**Mathematica** for Fourier Series and Transforms

## Fourier Series

### Periodic odd step function

Use built-in function "UnitStep" to define. "Mod" allows one to make the function periodic, with the "-Pi" shifting the fundamental region of the Mod to -Pi to Pi (rather than 0 to 2Pi). The period is taken to be 2 Pi, symmetric around the origin, so the function is even. "Exclusions->None" makes the plot include the steps.

In[5]:= ff0[x_] = -1/2 + UnitStep[Mod[x, 2 Pi, -Pi]]

In[6]:= ff0plot = Plot[ff0[x], {x, -3 Pi, 3 Pi}, Ticks -> {(-3 Pi, -2 Pi, -Pi, 0, Pi, 2 Pi, 3 Pi)}, PlotStyle -> {Thick, Red}, Exclusions -> None]

Truncated Fourier Series---since function is odd one can use "FourierSinSeries"

In[7]:= ft6ff0 = FourierSinSeries[ff0[x], x, 6]

Out[7]= \[
\frac{2 \sin[x]}{\pi} + \frac{2 \sin[3 x]}{3 \pi} + \frac{2 \sin[5 x]}{5 \pi}
\]

Faster for evaluation to define Fourier Series directly (given that we've worked it out by hand in class notes):

In[8]:= ftff0[x_, nmax_] := Sum[\sin[(2 n + 1) x] / (2 n + 1), \{n, 0, nmax\}] 2 / Pi
Comparing step to truncated Fourier series:

```
In[10]:= stepplot[n_] :=
    Plot[{ff0[x], ff0[x, n]}, {x, -Pi, Pi}, Ticks -> {(-Pi, -Pi/2, 0, Pi/2, Pi)},
    PlotStyle -> {Thick, Red}, {Thick, Blue}, Exclusions -> None]
```

```
In[11]:= stepplot[1]
```

Make animation showing convergence:

```
In[12]:= Manipulate[stepplot[n], {n, 0, 20, 1}]
```

Comparing 3, 21 and 101 term Fourier Series to original function

```
Out[10]=
```

```
Out[11]=
```

```
Out[12]=
```
In[13]:= Plot[{ff0[x], fftf0[x, 2], fftf0[x, 20], fftf0[x, 100]}, 
{x, -Pi, Pi}, Ticks -> {{-Pi, -Pi/2, 0, Pi/2, Pi}}, PlotStyle -> 
{{Thick, Red}, {Thick, Blue}, {Thick, Green}, {Thick, Black}}, Exclusions -> None]

Out[13]=

Zoom in to see Gibbs phenomenon: lack of convergence near step. 
Here use 21, 101 and 501 term series.

In[14]:= Plot[{ff0[x], fftf0[x, 20], fftf0[x, 100], fftf0[x, 500]}, 
{x, -0.1, +0.1}, PlotRange -> {(-.1, 0.1), (0.4, .6)}, PlotStyle -> 
{{Thick, Red}, {Thick, Green}, {Thick, Black}, {Thick, Orange}}, Exclusions -> None]

Out[14]=

Using complex fourier series

In[15]:= ft6ff0complex = FourierSeries[ff0[x], x, 6]

Out[15]= \[\frac{i e^{-ix}}{\Pi} - \frac{i e^{ix}}{\Pi} + \frac{i e^{-3ix}}{3 \Pi} - \frac{i e^{3ix}}{3 \Pi} + \frac{i e^{-5ix}}{5 \Pi} - \frac{i e^{5ix}}{5 \Pi}\]

Simplify doesn't help, but FullSimplify brings result back into Sin form found above

In[16]:= FullSimplify[ft6ff0complex]

Out[16]= \[\frac{2 (15 \text{Sin}[x] + 5 \text{Sin}[3 x] + 3 \text{Sin}[5 x])}{15 \Pi}\]
Even "barrier" function

Use built-in function "UnitStep" to define. "Mod" allows one to make the function periodic, with the "-Pi" shifting the fundamental region of the Mod to -Pi to Pi (rather than 0 to 2Pi). The period is taken to be 2Pi, symmetric around the origin, so the function is even.

\[
\text{ff1}[x_] = \text{UnitStep}[\text{Mod}[x + \frac{\pi}{2}, 2\pi, -\pi]] \text{UnitStep}[\text{Mod}[\frac{\pi}{2} - x, 2\pi, -\pi]]
\]

For an even function can use Cosine Fourier Series, here up to Cos[6x]. We will call this a 4 term series.

\[
\frac{1}{2} + \frac{2 \cos[x]}{\pi} - \frac{2 \cos[3x]}{3 \pi} + \frac{2 \cos[5x]}{5 \pi}
\]

The function does a pretty poor job of representing the "barrier" at this order.
Since the general form is known, it is faster (for evaluation) to write out the Fourier Transform explicitly. The number of terms is $2 \cdot \text{nmax} + 2$ (including the constant).

```math
In[21]:= \text{ftff1}[x_, \text{nmax}_] := 1/2 \cdot \text{Sum}((-1)^n \cos((2n + 1)x) / (2n + 1), \{n, 0, \text{nmax}\}) / \pi
```

Making and manipulating plots:

```math
In[22]:= \text{barrierplot}[n_] :=
    \text{Plot}[[\text{ff1}[x], \text{ftff1}[x, n]], \{x, -\pi, \pi\}, \text{Ticks} \to \{-\pi, -\pi/2, 0, \pi/2, \pi\},
    \text{PlotStyle} \to \{\{\text{Thick, Red}\}, \{\text{Thick, Blue}\}\}, \text{Exclusions} \to \text{None}]
```

```math
In[23]:= \text{barrierplot}[1]
```

Out[23]=

```
In[24]:= \text{Manipulate}[\text{barrierplot}[n], \{n, 0, 20\}, 1]
```

Out[24]=

Comparing the 4, 22 and 102 term Cosine Series to the function itself (which is no longer visible):
Zooming in on the edge of the step, showing the 22, 102 and 502 term series. (Colors match those in the previous plot.)

Note the oscillation near the edge with the fixed 9% overshoot (called the Gibbs phenomenon).

Using Complex Fourier Series

\[
\text{ft6ff1complex} = \text{FourierSeries[ff1[x], x, 6]}
\]

\[
\frac{1}{2} + \frac{e^{-ix}}{\pi} + \frac{e^{ix}}{\pi} - \frac{e^{-3ix}}{3\pi} - \frac{e^{3ix}}{3\pi} + \frac{e^{-5ix}}{5\pi} + \frac{e^{5ix}}{5\pi}
\]

\[
\text{FullSimplify[ft6ff1complex]}
\]

\[
\frac{15\pi + 60\cos|x| - 20\cos3x + 12\cos5x}{30\pi}
\]

\[
\text{Out[28]} = \frac{15\pi + 60\cos|x| - 20\cos3x + 12\cos5x}{30\pi}
\]
General real function (neither odd nor even)

\[ ff4[x_] := \text{Piecewise}[\{ \{3 \text{Mod}[x, 2 \pi, -\pi] + \pi) / \pi - 1, \text{Mod}[x, 2 \pi, -\pi] < -\pi / 2], \{-\text{Mod}[x, 2 \pi, -\pi] / \pi, \text{Mod}[x, 2 \pi, -\pi] \geq -\pi / 2\}] \]

Use "FourierTrigSeries" to get Cosine/Sine series

Here's the complex version:
\texttt{In[6]} = \texttt{ft6ff4complex = FourierSeries[ff4[x], x, 6]}

\texttt{Out[6]} = -\frac{1}{4} + \frac{(2 - 2 \ i) e^{-i x}}{\pi^2} + \frac{(2 + 2 \ i) e^{i x}}{\pi^2} - \frac{e^{-2 i x}}{\pi^2} - \frac{e^{2 i x}}{\pi^2} + \frac{\left(\frac{2}{9} + \frac{2 i}{9}\right) e^{3 i x}}{\pi^2} + \frac{\left(\frac{2}{25} - \frac{2 i}{25}\right) e^{5 i x}}{\pi^2} - \frac{\left(\frac{2}{25} + \frac{2 i}{25}\right) e^{-5 i x}}{\pi^2} - \frac{\left(2 - \frac{2 i}{9}\right) e^{3 i x}}{\pi^2} + \frac{\left(2 - \frac{2 i}{25}\right) e^{5 i x}}{\pi^2} - \frac{\left(2 + \frac{2 i}{25}\right) e^{-5 i x}}{\pi^2} - \frac{\left(2 + \frac{2 i}{9}\right) e^{3 i x}}{9 \pi^2} + \frac{\left(2 - \frac{2 i}{9}\right) e^{-3 i x}}{9 \pi^2}

\texttt{In[7]} = \texttt{FullSimplify[ft6ff4complex]}

\texttt{Out[7]} = \frac{1}{900 \pi^2} \left(3600 \cos[x] - 1800 \cos[2 x] + 400 \cos[3 x] + 144 \cos[5 x] - 25 \left(9 \pi^2 + 8 \cos[6 x] + 144 \sin[x] - 16 \sin[3 x]\right) - 144 \sin[5 x]\right)

\textbf{Even function with discontinuity in derivative}

Here we consider an example with a discontinuity in derivative but not in the function itself.

\texttt{In[8]} = \texttt{ff3[x_] = Piecewise[\{\{\cos[x], \ -\pi/2 < \text{Mod}[x, \ 2 \pi], \ -\pi < \pi/2\}\\}]

\texttt{Out[8]} = \begin{cases} 
\cos[x] & \pi/2 < \text{Mod}[x, 2 \pi] < -\pi < \pi/2 \\
0 & \text{True}
\end{cases}

\texttt{In[9]} = \texttt{ff3plot = Plot[ff3[x], \{x, \ -3 \pi, \ 3 \pi\}, \text{Ticks} \to \{\{-3 \pi, -2 \pi, -\pi, 0, \pi, 2 \pi, 3 \pi\}, \text{PlotStyle} \to \{\text{Thick, Red}\}, \text{Exclusions} \to \text{None}\}}

\texttt{Out[9]} = \text{(Plot graph)}

Starting with the \(\cos[2x]\) term, the number in the denominator is \(n^2 - 1\).

So the coefficients fall like \(1/n^2\), compared to the \(1/n\) we saw with a discontinuity in the function itself.

This is general.

\texttt{In[10]} = \texttt{ft6ff3[x_] = FourierCosSeries[ff3[x], x, 6]}

\texttt{Out[10]} = \begin{align*}
&\frac{\cos[x]}{\pi} + \frac{2 \cos[2 x]}{2} - \frac{2 \cos[4 x]}{3 \pi} + \frac{2 \cos[6 x]}{15 \pi} \end{align*}
\textbf{In[11]}:= \texttt{ft20ff3[x_] = FourierCosSeries[ff3[x], x, 20]}

\[
\frac{1}{\pi} + \frac{\cos(x)}{2} + \frac{2\cos(2x)}{3\pi} - \frac{2\cos(4x)}{15\pi} + \frac{2\cos(6x)}{35\pi} - \frac{2\cos(8x)}{63\pi} + \frac{2\cos(10x)}{99\pi} - \frac{2\cos(12x)}{143\pi} + \frac{2\cos(14x)}{195\pi} - \frac{2\cos(16x)}{255\pi} + \frac{2\cos(18x)}{323\pi} - \frac{2\cos(20x)}{399\pi}
\]

\textbf{In[12]}:= \texttt{ft100ff3[x_] = FourierCosSeries[ff3[x], x, 100];}

From this view, 20 terms is enough to do a good job, with 6 showing oscillations:

\textbf{In[13]}:= \texttt{ff3plot = Plot[{ff3[x], ft6ff3[x], ft20ff3[x]},
                   {x, -3\pi, 3\pi}, Ticks -> {{-3\pi, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi}},
                   PlotStyle -> {{Thick, Red}, {Thick, Blue}, {Thick, Green}}, Exclusions -> None]}

Zooming in, we need > 100 terms to approach the discontinuity in derivative. But there is no Gibbs phenomenon here.

\textbf{In[14]}:= \texttt{ff3plot = Plot[{ff3[x], ft6ff3[x], ft20ff3[x], ft100ff3[x]},
                    {x, Pi/2 - 0.1, Pi/2 + 0.2}, PlotStyle ->
                    {{Thick, Red}, {Thick, Blue}, {Thick, Green}, {Thick, Black}}, Exclusions -> None]}

Integrals needed in discussion of Gibbs phenomenon (see lecture notes)

\[
\text{In[15]:= } \int \frac{\sin x}{x} \, dx, \quad \text{Out[15]= } \frac{1}{2}
\]

\[
\text{In[16]:= } \int \frac{\sin x}{x}, \quad \text{Out[16]= } 0.58949
\]

Showing that the sum indeed asymptotes to the integral above:

\[
\text{In[17]:= } \text{ss}[n] := \text{Sum}[\frac{\sin(x) \cdot (2n+1)}{(2n+1)}, \quad \text{Out[17]= } \text{ListPlot[Table[ss[n], {n, 5, 500}]]}
\]

Fourier transforms

Section 7.12 problem number 10

This is one way of writing the function in Mathematica

\[
\text{In[19]:= } f10[t_] = \text{Piecewise}\left[\{\{(-2 + t), -a < t < 0\}, \{2 (a - t), 0 < t < a\}\}\right];
\]

Plotting: note that the "/. a->1" command tells Mathematica to set the parameter a to 1 in the function.
In[20]:= \[f_{10}\text{plot} = \text{Plot}[f_{10}[t] \/. a \rightarrow 1, \{t, -2, 2\}, \text{PlotStyle} \rightarrow \{\text{Thick, Red}\}]

Out[20]=

Obtaining the Fourier Transform. The "FourierParameters" must be set to match our conventions (see the Documentation for their definitions). The "Assumptions -> a>0" allows Mathematica to ignore the case where a<0 in which case, given the definitions above, the function vanishes. “FullSimplify” leads to a more compact answer than “Simplify”

In[21]:= \[f_{10}\text{alp}_\_] = \text{FullSimplify}[\text{FourierTransform}[f_{10}[t], t, \text{alp}, \text{FourierParameters} \rightarrow (-1, -1), \text{Assumptions} \rightarrow a > 0]]

Out[21]= \[
\frac{2 \, i \, (-a \, \text{alp} + \sin[a \, \text{alp}])}{\text{alp}^2 \, \pi}
\]

Since our function is odd, we can also use the "FourierSinTransform", which gives the imaginary part of the previous result

In[22]:= \[\text{img}_{10}\text{alp}_\_] = \text{FourierSinTransform}[f_{10}[t], t, \text{alp}, \text{FourierParameters} \rightarrow (-1, -1), \text{Assumptions} \rightarrow a > 0]

Out[22]= \[
\frac{2 \, (-a \, \text{alp} + \sin[a \, \text{alp}])}{\text{alp}^2 \, \pi}
\]

In[23]:= \[\text{Plot}[\text{img}_{10}\text{alp}_\_] \/. a \rightarrow 1, \{\text{alp}, -15, 15\}, \text{PlotStyle} \rightarrow \{\text{Thick, Blue}\}]

Out[23]=

Here is the inverse Fourier Transform which brings us back to the original function (expressed in a
different way).

\[ \text{In[24]}= \text{invft10[t]} = \text{FullSimplify[} \]
\[ \text{InverseFourierTransform[ft10[\alpha], \alpha, t, FourierParameters \to \{-1, -1\}] } \]
\[ \text{Out[24]}= (a-t) \text{Sign}(a-t) + 2 a \text{Sign}(t) - (a+t) \text{Sign}(a+t) \]

\[ \text{In[25]}= \text{Plot[invft10[t] /. a \to 1, \{t, -2, 2\}, PlotStyle \to \{Thick, Red\}] } \]

\[ \text{Out[25]}= \]