu$ (where the first index labels the row and the second labels the column, as usual for matrix components). Then the linear transformation (2.5.1) may be compactly rewritten as

$$x^\mu = \Lambda^\mu{}_\nu \, x'^\nu \, . \tag{2.7.5}$$

The inverse transformation, expressing primed coordinates in terms of unprimed ones, is

$$x'^\mu = (\Lambda^{-1})^\mu{}_\nu \, x^\nu \, , \tag{2.7.6}$$

where $(\Lambda^{-1})^\mu{}_\nu$ are the components of the inverse matrix Λ^{-1}.[2] The components of *any* 4-vector transform in exactly the same fashion when one changes reference frames.

The Lorentz transformation matrix also relates the basis vectors in the two frames (note the order of indices),

$$\hat{e}'_\nu = \hat{e}_\mu \, \Lambda^\mu{}_\nu \, . \tag{2.7.7}$$

In other words, if you view the list $(\hat{e}_0, \hat{e}_1, \hat{e}_2, \hat{e}_3)$ as a row-vector, then it is multiplied on the right by a Lorentz transformation matrix Λ. The transformation of basis vectors must have precisely this form so that the complete spacetime vector is frame independent, as initially asserted,

$$x = \hat{e}'_\mu \, x'^\mu = \hat{e}_\nu \, x^\nu \, . \tag{2.7.8}$$

Recall that the *dot product* of two spatial vectors, $\vec{a} \cdot \vec{b}$, may be defined geometrically, without reference to any coordinate system, as the product of the length of each vector times the cosine of the angle between them. One can then show that this is the same as the component-based definition, $\vec{a} \cdot \vec{b} = \sum_i a^i \, b^i$, for any choice of Cartesian coordinates. It is this frame (or rotation) independence that ensures that the dot product of spatial vectors is a scalar.

What is the appropriate generalization of dot products for spacetime vectors? This should be some operation which, given two 4-vectors a and b, produces a single number. The operation should be *symmetric*, so that $a \cdot b = b \cdot a$, and *linear*, so that $a \cdot (b + c) = a \cdot b + a \cdot c$. The result should be *independent* of the choice of (inertial) reference frame one uses to specify the components of these vectors. And it should reduce to the usual spatial dot product if both a and b lie within a common surface of simultaneity. There is a unique solution to these requirements: given two spacetime vectors a and b whose components in some inertial frame are a^μ and b^μ, the dot product of these vectors is

$$a \cdot b \equiv -a^0 \, b^0 + a^1 \, b^1 + a^2 \, b^2 + a^3 \, b^3 \, , \tag{2.7.9}$$

[2] For boost matrices of the form (2.5.7) or (2.5.9), changing the sign of v converts Λ into its inverse. Note that this changes the sign of the off-diagonal components in the first row and column, leaving all other components unchanged. For transformations which also include spatial rotations, to convert the transformation to its inverse one must transpose the matrix in addition to flipping the sign of these "time-space" components.

or with an implied summation on spatial indices, $a \cdot b = -a^0\, b^0 + a^i\, b^i$. This differs from the normal definition of a dot product (in four dimensional Euclidean space) merely by the change in sign of the time component term. This definition satisfies the required linearity and reduces to the usual spatial dot product if the time components of both four vectors vanish.[3]

To see that the dot product definition (2.7.9) is frame-independent, and thus defines a scalar, it is sufficient to check the effect of a boost of the form (2.5.7) (since we already know that a rotation of coordinates does not affect the three-dimensional dot product). Transforming the components of the 4-vectors a and b to a primed frame, as in Eq. (2.7.6), using the boost (2.5.7) gives

$$a'^0 = \gamma\left(a^0 - \tfrac{v}{c}\, a^1\right), \qquad a'^1 = \gamma\left(a^1 - \tfrac{v}{c}\, a^0\right), \qquad a'^2 = a^2\,, \qquad a'^3 = a^3\,, \tag{2.7.10a}$$

$$b'^0 = \gamma\left(b^0 - \tfrac{v}{c}\, b^1\right), \qquad b'^1 = \gamma\left(b^1 - \tfrac{v}{c}\, b^0\right), \qquad b'^2 = b^2\,, \qquad b'^3 = b^3\,. \tag{2.7.10b}$$

Hence

$$\begin{aligned}
-a'^0\, b'^0 + a'^1\, b'^1 + a'^2\, b'^2 + a'^3\, b'^3 \\
&= \gamma^2\left[-\left(a^0 - \tfrac{v}{c}\, a^1\right)\left(b^0 - \tfrac{v}{c}\, b^1\right) + \left(a^1 - \tfrac{v}{c}\, a^0\right)\left(b^1 - \tfrac{v}{c}\, b^0\right)\right] + a^2\, b^2 + a^3\, b^3 \\
&= \gamma^2\, \left[1 - (v/c)^2\right]\left(-a^0 b^0 + a^1 b^1\right) + a^2\, b^2 + a^3\, b^3 \\
&= -a^0 b^0 + a^1 b^1 + a^2\, b^2 + a^3\, b^3\,,
\end{aligned} \tag{2.7.11}$$

where the last step used $\gamma^2 \equiv 1/[1-(v/c)^2]$. Therefore, as claimed, the value of the dot product (2.7.9) is independent of the specific inertial frame one uses to define the vector coefficients.

The spacetime dot product (2.7.9) is a useful construct in many applications. As a preview of things to come, consider some plane wave (acoustic, electromagnetic, or any other type) propagating with angular frequency ω and wave-vector $\vec{k}$. One normally writes the complex amplitude for such a wave as some overall coefficient times the complex exponential $e^{-i\omega t + i\vec{k}\cdot\vec{x}}$. Having already defined the spacetime position vector x whose time component $x^0 \equiv ct$, if we also define a spacetime wave-vector k whose time component $k^0 \equiv \omega/c$ then this ubiquitous phase factor may be written compactly as a spacetime dot product,

$$e^{-i\omega t + i\vec{k}\cdot\vec{x}} = e^{ik\cdot x}\,. \tag{2.7.12}$$

Similarly, in quantum mechanics the wave function of a particle with definite momentum $\vec{p}$ and energy E moving in empty space is proportional to $e^{-iEt/\hbar + i\vec{p}\cdot\vec{x}/\hbar}$. If we define a spacetime momentum vector p with time component $p^0 \equiv E/c$, then this phase factor may also be written as a spacetime dot product,

$$e^{-iEt/\hbar + i\vec{p}\cdot\vec{x}/\hbar} = e^{ip\cdot x/\hbar}\,. \tag{2.7.13}$$

2.8 Minkowski spacetime

In Euclidean space, the dot product of a vector with itself gives the square of the norm (or length) of the vector, $\vec{v}\cdot\vec{v} \equiv |\vec{v}|^2$. Proceeding by analogy, we will *define* the square of a spacetime vector

[3]Regrettably some physicists, and some textbooks, define the dot product of spacetime vectors with an overall minus sign relative to (2.7.9), so that $a \cdot b \overset{\text{bad}}{\underset{\text{choice}}{\equiv}} a^0\, b^0 - a^i\, b^i$. This makes spacetime dot products reduce to *minus* the usual three-dimensional dot product when time components vanish. As long as one uses a single convention consistently, no physical result can depend on the choice. However, this author strongly recommends using, exclusively, the “mostly plus” convention (2.7.9).

using the dot product (2.7.9), so that

$$(a)^2 \equiv a \cdot a = -(a^0)^2 + (a^1)^2 + (a^2)^2 + (a^3)^2 \,. \tag{2.8.1}$$

If Δx is a spacetime vector representing the separation between two events, then the square of Δx is called the *invariant interval* separating these events. This is usually denoted by s^2, so that

$$s^2 \equiv -(\Delta x^0)^2 + (\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2 \,. \tag{2.8.2}$$

Spacetime in which the "distance" between events is defined by this expression is called *Minkowski spacetime*.[4] [5] [6]

The definition of the invariant interval (2.8.2), or the square of a vector (2.8.1), differ from the usual Euclidean space relations merely due to the minus sign in front of the time component terms. But this is a fundamental change. Unlike Euclidean distance, the spacetime interval s^2 can be positive, negative, or zero. Let Δx be the spacetime displacement from some event X to another event Y. If the interval $s^2 = (\Delta x)^2$ vanishes, then the spatial separation between these events equals their separation in time multiplied by c,

$$s^2 = 0 \quad \Longrightarrow \quad (\Delta\vec{x})^2 = (\Delta x^0)^2 = (c\,\Delta t)^2 \qquad (\textit{lightlike separation}). \tag{2.8.3}$$

This means that light could propagate from X to Y (if $\Delta t > 0$), or from Y to X (if $\Delta t < 0$). In other words, event Y is on the lightcone of X, or vice-versa. In this case, one says that the separation between X and Y is *lightlike*.

If the interval s^2 is negative, then the spatial separation is less than the time separation (times c),

$$s^2 < 0 \quad \Longrightarrow \quad (\Delta\vec{x})^2 < (\Delta x^0)^2 = (c\,\Delta t)^2 \qquad (\textit{timelike separation}). \tag{2.8.4}$$

This means that some particle moving slower than light could propagate from X to Y (if $\Delta t > 0$), or from Y to X (if $\Delta t < 0$). In other words, event Y is in the interior of the lightcone of X, or vice-versa. In this case, one says that the separation between X and Y is *timelike*.

Finally, if the interval s^2 is positive, then the spatial separation is greater than the time separation (times c),

$$s^2 > 0 \quad \Longrightarrow \quad (\Delta\vec{x})^2 > (\Delta x^0)^2 = (c\,\Delta t)^2 \qquad (\textit{spacelike separation}). \tag{2.8.5}$$

In other words, event Y is outside the lightcone of X, and vice-versa. In this case, one says that the separation between X and Y is *spacelike*. These possibilities are shown pictorially in Figure 2.7 .

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[4] Those authors who choose to define spacetime dot products with an overall minus sign ("mostly minus" convention), as discussed in footnote 3 , also define the square of 4-vectors and the spacetime interval with an overall minus sign relative to our definitions (2.8.1) and (2.8.2). Our sign conventions are more convenient — use them, but beware of differing conventions in the literature.

[5] Minkowski spacetime is the domain of special relativity, in which gravity is neglected. Correctly describing gravitational dynamics leads to general relativity, in which spacetime can have curvature and the interval between two arbitrary events need not have the simple form (2.8.2). We will largely ignore gravity.

[6] A further word about index conventions may be appropriate. It is standard in modern physics to write the components of 4-vectors with superscripts, like a^μ or x^ν, as we have been doing. Although we will not need this, it is also conventional to define subscripted components which, in Minkowski space, differ merely by flipping the sign of the time component, so that $a_0 \equiv -a^0$ for any 4-vector a. This allows one to write the dot product of two 4-vectors a and b as $a_\mu b^\mu$ (with the usual implied sum). More generally, in curved space one defines a metric tensor $g_{\mu\nu}$ via a differential relation of the form $ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu$, and then defines $a_\mu \equiv g_{\mu\nu}\, a^\nu$ so that $a \cdot b = a_\mu b^\mu = a^\mu b_\mu = g_{\mu\nu}\, a^\mu\, b^\nu$. In flat Minkowski spacetime, the metric tensor is diagonal, $\|g_{\mu\nu}\| = \mathrm{diag}(-1, +1, +1, +1)$.

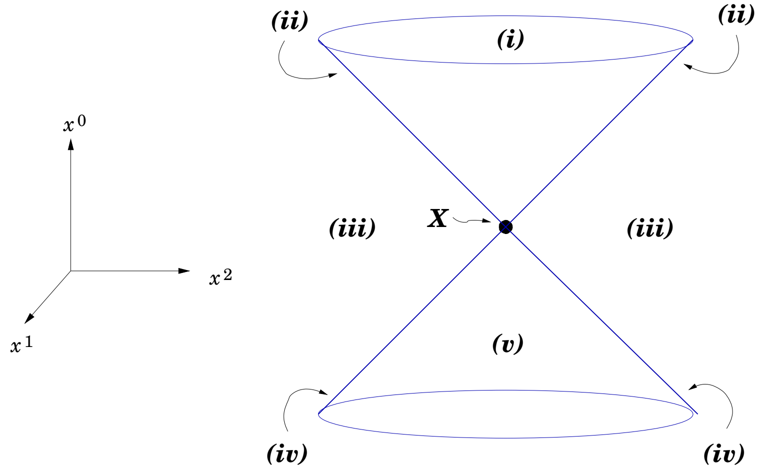


Figure 2.7: The past and future lightcones of an event X separate spacetime into those events which are: (*i*) timelike separated and in the future of X, (*ii*) lightlike separated and in the future of X, (*iii*) spacelike separated, (*iv*) lightlike separated and in the past of X, and (*v*) timelike separated and in the past of X.

2.9 The pole and the barn

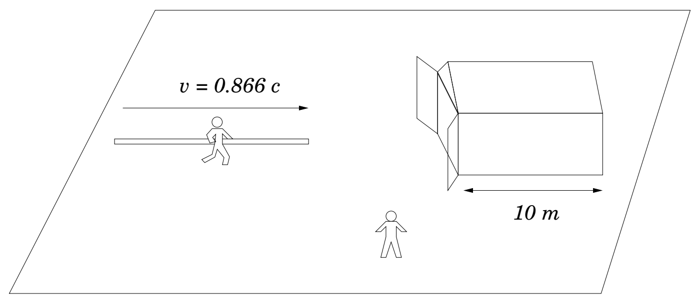


Figure 2.8: A relativistic runner, carrying a long pole, approaches and passes through a barn. Does the pole fit within the barn?

A classic puzzle illustrating basic aspects of special relativity is the pole and the barn, sketched in Figure 2.8. You are standing outside a barn whose front and back doors are open. A (very fast!) runner carrying a long horizontal pole is approaching the barn. The length of the barn is 10 meters. The length of the pole, when measured at rest, is 20 meters. But the relativistic runner is moving at a speed of $\frac{\sqrt{3}}{2}\,c \simeq 0.866\,c$, and hence the pole (in your frame) is Lorentz contracted by a factor of $1/\gamma = \sqrt{1-(v/c)^2} = 1/2$. Consequently, the pole just fits within the barn; when the front of the pole emerges from one end of the barn, the back of the pole will have entered the barn through the other door.

But now consider the runner's perspective. In his (or her) co-moving frame, the pole is 20 meters long. The barn is coming toward the runner at a speed of $-\frac{\sqrt{3}}{2}\,c$, and hence the barn which is 10 meters long in its rest frame is Lorentz contracted to a length of only 5 meters. The pole cannot possibly fit within the barn!

Surely the pole either does, or does not, fit within the barn. Right? Which description is correct?

This puzzle, like most apparent paradoxes in special relativity, is most easily resolved by drawing a spacetime diagram which clearly displays the relevant worldlines and events of interest. It is also often helpful to draw contour lines on which the invariant interval s^2 relative to some key event is constant. For events within the x^0–x^1 plane, the invariant interval from the origin is just $s^2 = -(x^0)^2 + (x^1)^2$. Therefore, the set of events in the x^0–x^1 plane which are at some fixed interval s^2 from the origin lie on a hyperbola.[7]

[7]Recall that the equation $x^2 - y^2 = s^2$ defines a hyperbola in the (x, y) plane whose asymptotes are the 45° lines

Let us create a spacetime diagram for this puzzle working in the reference frame of the barn. (This is an arbitrary choice. We could just as easily work in runner's frame.) Try doing this yourself before reading the following step-by-step description of Figure 2.9 .

Orient coordinates so that the ends of the barn are at $x^1 = 0$ and $x^1 = 10\,\mathrm{m}$. Therefore, the worldline of the left end of the barn (w_{L}) is a vertical line at $x^1 = 0$, while the worldline of the right end of the barn (w_{R}) is a vertical line at $x^1 = 10\,\mathrm{m}$. Since the pole is moving at velocity $\frac{\sqrt{3}}{2}\,c$ (in the x^1 direction), the worldlines of the ends of the pole are straight lines in the x^0–x^1 plane with a slope of $c/v = 2/\sqrt{3} \simeq 1.155$. Call the moment when the back end of the pole passes into the barn time zero. So the worldline of the back end of the pole (w'_{B}) crosses the worldline of the left end of the barn at event A with coordinates $(x^0, x^1) = (0, 0)$. In the frame in which we're working, the pole is Lorentz contracted to a length of 10 meters. Hence, the worldline of the front end of the pole (w'_{F}) must cross the x^1 axis at event B with coordinates $(x^0, x^1) = (0, 10\,\mathrm{m})$. This event lies on the worldline w_{R} of the right end of the barn, showing that in this reference frame, at time $t = 0$, the Lorentz contracted pole just fits within the barn.

Now add to the diagram the surface of simultaneity of event A in the runner's frame. From section 2.4 we know that this surface, in the frame in which we are a drawing our diagram, is tilted upward so that its slope is $v/c \simeq 0.866$ (and the 45° lightcone of event A bisects the angle between this surface and the worldline w'_{B}). The worldline w_{R} of the right end of the barn intersects this surface of simultaneity at event C, while the worldline w'_{F} of the front of the pole intersects this surface at event D. This surface of simultaneity contains events which, in the runner's frame, occur at the same instant in time. From the diagram it is obvious that event C lies between events A and D. In other words, in the runner's frame, at the moment when the back end of the pole passes into the barn, the front end of the pole is far outside the other end of the barn — the pole does not fit in the barn.

The essential point of this discussion, and the spacetime diagram in Figure 2.9 , is the distinction between events which are simultaneous in the runner's frame (events A, C, and D), and events which are simultaneous in the barn's frame (A and B). *Both* descriptions given initially were correct. The only fallacy was thinking that it was meaningful to ask whether the pole does (or does not) fit within the barn, without first specifying a reference frame.

To complete our discussion of this spacetime diagram, consider the invariant interval between event A (which is our spacetime origin) and each of the events B, C, and D. Within the two-dimensional plane of the figure, the invariant interval from the origin is $s^2 = -(x^0)^2 + (x^1)^2$. We know that event B has coordinates $(x^0, x^1) = (0, 10\,\mathrm{m})$ so it is immediate that $s^2_{AB} = (10\,\mathrm{m})^2$. We could work out the (x^0, x^1) coordinates of events C and D, and from those coordinates evaluate their interval from event A. But this is not necessary since we can use the fact that events C and D lie on the runner's frame surface of simultaneity of event A. We are free to evaluate intervals from event A using the runner's frame coordinates, instead of barn frame coordinates. Within the plane of the figure, $s^2 = -(x'^0)^2 + (x'^1)^2$. Events A, C, and D are simultaneous in the runner's frame, so all their x'^0 coordinates vanish. And in this frame (the rest frame of the pole) we know that the pole's length is $20\,\mathrm{m}$, while the barn's length is Lorentz contracted to $5\,m$. Hence $s^2_{AC} = (5\,\mathrm{m})^2$ and $s^2_{AD} = (20\,\mathrm{m})^2$. Therefore, event C must lie on the hyperbola whose intersection with the x^1 axis is at $5\,\mathrm{m}$, while event D must lie on the hyperbola whose intersection with the x^1 axis is at $20\,\mathrm{m}$, as shown.

$y = \pm x$. If $s^2 > 0$ then one branch opens toward the right and the other opens toward the left. If $s^2 < 0$ then one branch opens upward and one opens downward.

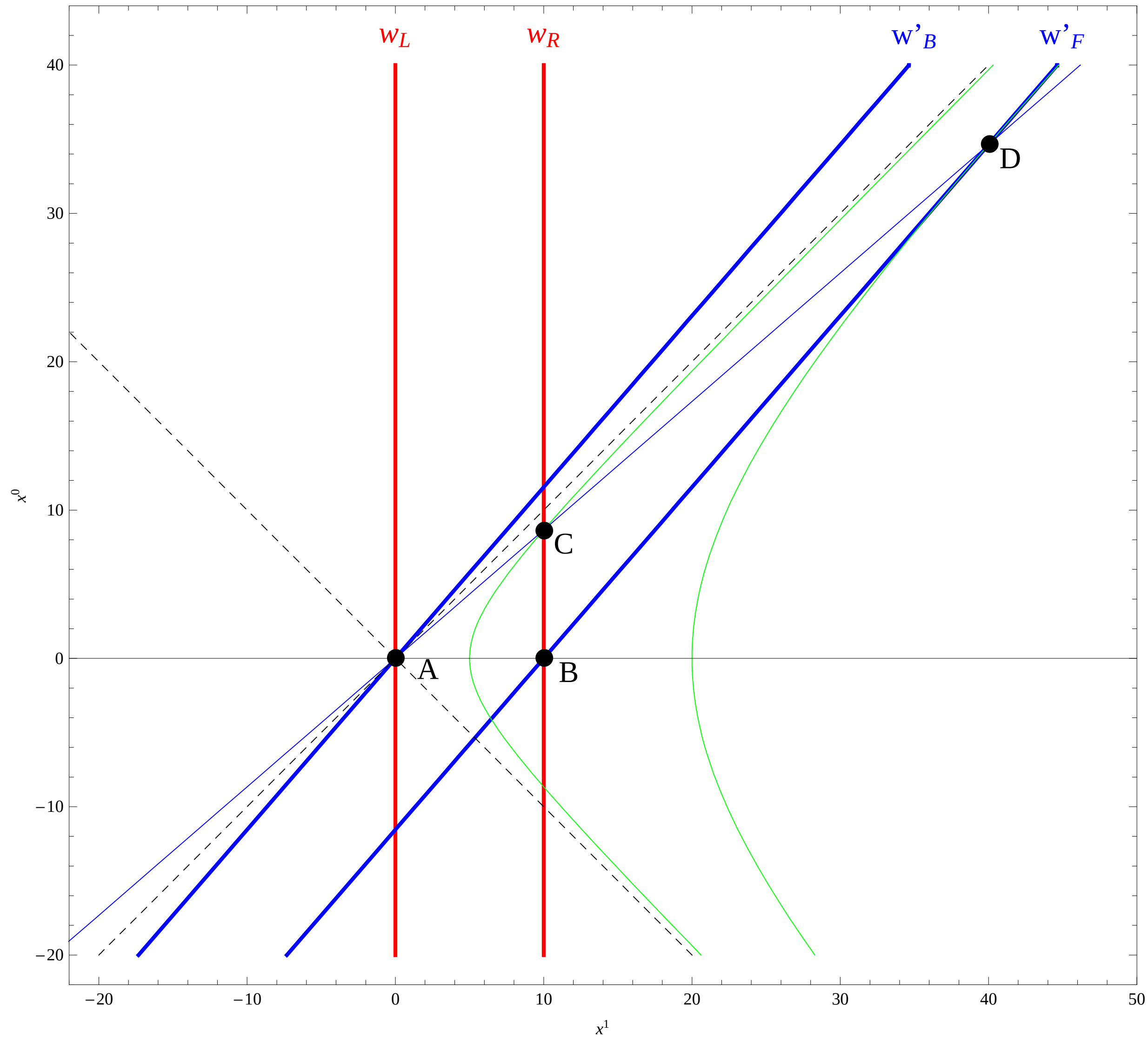


Figure 2.9: A spacetime diagram of the pole and the barn, showing events in the rest frame of the barn. The red vertical lines are the worldlines $w_{\rm L}$ and $w_{\rm R}$ of the left and right ends of the barn. The blue lines labeled $w'_{\rm F}$ and $w'_{\rm B}$ are the worldlines of the front and back of the pole, respectively. The thin blue line passing through events A, C, and D is a surface of simultaneity in the runner's reference frame. The green hyperbola passing through event C shows events at invariant interval $s^2 = (5\,\mathrm{m})^2$ relative to event A. This hyperbola intercepts the x^1 axis at 5 m. The other green hyperbola passing through event D shows events at invariant interval $s^2 = (20\,\mathrm{m})^2$ relative to event A. Note that this hyperbola intercepts the x^1 axis at 20 m.

2.10 Causality

Consider any two spacetime events A and B which are spacelike separated. A basic consequence of the fact that surfaces of simultaneity are observer dependent is that different observers can *disagree* about the temporal ordering of spacelike separated events. For example, in the unprimed reference frame illustrated in Fig. 2.10, event B lies in the future of event A — its x^0 coordinate is bigger. But event B lies below the $x'^0 = 0$ surface of simultaneity which passes through event A. This means that event B lies in the past of event A in the primed reference frame.

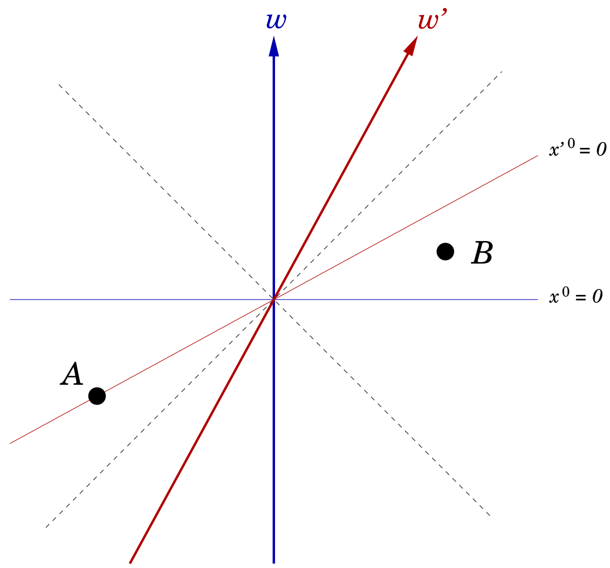


Figure 2.10: Two spacelike separated events A and B. In the unprimed frame, B is in the future of A, but in the primed frame B is in the past of A.

This should seem bizarre. If observers at rest in the unprimed frame were to see some particle or signal travel from event A to event B, then this signal would be traveling backwards in time from the perspective of observers at rest in the primed frame. This is inconsistent with causality — the fundamental idea that events in the past influence the future, but not vice-versa.

An idealized view of the goal of physics is the prediction of future events based on knowledge of the past state of a system. But if different observers disagree about what events are in the future and what events are in the past, how can the laws of physics possibly take the same form in all reference frames? Are our two relativity postulates fundamentally inconsistent?

If it is possible for some type of signal to travel between events A and B then, because these two events are outside each others lightcones, this would be superluminal propagation of information. The only way that our postulates can be consistent is if it is simply *not possible* for any signal to travel between spacelike separated events. In other words, a necessary consequence of our postulates is that no signal whatsoever can travel faster than light. For fans of science fiction this is a sad state of affairs, but it is an inescapable conclusion.

2.11 Example problems

2.11.1 Proper time intervals[8]

The time interval between two events is called a *proper time interval* in some given inertial frame if the two events occur at the same spatial location in that frame. Consider two frames of reference: the rest frame (frame S) of the Earth and the rest frame (frame S') of a spaceship moving with velocity $v = 0.6\,c$ with respect to Earth. The spaceship skims the surface of the Earth at some instant — call this event 1. Assume that coordinates and clocks are adjusted so that event 1 has coordinates

[8] Adapted from Kogut problem 2-1.

$t_1 = 0$, $x_1 = 0$ in frame S, and $t'_1 = 0$, $x'_1 = 0$ in frame S'. Event 2 marks the emission of a pulse of light from the Earth towards the spaceship at $t_2 = 10$ minutes. Event 3 marks the detection of the light pulse by observers in the spaceship. (Neglect the sizes of both the Earth and the spaceship.)

(a) Q: Is the time interval between events 1 and 2 a proper time interval in the spaceship frame? In the Earth frame?

A: Events 1 and 2 occur at the same spatial location in frame S (*i.e.*, on the Earth), but not at the same location in frame S' on the spaceship. Hence the time interval between events 1 and 2 is a proper time interval in the Earth frame, but not in the spaceship frame.

(b) Q: Is the time interval between events 2 and 3 a proper time interval in the spaceship frame? In the Earth frame?

A: Events 2 and 3 occur at different locations in both frames. Hence the time interval between events 2 and 3 is not a proper time interval in either frame.

(c) Q: Is the time interval between events 1 and 3 a proper time interval in the spaceship frame? In the Earth frame?

A: Events 1 and 3 occur at the same location on the spaceship (frame S'), but not at the same point on the Earth. Hence the time interval between events 1 and 3 is a proper time interval in the spaceship frame, but not in the Earth frame.

(d) Q: What is the time of event 2 as measured on the spaceship?

A: We want to determine the time t'_2 of the light emission in frame S'. This time interval (from event 1) is not a proper time interval and we must account for time dilation (with respect to the proper time interval in frame S). We have $\gamma = 1/\sqrt{1-(v/c)^2} = 1/\sqrt{1-(0.6)^2} = 1.25$, and hence $t'_2 = \gamma\, t_2 = 1.25 \times 10\,\text{min} = 12.5\,\text{min}$.

(e) Q: In the spaceship frame, how far away is the Earth when the light pulse is emitted?

A: We need to determine the distance to the Earth from the spaceship at the moment (in frame S') when the light is emitted. This is just the distance traveled at velocity v during the time interval $\Delta t' = t'_2 - t'_1$ (as measured in frame S') between events 1 and 2. Using the value for t'_2 from part (d), and $t'_1 = 0$, we have:

$$l'_2 = v\,\Delta t' = 0.6 \times (3.0 \times 10^8\,\text{m/s}) \times (12.5\,\text{min}) \times (60\,\text{s/min}) = 1.35 \times 10^{11}\,\text{m}\,.$$

(f) Q: From your answers in parts (d) and (e), what does the spaceship clock read when the light pulse arrives?

A: We need the time of event 3 in frame S'. We already know both the time and distance to the Earth at the emission of the pulse, and we know that light travels at c in *all* frames. Thus, we merely need to add the light travel time to the emission time (all in frame S'),

$$t'_3 = t'_2 + l'_2/c = 12.5\,\text{min} + (1.35 \times 10^{11}\,\text{m})/(3 \times 10^8\,\text{m/s}) \times (1\,\text{min}/60\,\text{s}) = 20\,\text{min}\,.$$

(g) Q: Analyzing everything in the Earth frame, find the time of event 3 according to Earth's clock.

A: The light pulse is emitted (in frame S) at $t_2 = 10$ min. At that moment the distance to the spaceship, in Earth's frame, is $l_2 = vt_2 = 0.6 \times (3 \times 10^8\,\text{m/s}) \times 10\,\text{min} \times 60\,\text{s/min} = 1.08 \times 10^{11}$ m.

Between events 2 and 3, the light pulse moves away from Earth at speed c while the spaceship continues to recede at speed v. The light reaches the spaceship when $c\,(t_3 - t_2) = l_2 + v\,(t_3 - t_2)$, so

$$t_3 - t_2 = \frac{l_2}{c - v} = \frac{1.08 \times 10^{11}\ \text{m}}{1.2 \times 10^{8}\ \text{m/s}} = 900\,\text{s} = 15\,\text{min}\,.$$

Finally, $t_3 = t_2 + (t_3 - t_2) = 10\,\text{min} + 15\,\text{min} = 25\,\text{min}$.

(h) Q: Are your answers to parts (f) and (g) consistent with conclusions from parts (a), (b) and (c)?

A: We noted in (c) that the time interval between events 1 and 3 is a proper time interval in frame S'. In frame S, it will appear that the spaceship clock is running slow, due to time dilation, so that

$$t_3 = \gamma\, t'_3 = 1.25 \times 20\,\text{min} = 25\,\text{min}\,.$$

Reassuringly, this agrees with our result from (g).

2.11.2 Passing in the night[9]

Two rockets, A and B, pass each other while moving in opposite directions. The rockets have identical proper lengths (*i.e.*, lengths in their respective rest frames) of 100 m. Consider two events: Event 1 is when the front of B passes the front end of A. Event 2 is when the front of B passes the back end of A. In frame A (the rest frame of rocket A), the time interval Δt_{A} between the two events is 1.5×10^{-6} s.

(a) Q: What is the relative velocity of the two rockets?

A: We know the length of rocket A in its rest frame, 100 m, and the time for the front of rocket B to travel that distance (as measured in frame A). The distance/time ratio gives the velocity of B as measured in frame A, and this is the relative velocity of the two rockets. Hence,

$$v_{\text{rel}} = \frac{100\ \text{m}}{1.5 \times 10^{-6}\ \text{s}} = 6.667 \times 10^{7}\ \text{m/s}\,.$$

(b) Q: According to the clocks on rocket B, how long does the front end of A take to pass the entire length of rocket B?

A: The passing of rocket A viewed from B will be exactly equivalent to the passing of B as viewed from A. Hence, the time Δt_{B} for the front of A to pass the entire length of rocket B, as measured in frame B, is again 1.5×10^{-6} s.

(c) Q: According to the clocks on rocket B, how much time passes between events 1 and 2 (*i.e.*, between the passage of the front of B by the front of A, and the passage of the front of B by the rear of A)? Does this agree with your answer to (b)? Should it?

A: In frame B, the length of rocket A is Lorentz contracted, $L_{\text{A in B}} = (100\ \text{m})/\gamma$, with $\gamma = 1/\sqrt{1 - (v_{\text{rel}}/c)^2} = 1.0257$. So $L_{\text{A in B}} = 97.50$ m, and

$$\Delta t = \frac{L_{\text{A in B}}}{v_{\text{rel}}} = \frac{\Delta t_{\text{B}}}{\gamma} = 1.46 \times 10^{-6}\ \text{s}\,.$$

This results does not, and should not, agree with Δt_{B}.

[9] Adapted from Kogut problem 2-2.

2.11.3 Emission and absorption[10]

Q: The emission and absorption of a light ray define two distinct spacetime events, which are separated by a distance ℓ in the common rest frame of the emitter and the absorber. Find the spatial and temporal separation of these events as observed in a boosted reference frame traveling with velocity v parallel to the direction from the emitter to the absorber.

A: Three different methods for solving the problem, each of which are instructive, are presented:

Method #1: Thought-experiment

Choose the x^1 direction to coincide with the direction of the light ray. In the original frame, the light ray travels a distance $\Delta x^1 = \ell$ in a time $\Delta t = \ell/c$. Now consider the emission and absorption process in a frame moving with speed v along the x^1 direction of the original frame. Without loss of generality, assume that the origin of the boosted frame coincides with the emission event. As seen in the boosted frame, the original frame is moving with velocity $-v$ along the x'^1 direction. Call the time between emission and absorption events (in the boosted frame) $\Delta t'$, so in this frame the light ray travels a distance $c\,\Delta t'$. Since the distance between the emission and absorption locations equals ℓ in the original frame, that separation is now ℓ/γ in the boosted frame due to Lorentz contraction. But it is *essential* to realize that while the emission and absorption locations are fixed in the original frame, they are moving in the boosted frame. In particular, the location of the absorption event moves a distance $-v\,\Delta t'$ while the light is traveling, which must be added to ℓ/γ to obtain the net distance traveled by the light in this frame. Therefore, $c\,\Delta t' = \ell/\gamma - v\Delta t'$. Solve for $c\Delta t'$:

$$c\,\Delta t' = \frac{\ell/\gamma}{1+v/c} = \ell\sqrt{\frac{1-v/c}{1+v/c}}\,.$$

This is the distance between emission and absorption events in the boosted frame; the time between these events (in the boosted frame) is just $\Delta t' = \frac{\ell}{c}\sqrt{\frac{1-v/c}{1+v/c}}$, since the speed of light is frame-independent. Notice that this result is not just given by time dilation. For positive v, the time interval between emission and absorption in the boosted frame is less than in the original frame, while for negative v, the boosted frame time interval is greater.[11]

Method #2: Lorentz transformation

In the original frame, the emission event may be placed at the origin of the Minkowski diagram of spacetime. The absorption event then has coordinates $(x^0, x^1) = (\ell, \ell)$ which lies on the lightcone. Under a boost, the origin is mapped to the origin so the emission event also occurs at the origin of

[10] Adapted from Kogut problem 2-6.

[11] This result allows us to make a nice connection with the discussion in Chapter 1. Suppose that the light ray, instead of being absorbed, is reflected back and detected at the emitter. The total time interval between emission and detection (in the original frame), $\Delta t_{\rm tot} = 2\ell/c\,,$ is just the time between ticks of the clock discussed in Chapter 1. Taking into account the different direction of motion of the light after reflection, the total time interval as observed in the moving frame (in the configuration of Figure 1.4) is

$$\Delta t'_{\rm tot} = \frac{\ell}{c}\sqrt{\frac{1-v/c}{1+v/c}} + \frac{\ell}{c}\sqrt{\frac{1+v/c}{1-v/c}} = \frac{2\ell}{c}\,\frac{1}{\sqrt{1-(v/c)^2}} = \gamma\,\Delta t\,,$$

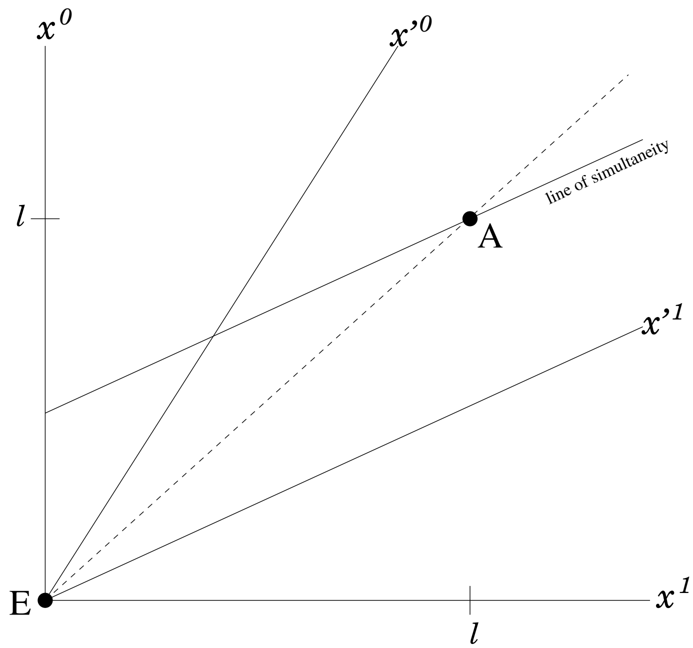
in agreement with the original time dilation result.

the boosted frame (since we assumed that this was the synchronizing event). The absorption event coordinates, in the boosted frame, are given by

$$\begin{pmatrix} x'^0 \\ x'^1 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma \frac{v}{c} \\ -\gamma \frac{v}{c} & \gamma \end{pmatrix} \begin{pmatrix} \ell \\ \ell \end{pmatrix}.$$

The spatial separation is given by $x'^1 = \gamma\ell\,(1-v/c)$, which simplifies to the same answer given above for $c\,\Delta t'$, namely $\ell\sqrt{\frac{1-v/c}{1+v/c}}$. Since the events lie on the lightcone, the time separation (times c) and spatial separation are identical.

Method #3: Spacetime diagram



In the diagram we have drawn the lines of simultaneity for the boosted observer that intersect the emission and absorption events, labeled E and A, respectively. The upper line of simultaneity is described by the equation $(x^0 - \ell)/(x^1 - \ell) = v/c$ which, written in more familiar slope-intercept form, is $x^0 = (v/c)x^1 + \ell\,(1-v/c)$. The x^0-intercept is $\ell\,(1-v/c)$ and, as you can see from the diagram, it gives the time between emission and absorption events for the boosted observer (times c). Well, almost. We must realize that the orthogonal axes of the diagram are drawn in the original frame, not the boosted one. So the time we have just extracted is the time measured by clocks in the original frame, not those in the boosted frame. But we already know how to convert time intervals between frames in relative motion—use time dilation. A clock carried by the boosted observer will run slower than that carried by the observer at rest. So we again obtain the same result $x'^0 = \gamma x^0 = \gamma\ell\,(1-v/c) = \ell\sqrt{(1-v/c)/(1+v/c)}$.

2.11.4 Changing frame (I)[12]

Q: An event has coordinates $(x')^\mu = (c \times 9 \times 10^{-8}\,\mathrm{s},\ 100\,\mathrm{m},\ 0,\ 0)$ in frame S'. Frame S' moves with velocity $v/c = 4/5$ along the x^1 axis with respect to the S frame. Determine the location of the event in frame S.

A: Assume, for convenience, that the spacetime origins of the two frames coincide. The boost factor relating the frames is $\gamma = 1/\sqrt{1-(v/c)^2} = 5/3$, and hence the relevant Lorentz transformation is:

$$x = \Lambda(v)\,x' = \begin{pmatrix} 5/3 & 4/3 & 0 & 0 \\ 4/3 & 5/3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 27\,\mathrm{m} \\ 100\,\mathrm{m} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 178.3\,\mathrm{m} \\ 202.7\,\mathrm{m} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} c \times 59.4 \times 10^{-8}\,\mathrm{s} \\ 202.7\,\mathrm{m} \\ 0 \\ 0 \end{pmatrix}.$$

[12]Adapted from Kogut problem 4-3.

2.11.5 Changing frame (II)[13]

Q: Two events have coordinates $(x_1)^\mu = (L,\, L,\, 0,\, 0)$ and $(x_2)^\mu = (L/2,\, 2L,\, 0,\, 0)$ in frame S. The two events are simultaneous in frame S'. Find the velocity $\vec{v}$ of frame S' as seen from frame S. Assume the spacetime origins of both frames coincide. When do these events occur in frame S'?

A: We have $\Delta x^0 = L/2$ and $\Delta x^1 = -L$ (with $\Delta x \equiv x_1 - x_2$). Coordinates in frame S' will be related to those in frame S by some boost in the x^1 direction with velocity $-\vec{v} = -v\,\hat{e}_1$, $x' = \Lambda(-v)\,x$. (It is $\Lambda(-v) = \Lambda(v)^{-1}$ since we have interchanged x and x' relative to Eq. 2.5.1.) Using the explicit form (2.5.7), with $v \to -v$, we have $\Delta x'^0 = \gamma\,(\Delta x^0 - \frac{v}{c}\,\Delta x^1) = \gamma L(\frac{1}{2} + \frac{v}{c})$. For this to vanish, we must have $v/c = -1/2$, implying that frame S' moves with velocity $\vec{v} = -(c/2)\,\hat{e}_1$ as seen in frame S. The common time of the two events in the S' frame is $t' = \gamma\,(ct_1 - \frac{v}{c}\,x_1)/c = \frac{2}{\sqrt{3}}\,(L + \frac{1}{2}L)/c = \sqrt{3}L/c$. As a check, the same result is obtained using the second event's coordinates.

[13] Adapted from Kogut problem 4-4.