 Name _____solutions______________________

Electrodynamics, Physics 321 **Final exam** **2.30 pm Wed 15 March**
Winter 2006 Instructor: David Cobden

Do not turn this page until I say 'go' at 2.30 pm. You have 110 minutes. Hand your exam to the moderator before he leaves the room at 4.25 pm.

This exam contains 200 points. Be sure to attempt all the questions.

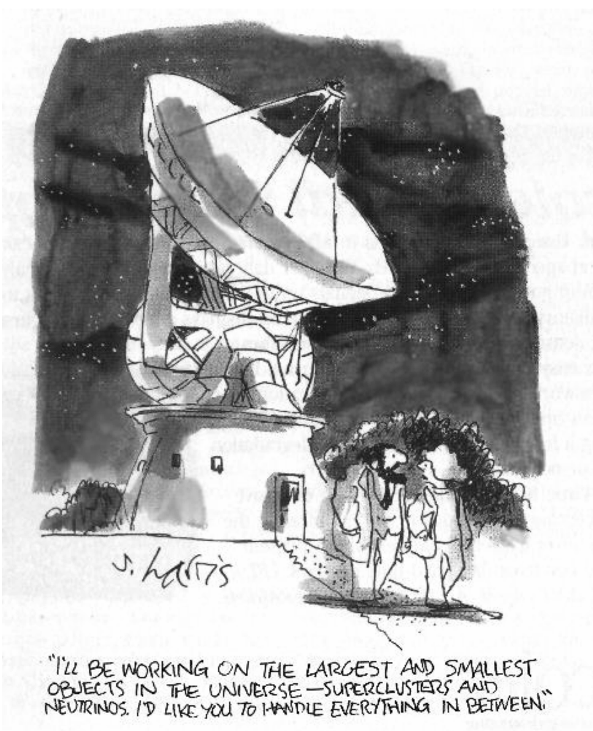
Please write your name on every page and your SID on the first page.

Write all your working on these question sheets. Use the front and back pages for extra working. It is important to show your calculation or derivation. You probably won't get full marks just for stating the correct answer if you don't show how you get it.

Watch the blackboard/overhead for corrections or clarifications during the exam.

This is a closed book exam. No books, notes or calculators allowed.

$$V_{dip}(\mathbf{r}) = \frac{\mathbf{p}.\hat{\mathbf{r}}}{4\pi\varepsilon_0 r^2} . \qquad \nabla\left(\frac{\mathbf{p}.\hat{\mathbf{r}}}{r^2}\right) = \frac{3(\mathbf{p}.\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}}{r^3} \qquad \frac{1}{(1-2su+s^2)^{1/2}} = \sum_{n=0}^{\infty} P_n(u) s^n$$



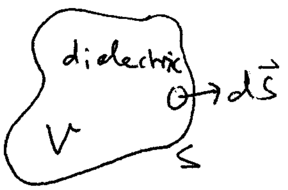
Name solutions

1. [10] Give expressions relating the polarization density **P** to the bound charge density on the surface and in the bulk of a dielectric.

$$\rho_b = -\vec{\nabla}\cdot\vec{P} \qquad \sigma_b = \vec{P}\cdot\hat{n}$$

2. [10] Using these expressions, show that the total bound charge (the sum of total surface charge and total volume charge) on any piece of dielectric is zero. Explain why this has to be true.

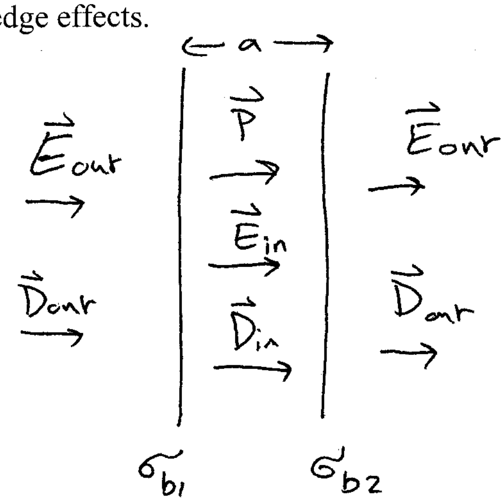
$$Q_b = \int_V \rho_b \, d^3r + \oint_S \sigma_b \, dS = \int_V -\vec{\nabla}\cdot\vec{P}\, d^3r + \oint_S \vec{P}\cdot\hat{n}\, dS$$



$$= \underbrace{-\oint_S \vec{P}\cdot d\vec{S}}_{\text{by Gauss}} + \oint_S \vec{P}\cdot d\vec{S} = 0$$

Every molecule/atom inside has zero net charge

3. [15] A thin sheet of dielectric material, in vacuum, has thickness *a* and *uniform* frozen-in polarization density **P** perpendicular to its faces. Find the charge density, electric field **E**, and displacement field **D**, inside, on, and outside the slab. Do so only near the middle of sheet, ie, neglect edge effects.

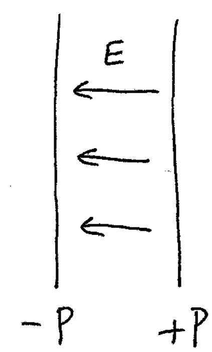


Bound charge:

$$\rho_b = -\vec{\nabla}\cdot\vec{P} = 0$$

$$\sigma_{b2} = \vec{P}\cdot\hat{n}_2 = P$$

$$\sigma_{b1} = \vec{P}\cdot\hat{n}_1 = -P$$



(It looks like a parallel plate capacitor.)

$\vec{E}_{out} = 0$ (assuming no external applied field)

$\vec{D}_{out} = 0 = \varepsilon_0 \vec{E}_{out}$

b.cs. at left surface:

$$E_{in} = E_{out} + \frac{\sigma}{\varepsilon_0} = \frac{-P}{\varepsilon_0}$$

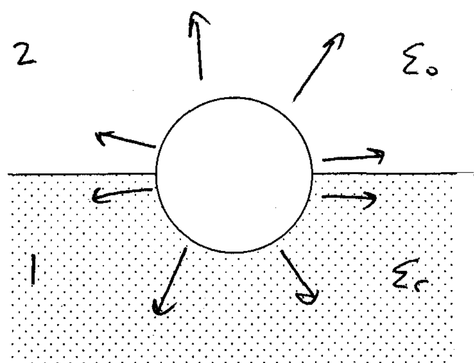
$$D_{in} = D_{out} + 0 \leftarrow \text{no free charge}$$

$$= 0$$

 Name solutions

4. [15] A conducting sphere of radius a held at voltage V_0 (relative to ground) is centered at the origin. Half the space around it, in the region $z < 0$, is filled with a uniform linear dielectric of relative permittivity ε_r, as indicated below. The rest of space, $z > 0$, is vacuum. Show that the electric field all around the sphere is the same as it would be if the dielectric were not present (ie, that this solution obeys all appropriate requirements).

Without dielectric $\vec{E} = \frac{V_0 a \hat{r}}{r^2}$ $\left(= \frac{Q\hat{r}}{4\pi\varepsilon_0 r^2} \text{ and } V_0 = \frac{Q}{4\pi\varepsilon_0 a} \right)$



We need to show that this obeys the b.c.'s in this problem:

at surface, require $E_1^{\parallel} = E_2^{\parallel}$ and $\varepsilon_{r1} E_1^{\perp} = \varepsilon_{r2} E_2^{\perp}$

OK

also OK because $E_1^{\perp} = 0 = E_2^{\perp}$

Also $V = V_0$ at sphere surface

$\therefore$ by uniqueness, it is the correct solution

5. [20] Find the energy stored at the applied voltage V_0, and hence its self-capacitance.

$$U = \int \tfrac{1}{2} \vec{D}\cdot\vec{E}\, d^3r = \underbrace{\int \tfrac{1}{2}\varepsilon_0 E^2 d^3r}_{\text{vacuum (region 2)}} + \underbrace{\int \tfrac{1}{2}\varepsilon_r\varepsilon_0 E^2 d^3r}_{\text{dielectric}}$$

$$= \tfrac{1}{2}\varepsilon_0(1+\varepsilon_r) \underbrace{\int E^2 dr}_{\text{vacuum}}$$

$$= \tfrac{1}{2}\varepsilon_0 \frac{(1+\varepsilon_r)}{2} \int_a^\infty \left(\frac{V_0 a}{r^2}\right)^2 4\pi r^2 dr$$

$$= \tfrac{1}{2}\cdot\tfrac{1}{2}\cdot\varepsilon_0(1+\varepsilon_r)\cdot V_0^2 a^2 \cdot 4\pi \cdot \int_a^\infty \frac{dr}{r^2}$$

$$= \varepsilon_0(1+\varepsilon_r) V_0^2 a^2 \pi \cdot \frac{1}{a} = \pi a V_0^2 \varepsilon_0 (1+\varepsilon_r)$$

$U = \tfrac{1}{2} C V_0^2$

$$\therefore C = \frac{2U}{V_0^2} = 2\pi\varepsilon_0(1+\varepsilon_r)a$$

(check: for $\varepsilon_r = 1$, $C = 4\pi a \varepsilon_0$ as for isolated sphere)

 Name ________________________

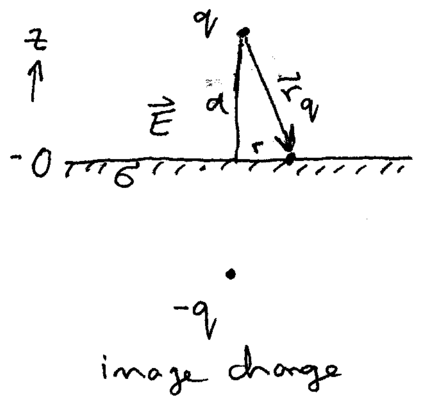
A particle of charge q is a distance a above a planar conducting surface at $z = 0$ in vacuum.

6. [5] What is the force on the particle?

Particle experiences field equiv. to that of image charge:

$$F = \frac{q^2}{4\pi\varepsilon_0 (2a)^2} \text{ towards conductor.}$$

7. [20] Show that the (free) charge density on the conductor surface is $\sigma = \dfrac{-qa}{2\pi(r^2+a^2)^{3/2}}$, where r is the distance from the origin which is directly beneath the particle in the $z = 0$ plane.



$$\vec{E} = \vec{E}_q + \vec{E}_{-q} \qquad \sigma = \varepsilon_0 E_\perp = \varepsilon_0 E_z\Big|_{z=0}$$

$$E_z\Big|_{z=0} = 2 \times E_z(\text{due to } q \text{ only})\Big|_{z=0}$$ contributions from q and image are equal

$$= \frac{2q(\vec{r}_q)_z}{4\pi\varepsilon_0 r_q^3}\Bigg|_{z=0} \qquad r_q^2 = r^2 + a^2 \qquad (\vec{r}_q)_z = -a$$

$$= \frac{-qa}{2\pi\varepsilon_0 (r^2+a^2)^{3/2}}$$

The space around the point charge is now entirely filled with a linear dielectric material having relative permittivity ε_r.

8. [5] Does the *free* charge density on the conductor change? Why (not)?

No. The dielectric has no net charge. The particle's charge must be balanced by free charge on the conductor's surface whatever the dielectric.

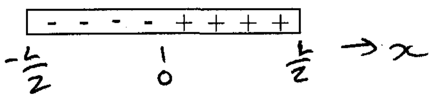
9. [5] Does the electrostatic force on the particle change? Why (not)? (This force must be balanced by a mechanical force from the dielectric to prevent the particle moving.)

Yes. Polarization of the dielectric reduces the electric field due to the conductor (image charge)

 Name solutions

Imagine that we have a polar molecule and an ion in a big and otherwise empty box. Taking the center of the molecule to be the origin and the x-axis to be oriented conveniently, this particular molecule can be thought of as a thin rigid bar of length L with positive line charge density λ_0 glued along one half ($x > 0$) and negative density $-\lambda_0$ glued along the other ($x < 0$), as indicated below.

10. [5] What is the dipole moment of the molecule?



$$\vec{p} = \int \rho \vec{r}\, d^3r \quad \therefore p = \int_{-\frac{L}{2}}^{\frac{L}{2}} \lambda x\, dx = \int_{-\frac{L}{2}}^{0} -\lambda_0 x\, dx + \int_{0}^{\frac{L}{2}} \lambda_0 x\, dx$$

$$= 2\lambda_0 \left[\frac{x^2}{2}\right]_0^{\frac{L}{2}} = \frac{\lambda_0 L^2}{4}$$

11. [10] The ion has charge $+Q$ and is located at point $\mathbf{r}$, where $r >> L$. What is the force exerted on the ion by the molecule?

$$\vec{F}_{ion} = +Q\vec{E}_{dip} = Q\vec{\nabla}\left(\frac{\vec{p}\cdot\hat{r}}{4\pi\varepsilon_0 r^2}\right) = \frac{Q\left[3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}\right]}{4\pi\varepsilon_0 r^3}$$

(see front page)

12. [10] What is the torque on the molecule due to the ion?

$$\vec{N} = \vec{p}\times\vec{E} = \frac{\lambda_0 L^2 \hat{x}}{4} \times \left(\frac{-Q\hat{r}}{4\pi\varepsilon_0 r^2}\right) = -\frac{Q\lambda_0 L^2\, \hat{x}\times\hat{r}}{16\pi\varepsilon_0 r^2}$$

Coulomb field of ion at $\vec{r}$.

13. [10] Show that the force exerted on the molecule by the ion is equal and opposite to the answer to Q. 11, as required by Newton's first law.

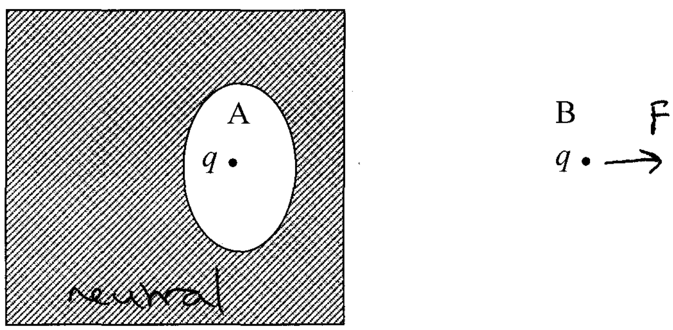
$$\vec{F}_{mol} = -\vec{\nabla} U_{dip} = -\vec{\nabla}(-\vec{p}\cdot E) = \vec{\nabla}\left[\vec{p}\cdot\left(\frac{-Q\hat{r}}{4\pi\varepsilon_0 r^2}\right)\right]$$

$$= -Q\vec{\nabla}\left(\frac{\vec{p}\cdot\hat{r}}{4\pi\varepsilon_0 r^2}\right) = -\vec{F}_{ion}$$

 Name solutions

A particle – let's call it particle A – of charge q is inside a cavity inside a *neutral* conducting cube of side a. Another identical particle, B, is outside the cube at a distance about a, as sketched below.

14. [5] Is particle B attracted or repelled by the cube, and why?



repelled. Flux out of cube is $\frac{q}{\varepsilon_0}$. Far away, cube looks like monopole charge q.

15. [5] The cube is now connected to ground. Is particle B now attracted or repelled by the cube, and why?

Attracted. Now cube is grounded conductor with some induced surface charge opposite in sign to q which attracts it.

16. [15] Expand $\dfrac{1}{|\mathbf{r}-\mathbf{r}_k|}$ in powers of (r_k/r), assuming $r > r_k$, defining θ_k as the angle between $\mathbf{r}$ and $\mathbf{r}_k$. (see cover page).

$$\frac{1}{|\vec{r}-\vec{r}_k|} = \frac{1}{\sqrt{(\vec{r}-\vec{r}_k)^2}} = \frac{1}{\sqrt{\vec{r}^2 + r_k^2 - 2\vec{r}\cdot\vec{r}_k}}$$

$$= \frac{1}{(r^2 - 2rr_k\cos\theta_k + r_k^2)^{1/2}} = \frac{1}{r\left(1 - 2\frac{r_k}{r}\cos\theta_k + \left(\frac{r_k}{r}\right)^2\right)^{1/2}}$$

$$= \frac{1}{r}\sum_{n=0}^{\infty} P_n(\cos\theta_k)\left(\frac{r_k}{r}\right)^n \quad \text{(from cover page)}$$

$s = \frac{r_k}{r}$ $\quad u = \cos\theta_k$

17. [10] Show that every term in the expansion obeys Laplace's equation and say why it must.

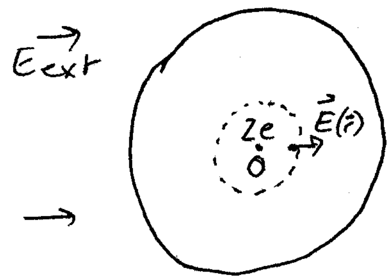
General Laplace solution in spherical polars is

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$

Each term in the answer to 16 has this form so it is a valid solution.

Name solutions

18. [15] Assume the electronic charge cloud in an atom of atomic number Z has approximate charge density $\rho(r) = \rho_0 \exp(-r/a)$. Show that *for* $r << a$ the electric field inside the charge cloud, experienced by the nucleus, is $E(r) = \rho_0 r/(3\varepsilon_0)$.



Symmetry $\rightarrow \vec{E} = E(r)\hat{r}$

Gauss : $4\pi r^2 E(r) = \frac{Q\text{(inside sphere rad. } r)}{\varepsilon_0}$

$$= \frac{1}{\varepsilon_0}\int_0^r \rho(r') 4\pi r'^2 dr'$$

$$= \frac{4\pi}{\varepsilon_0}\int_0^r \rho_0 e^{-r'/a} r'^2 dr'$$

$$= \frac{4\pi\rho_0}{\varepsilon_0}\int_0^r \left(1 - r'/a + \ldots\right) r'^2 dr'$$

$$\simeq \frac{4\pi\rho_0}{\varepsilon_0}\int_0^r r'^2 dr' \quad \text{for } r'/a << 1$$

$$= \frac{4\pi\rho_0}{\varepsilon_0}\frac{r^3}{3} \qquad \therefore E(r) = \frac{\rho_0 r}{3\varepsilon_0}$$

19. [10] Hence estimate the polarizability α of the atom in terms of the parameters given.

In equilibrium, $Ze E_{ext} = -Ze E(r)$

$$= -Ze \frac{\rho_0 r}{3\varepsilon_0}$$

$$= -\frac{p \rho_0}{3\varepsilon_0} \qquad p = Zer \text{ dipole moment of atom}$$

$$\therefore p = -\frac{3\varepsilon_0}{\rho_0} Ze E_{ext}$$

$p = \alpha E_{ext}$ $\qquad \therefore \alpha = \frac{3\varepsilon_0 Ze}{(-\rho_0)}$