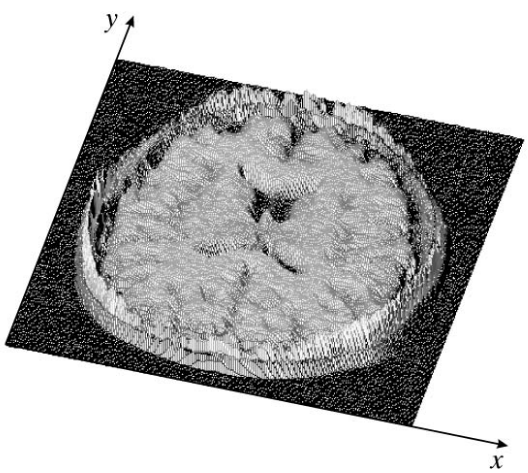
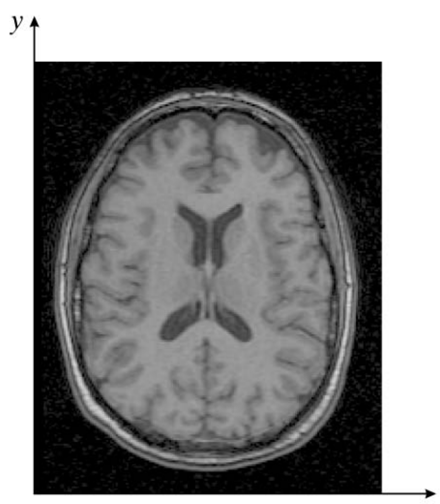
2D Signals and Systems

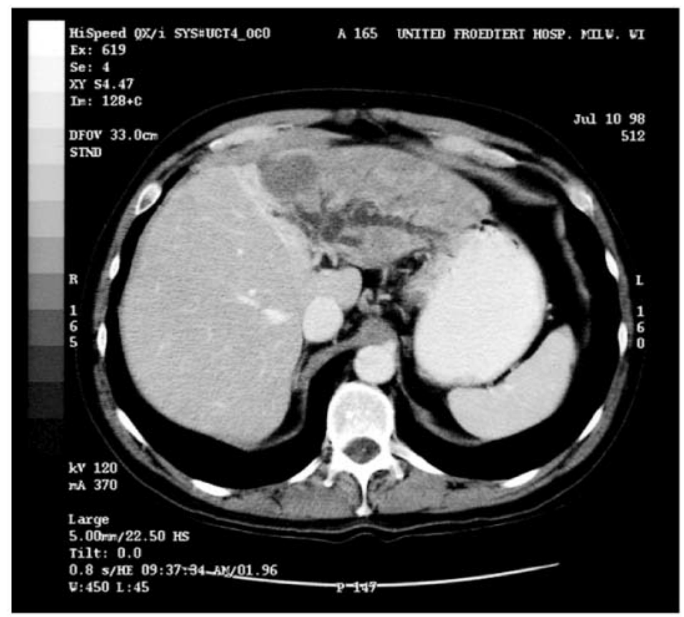
Signals

- A signal can be either continuous $f(x), f(x,y), f(x,y,z), f(\mathbf{x})$
- or discrete $f_{i,j,k}$ etc. where *i,j,k* index specific coordinates

- Digital images on computers are necessarily discrete sets of data
- Each element, or bin, or voxel, represents some value, either measured or calculated

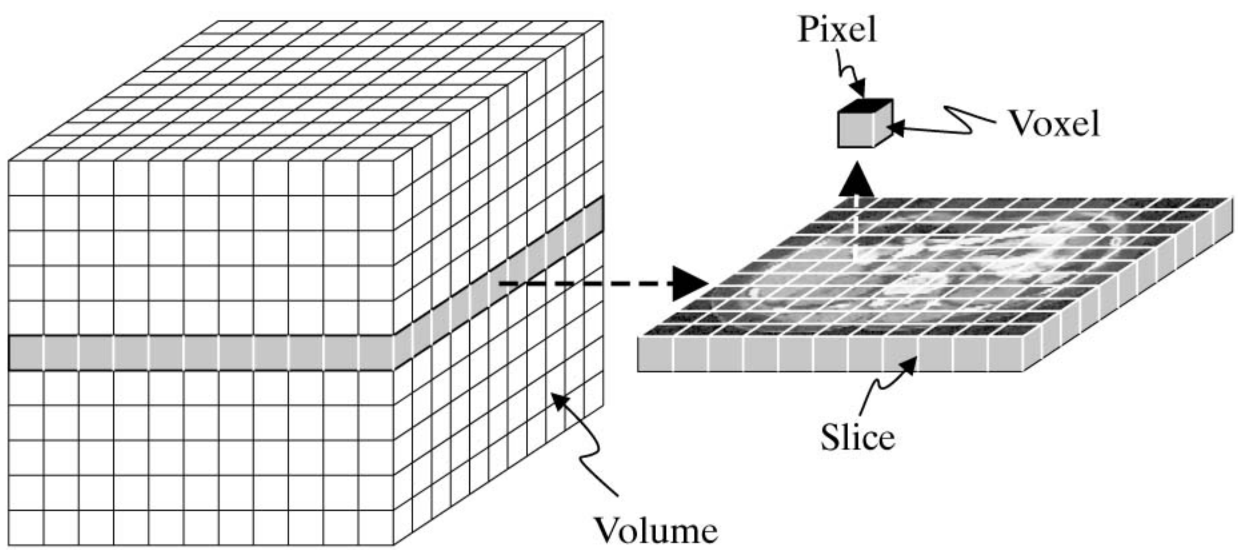






Digital Images

- Real objects are continuous (at least above the quantum level), but we represent them digitally as an approximation of the true continuous process (pixels or voxels)
- For image representation this is usually fine (we can just use smaller voxels as necessary)



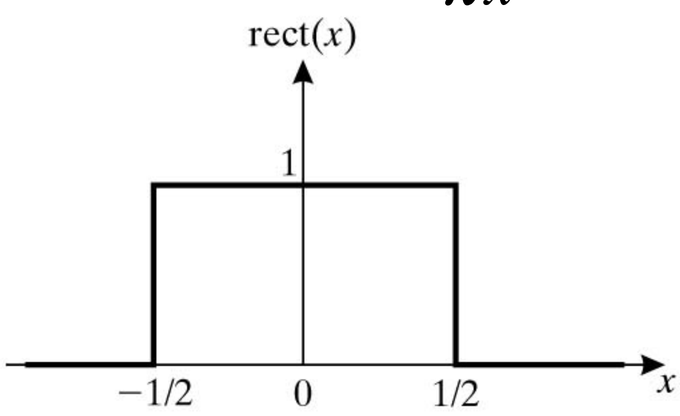
- For data measurements the element size is critical (e.g. Shannon's sampling theorem)
- For most of our work we will use continuous function theory for convenience, but sometimes the discrete theory will be required

Important signals - rect() and sinc() functions

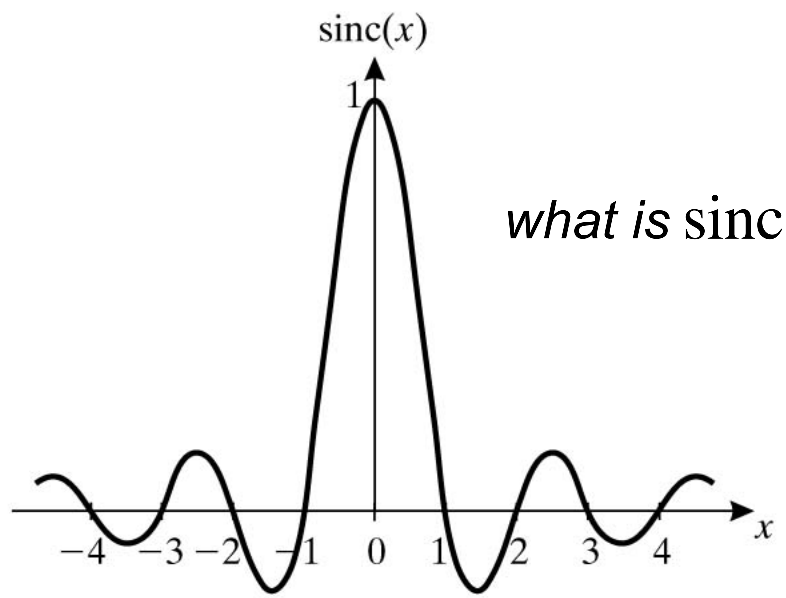
- 1D rect() and sinc() functions
  - both have unit area

$$\text{(a)} \quad \mathrm{rect}(x) = \begin{cases} 1, & \text{for } |x| < 1/2 \\ 0, & \text{for } |x| > 1/2 \end{cases}$$

$$\text{(b)} \quad \mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$



(a)



(b)

what is $\mathrm{sinc}(0)$?

Important signals - 2D rect() and sinc() functions

- 2D rect() and sinc() functions are straightforward generalizations

$$\text{(a)} \quad \text{rect}(x,y) = \begin{cases} 1, & \text{for } |x| < 1/2 \text{ and } |y| < 1/2 \\ 0, & \text{otherwise} \end{cases}$$

$$\text{(b)} \quad \text{sinc}(x,y) = \frac{\sin(\pi x)\sin(\pi y)}{\pi^2 xy}$$

- *Try to sketch these*
- 3D versions exist and are sometimes used
- Fundamental connection between rect() and sinc() functions and very useful in signal and image processing

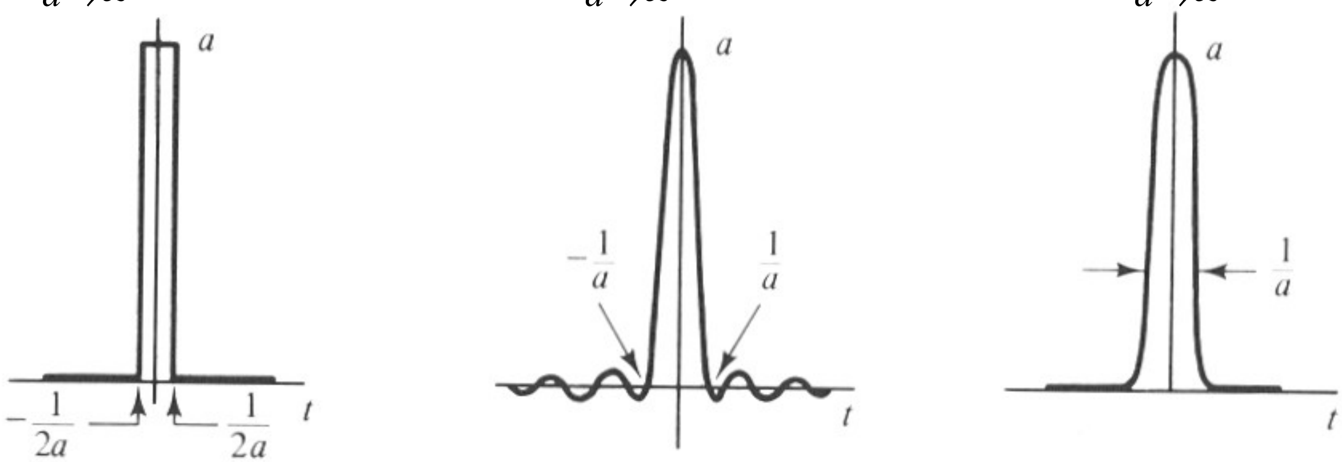
Important signals - Impulse function

- 1D Impulse (delta) function
- A 'generalized function'
  - operates through integration
  - has zero width and unit area
  - has important 'sifting' property
  - can be understood by considering:
- Ways to approach the delta function

$$\begin{cases} \delta(x) = 0, \quad x \neq 0, \\ \int_{-\infty}^{\infty} \delta(x)\,dx = 1 \end{cases}$$

$$\begin{cases} \int_{-\infty}^{\infty} f(x)\delta(x)\,dx = f(0) \\ \int_{-\infty}^{\infty} f(x)\delta(x-t)\,dx = f(t) \end{cases}$$

$$\delta(t) = \lim_{a\to\infty} a\,\mathrm{rect}(at) \qquad \delta(t) = \lim_{a\to\infty} a\,\mathrm{sinc}(at) \qquad \delta(t) = \lim_{a\to\infty} a e^{-\pi a^2 t^2}$$



Exponential and sinusoidal signals

- Recall Euler's formula,which connects trigonometric and complex exponential functions

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$ (not i)

- The exponential signal is defined as:

$$e^{j2\pi x} = \cos(2\pi x) + j\sin(2\pi x), \quad \text{where } j^2 = -1$$

- u_0 and v_0 are the fundamental frequencies in x- and y-directions, with units of 1/distance $e(x,y) = e^{j2\pi(u_0 x + v_0 y)}$

- We can write $e(x,y) = e^{j2\pi(u_0 x + v_0 y)}$

$$= \underbrace{\cos\left[2\pi\left(u_0 x + v_0 y\right)\right]}_{\text{real and even}} + \underbrace{j\sin\left[2\pi\left(u_0 x + v_0 y\right)\right]}_{\text{imaginary and odd}}$$

Exponential and sinusoidal signals

- Recall that $$\sin(2\pi x) = \frac{1}{2j}\left(e^{j2\pi x} - e^{-j2\pi x}\right)$$

$$\cos(2\pi x) = \frac{1}{2}\left(e^{j2\pi x} + e^{-j2\pi x}\right)$$

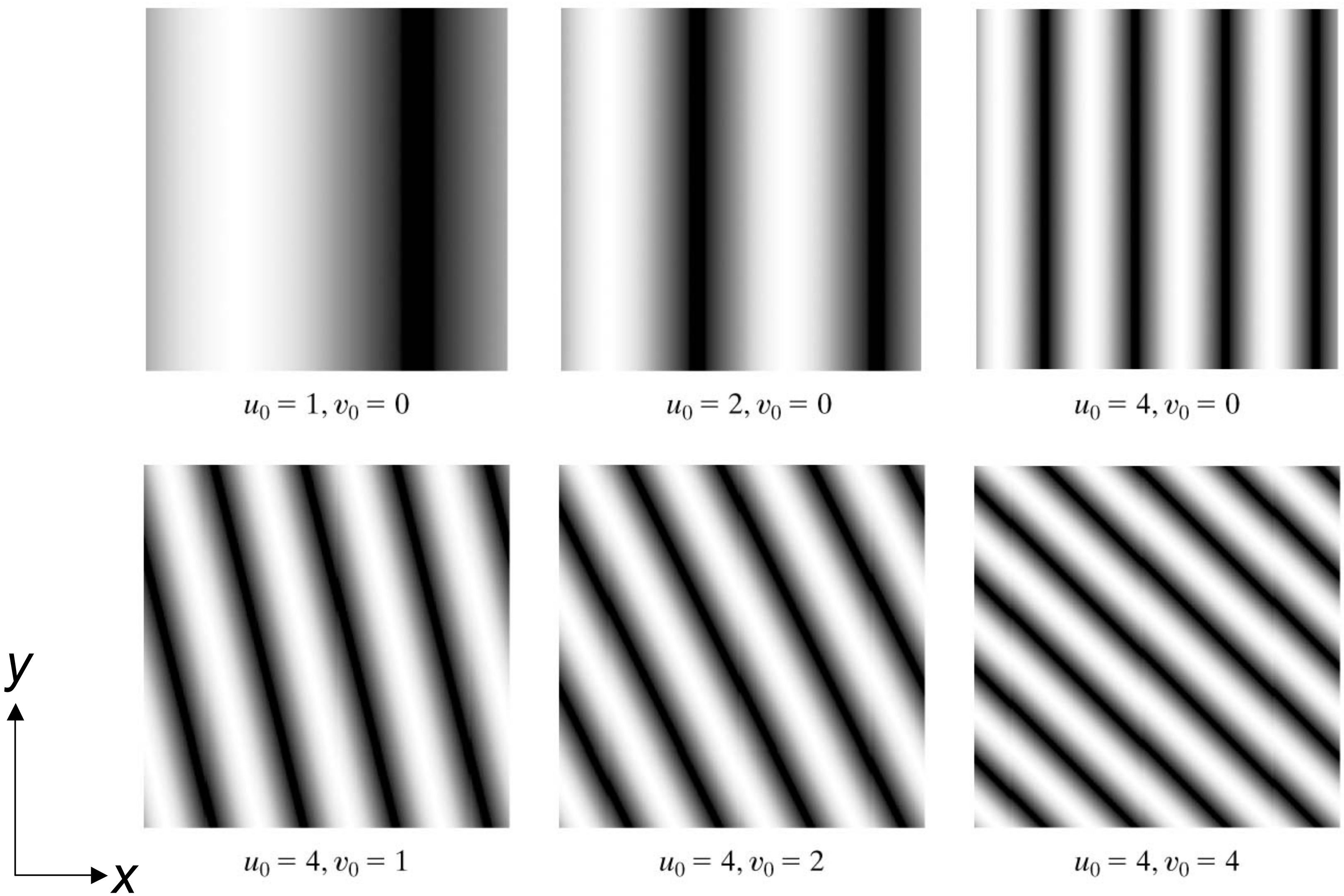
- so we have $$\sin\left[2\pi\left(u_0 x + v_0 y\right)\right] = \frac{1}{2j}\left(e^{j2\pi\left(u_0 x + v_0 y\right)} - e^{-j2\pi\left(u_0 x + v_0 y\right)}\right)$$

$$\cos\left[2\pi\left(u_0 x + v_0 y\right)\right] = \frac{1}{2}\left(e^{j2\pi\left(u_0 x + v_0 y\right)} + e^{-j2\pi\left(u_0 x + v_0 y\right)}\right)$$

- Fundamental frequencies u_0, v_0 affect the oscillations in x and y directions, E.g. small values of u_0 result in slow oscillations in the x-direction
- These are complex-valued and directional plane waves

Exponential and sinusoidal signals

- Intensity images for $s(x,y) = \sin\left[2\pi\left(u_0 x + v_0 y\right)\right]$



System models

- Systems analysis is a powerful tool to characterize and control the behavior of biomedical imaging devices
- We will focus on the special class of *continuous*, *linear*, *shift-invariant* (LSI) systems
- Many (all) biomedical imaging systems are not really any of the three, but it can be useful tool, as long as we understand the errors in our approximation
- "all models are wrong, but some are useful" -*George E. P. Box*
- Continuous systems convert a continuous input to a continuous output

$$g(x) = \mathscr{S}\left[f(x)\right] \qquad \left(g(t) = \mathscr{S}\left[f(t)\right]\right)$$

Linear Systems

- A system $\mathscr{S}$ is a linear system if: we have $\mathscr{S}\left[f(x)\right]=g(x)$

  then $$\mathscr{S}\left[a_1 f_1(x)+a_2 f_2(x)\right]=a_1 g_1(x)+a_2 g_2(x)$$

or in general $$\mathscr{S}\left[\sum_{k=1}^{K} w_k f_k(x)\right]=\sum_{k=1}^{K} w_k\,\mathscr{S}\left[f_k(x)\right]=\sum_{k=1}^{K} w_k g_k(x)$$

- Which are linear systems?

$$g(x)=e^{\pi} f(x)$$

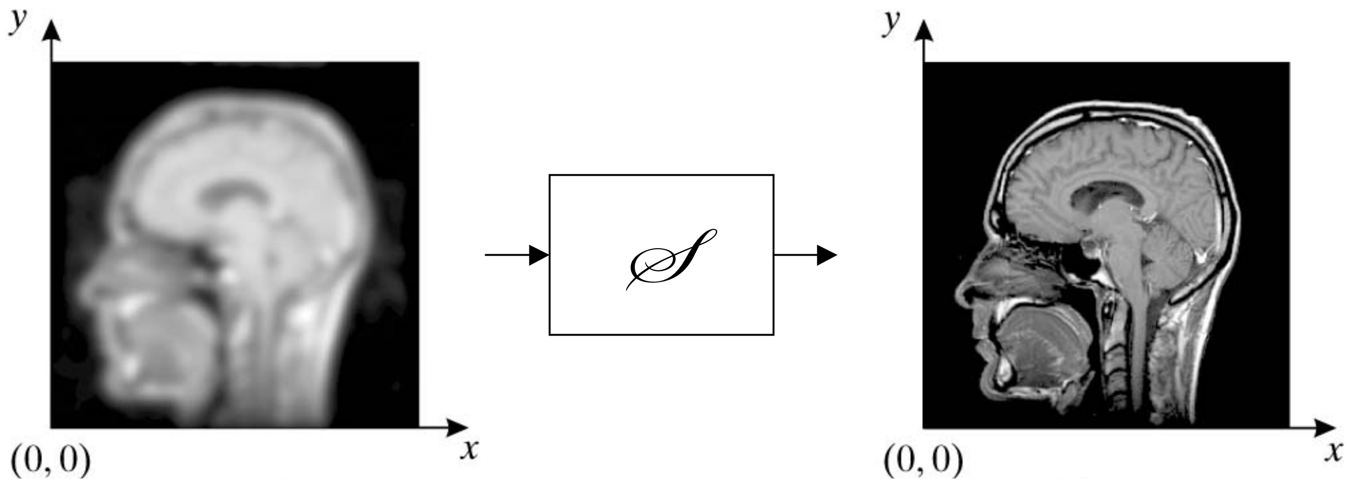
$$g(x)=f(x)+1$$

$$g(x)=x f(x)$$

$$g(x)=\left(f(x)\right)^2$$

2D Linear Systems

- Now use 2D notation
- Example: sharpening filter



$f(x,y)$ $\qquad$ $g(x,y)$

- In general

$$\mathscr{S}\left[\sum_{k=1}^{K} w_k f_k(x,y)\right] = \sum_{k=1}^{K} w_k \, \mathscr{S}\left[f_k(x,y)\right] = \sum_{k=1}^{K} w_k g_k(x,y)$$

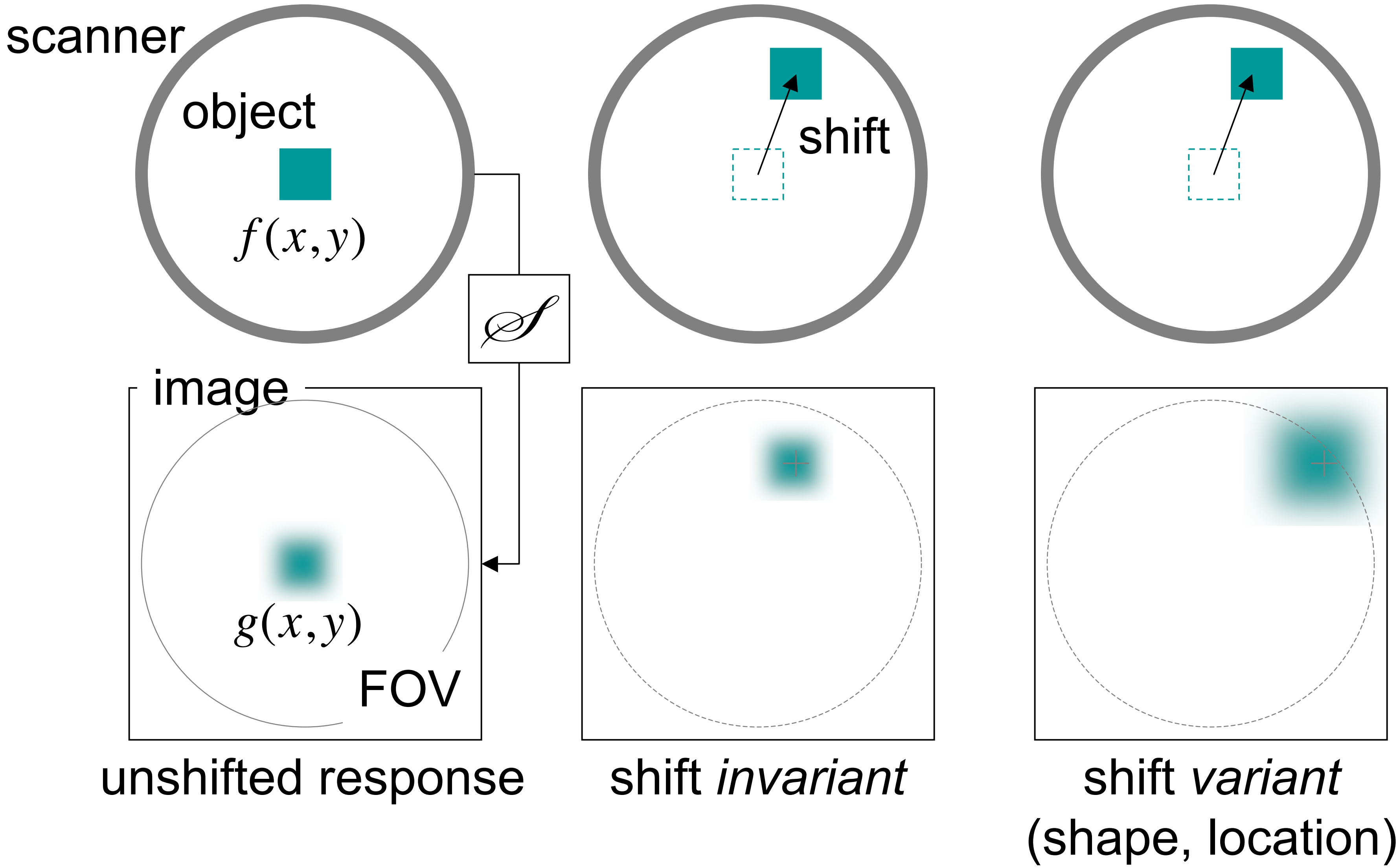
Shift-Invariant Systems

- Start by shifting the input $f_{x_0 y_0}(x,y) \triangleq f(x - x_0, y - y_0)$

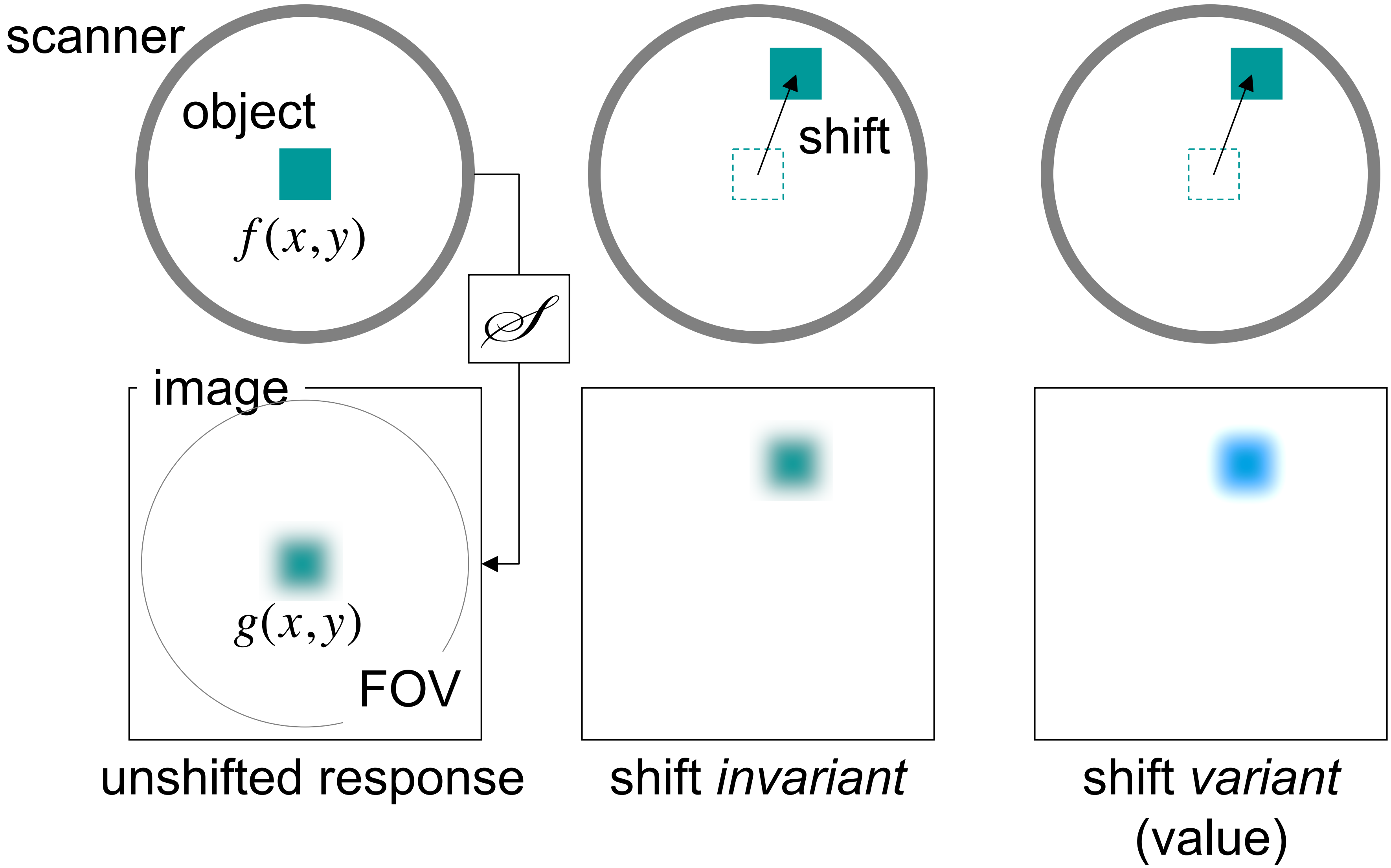
  then if
  $$g_{x_0 y_0}(x,y) = \mathscr{S}\left[f_{x_0 y_0}(x,y) \right] = g(x - x_0, y - y_0)$$
  the system is *shift-invariant*, i.e. response does not depend on location

- Shift-invariance is separate from linearity, a system can be
  - shift-invariant and linear
  - shift-invariant and non-linear
  - shift-variant and linear
  - shift-variant and non-linear
  - (what else have we forgotten?)

Shift invariant and shift-variant system response

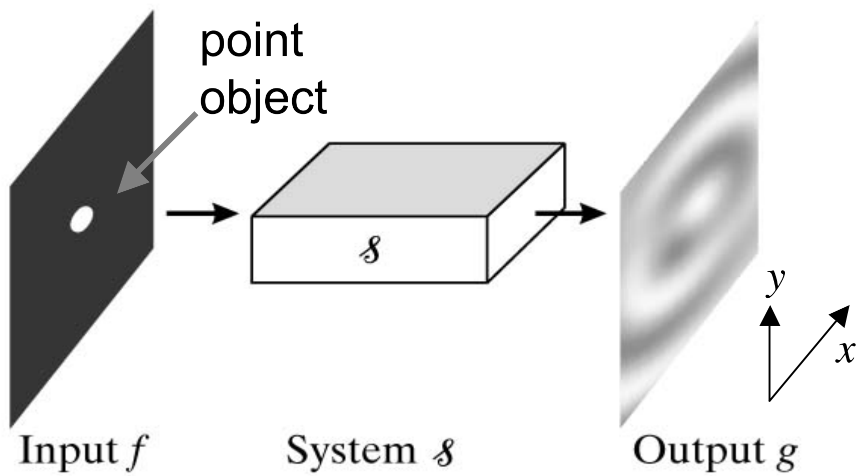


Shift invariant and shift-variant system response



Impulse Response

- Linear, shift-invariant (LSI) systems are the most useful
- First we start by looking at the response of a system using a point source at location (ξ,η) as an input



input $f_{\xi\eta}(x,y) \triangleq \delta(x-\xi, y-\eta)$

output $g_{\xi\eta}(x,y) \triangleq h(x,y;\xi,\eta)$

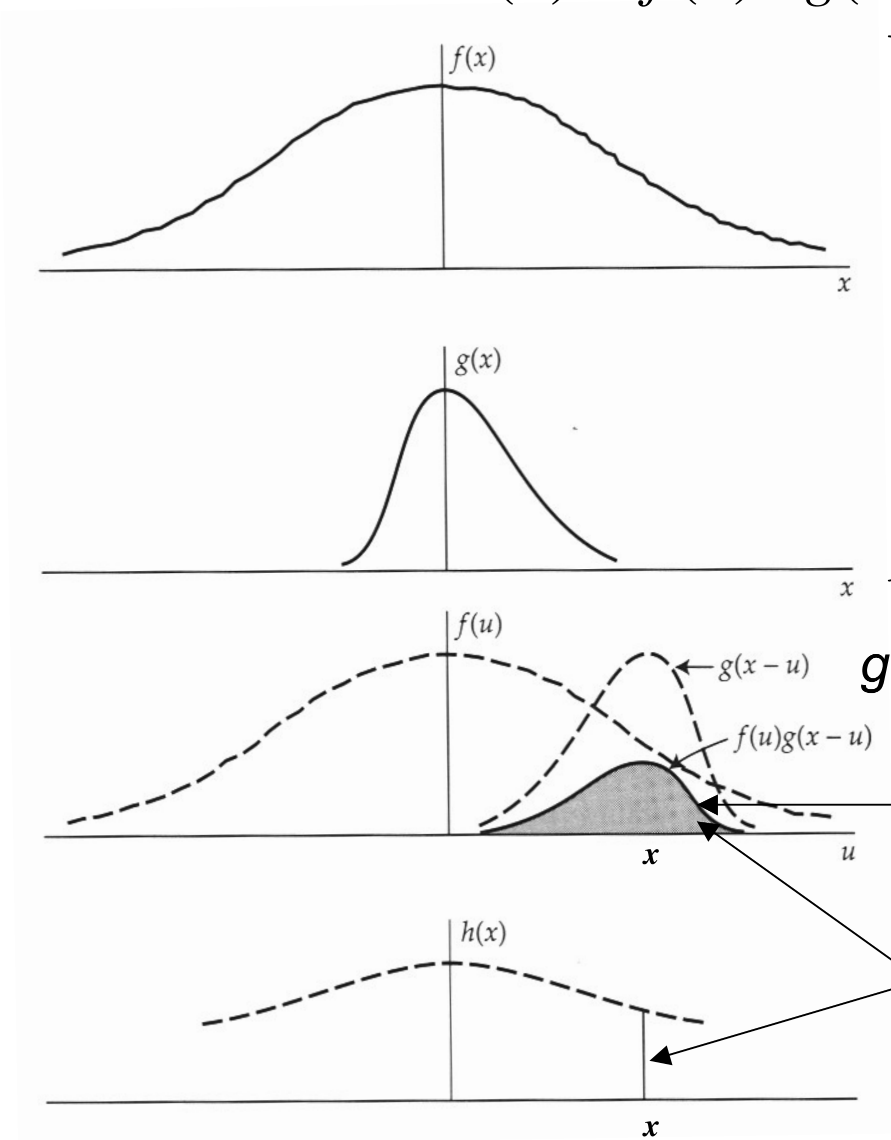
- The output *h()* depends on location of the point source (ξ,η) and location in the image (*x,y*), so it is a 4-D function
- Since the input is an impulse, the output is called the *impulse response function,* or the *point spread function* (PSF) - *why?*

Impulse Response of Linear Shift Invariant Systems

- For LSI systems
$$\mathscr{S}\left[f(x-x_0,y-y_0)\right]=g(x-x_0,y-y_0)$$
- So the PSF is
$$\mathscr{S}\left[\delta(x-x_0,y-y_0)\right]=h(x-x_0,y-y_0)$$
- Through something called the superposition integral, we can show that
$$g(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(\xi,\eta)\,h(x,y;\xi,\eta)\,d\xi\,d\eta$$
- And for LSI systems, this simplifies to:
$$g(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(\xi,\eta)\,h(\xi-x,\eta-y)\,d\xi\,d\eta$$
- The last integral is a convolution integral, and can be written as
$$g(x,y)=f(x,y)*h(x,y)\quad(\text{or}\,f(x,y)**h(x,y))$$

Review of convolution

- Illustration of $h(x) = f(x) * g(x) = \int_{-\infty}^{\infty} f(u) g(x-u) du$



original functions

g(x-u), reversed and shifted to *x*

curve = product of *f(u)g(x-u)*

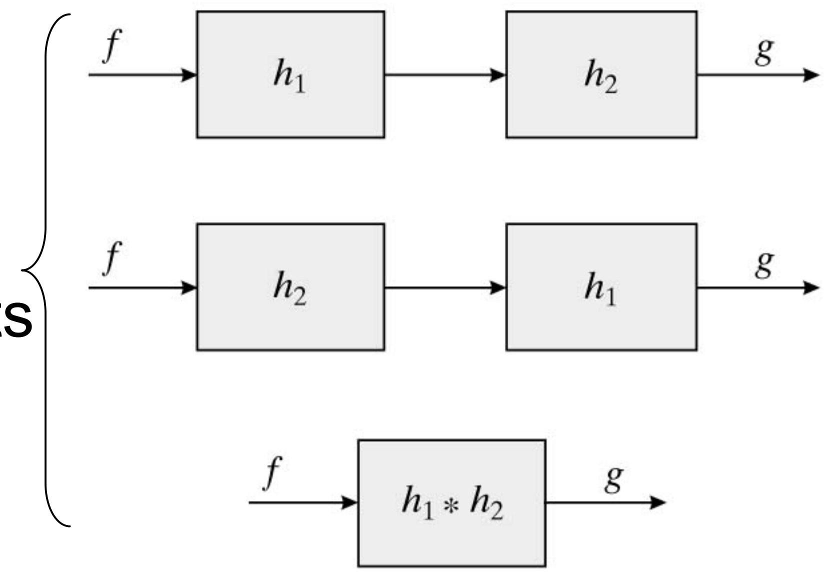
area = integral of *f(u)g(x-u)*

= value of *h()* at *x*

Properties of LSI Systems

- The convolution integral has the basic properties of
  1. Linearity (definition of a LSI system)
  2. Shift invariance (ditto)
  3. Associativity $$g(x,y) = h_2(x,y) * \left[h_1(x,y) * f(x,y) \right] = \left[h_2(x,y) * h_1(x,y) \right] * f(x,y)$$
  4. Commutativity $$h_1(x,y) * h_2(x,y) = h_2(x,y) * h_1(x,y)$$

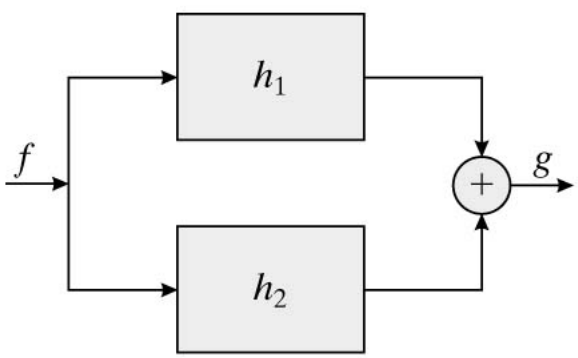
Equivalent arrangements

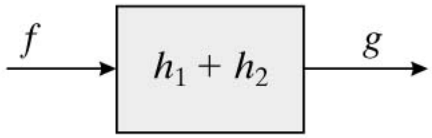


Combined LSI Systems

- Parallel systems have property of
  5. Distributivity

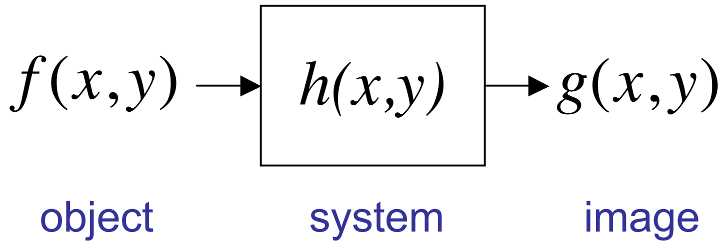
$$g(x,y) = h_1(x,y) * f(x,y) + h_2(x,y) * f(x,y)$$
$$= \left[h_1(x,y) + h_2(x,y)\right] * f(x,y)$$





Summary of advantages of Linear Shift Invariant Systems

- For LSI systems we have



$$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)\,h(\xi - x,\eta - y)\,d\xi\,d\eta$$
$$= f(x,y) ** h(x,y)$$

- Treating imaging systems as LSI significantly simplifies analysis
- In many cases of practical value, non-LSI systems can be approximated as LSI
- Allows use of Fourier transform methods that accelerate computation

2D Fourier Transforms

Fourier Transforms

- Recall from the sifting property (with a change of variables)

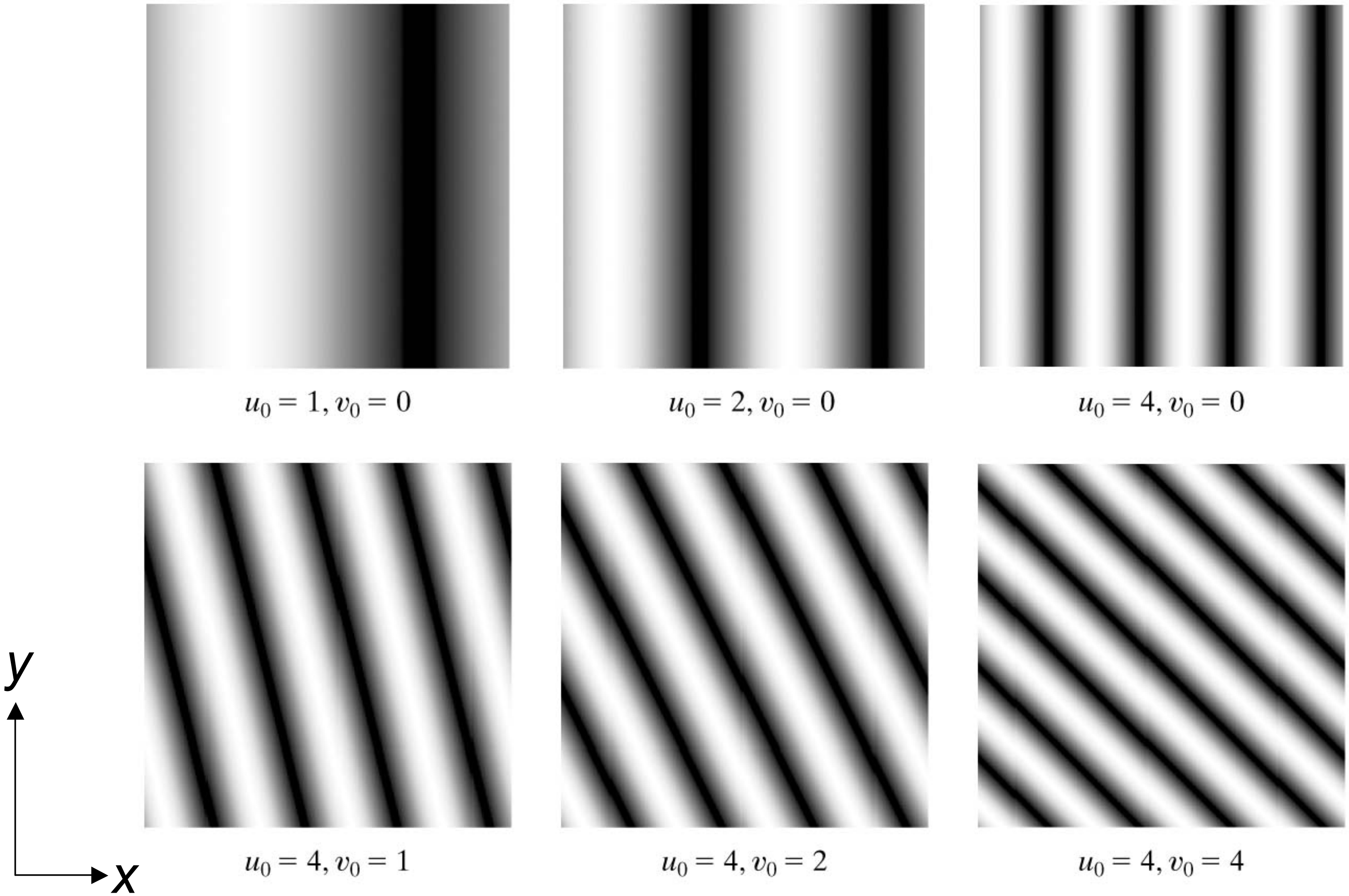
$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)\delta(\xi - x,\eta - y)\,d\xi\,d\eta$$

- Expresses *f(x,y)* as a weighted combination of shifted basis functions, *δ(x,y),* also called the superposition principle
- An alternative and convenient set of basis functions are sinusoids, which bring in the concept of frequency
- Using the complex exponential function allows for compact notation, with *u* and *v* as the frequency variables

$$e^{j2\pi(ux+vy)} = \cos\left[2\pi\left(ux+vy\right)\right] + j\sin\left[2\pi\left(ux+vy\right)\right]$$

Exponential and sinusoidal signals as basis functions

- Intensity images for $s(x,y) = \sin\left[2\pi\left(u_0 x + v_0 y\right)\right]$



Fourier Transforms

- Using this approach we write

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)\, e^{j2\pi(ux+vy)}\, du\, dv$$

- F(u,v) are the weights for each frequency, exp{ *j2π(ux+vy)*} are the basis functions
- It can be shown that using exp{ *j2π(ux+vy)*} we can readily calculate the needed weights by

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{-j2\pi(ux+vy)}\, dx\, dy$$

- This is the 2D Fourier Transform of *f(x,y),* and the first equation is the inverse 2D Fourier Transform

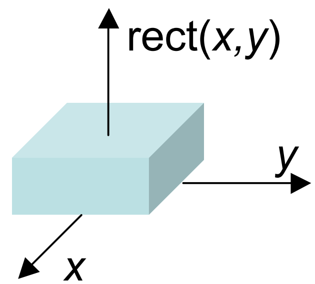
Fourier Transforms

- For even more compact notation we use

$$F(u,v) = \mathscr{F}_{\mathrm{2D}}\left\{f(x,y)\right\}, \ \text{and}\ f(x,y) = \mathscr{F}_{\mathrm{2D}}^{-1}\left\{F(u,v)\right\}$$

- Notes on the Fourier transform
  - *F(u,v)* can be calculated if *f(x,y)* is continuous, or has a finite number of discontinuities, and is absolutely integrable
  - (u,v) are the spatial frequencies
  - *F(u,v)* is in general complex-valued, and is called the <u>spectrum</u> of *f(x,y)*
- As we will see, the Fourier transform allows consideration of an LSI system for each separate sinusoidal frequency

Fourier Transform Example



- What is the Fourier transform of

$$\operatorname{rect}(x,y)=\begin{cases}1, & \text{for } |x|<1/2 \text{ and } |y|<1/2\\ 0, & \text{otherwise}\end{cases}$$

- First note that it is separable $\operatorname{rect}(x,y)=\operatorname{rect}(x)\operatorname{rect}(y)$
- So we compute

$$\mathscr{F}_{1\mathrm{D}}\{\operatorname{rect}(x)\}=\int_{-\infty}^{\infty}\operatorname{rect}(x)e^{-j2\pi ux}\,dx$$

$$=\int_{-1/2}^{1/2}e^{-j2\pi ux}\,dx=\frac{1}{j2\pi u}e^{-j2\pi ux}\Big|_{-1/2}^{1/2}$$

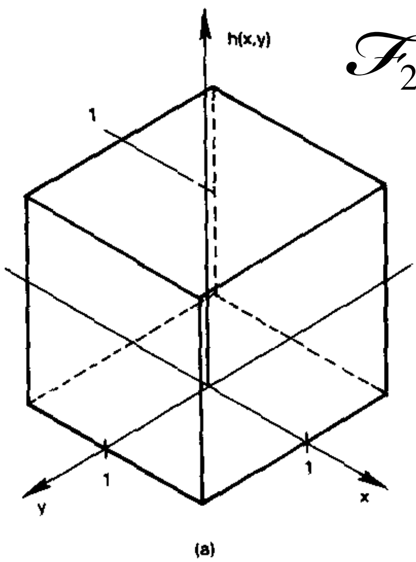
$$=\frac{1}{\pi u}\frac{e^{j\pi u}-e^{-j\pi u}}{j2}=\frac{\sin(\pi u)}{\pi u}$$

$$=\operatorname{sinc}(u)$$

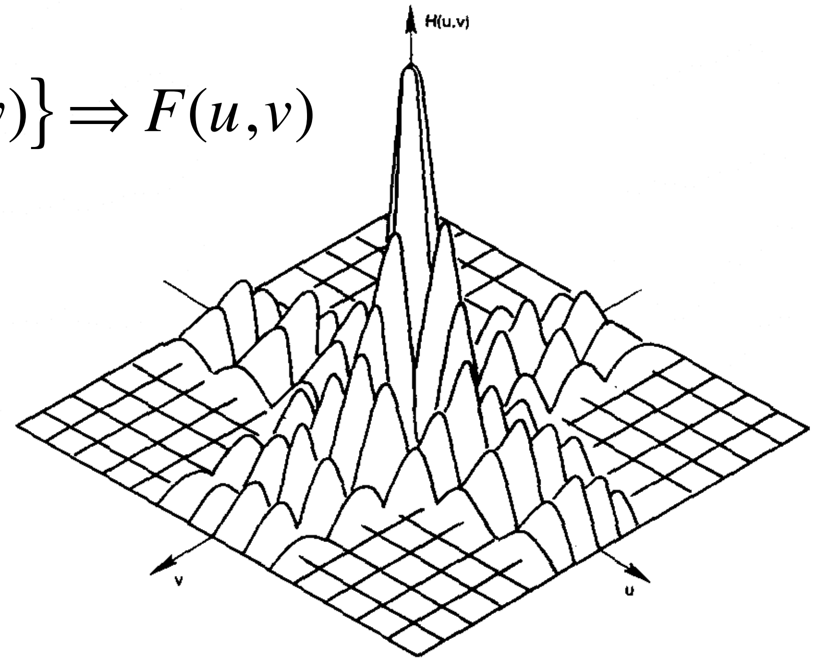
Thus $$\mathscr{F}_{2\mathrm{D}}\{\operatorname{rect}(x,y)\}=\operatorname{sinc}(u,v)$$

Fourier Transform Example

$$\mathscr{F}_{\mathrm{2D}}\left\{f(x,y)\right\} \Rightarrow F(u,v)$$



$\mathrm{rect}(x,y)$

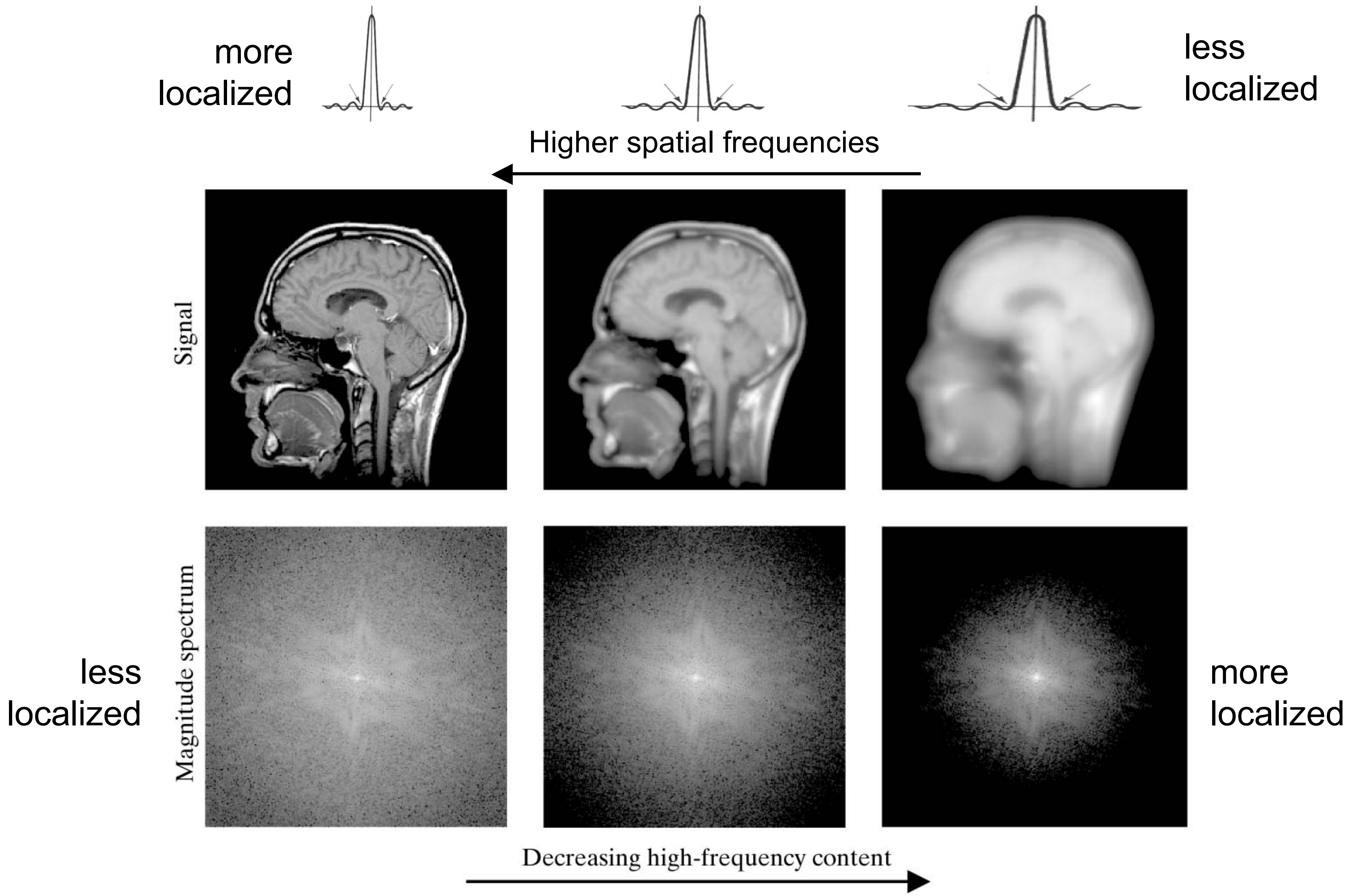


$\mathrm{sinc}(u,v)$

Two Key Properties of the 2D Fourier Transform

- Linearity $$\mathcal{F}_{2D}\left\{a_1 f(x,y)+a_2 g(x,y)\right\}=a_1 F(u,v)+a_2 G(u,v)$$
- Scaling $$\mathcal{F}_{2D}\left\{f(ax,by)\right\}=\frac{1}{|ab|}F\left(\frac{u}{a},\frac{v}{b}\right)$$

Signal localization in image versus frequency space



Fourier Transforms and Convolution

- Very useful! $\mathcal{F}_{\text{2D}}\left\{f(x,y)*g(x,y)\right\} = F(u,v)G(u,v)$
- Proof (1-D)

$$\mathcal{F}\left\{f(x)*g(x)\right\} = \int_{-\infty}^{\infty}\left(f(x)*g(x)\right)e^{-j2\pi ux}\,dx$$

$$= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(\xi)g(x-\xi)d\xi\right)e^{-j2\pi ux}\,dx = \int_{-\infty}^{\infty} f(\xi)\left(\int_{-\infty}^{\infty} g(x-\xi)\,e^{-j2\pi ux}\,dx\right)d\xi$$

$$= \int_{-\infty}^{\infty} f(\xi)\left(\int_{-\infty}^{\infty} \mathcal{F}\left\{g(x-\xi)\right\}\right)d\xi = \int_{-\infty}^{\infty} f(\xi)\left(e^{-j2\pi u\xi}\,G(u)\right)d\xi$$

$$= G(u)\int_{-\infty}^{\infty} f(\xi)e^{-j2\pi u\xi}\,d\xi = F(u)G(u)$$

Fourier transform pairs

Signal	Fourier Transform
1	$\delta(u, v)$
$\delta(x, y)$	1
$\delta(x - x_0, y - y_0)$	$e^{-j2\pi(ux_0 + vy_0)}$
$\delta_s(x, y; \Delta x, \Delta y)$	$\text{comb}(u\Delta x, v\Delta y)$
$e^{j2\pi(u_0 x + v_0 y)}$	$\delta(u - u_0, v - v_0)$
$\sin[2\pi(u_0 x + v_0 y)]$	$\frac{1}{2j}[\delta(u - u_0, v - v_0) - \delta(u + u_0, v + v_0)]$
$\cos[2\pi(u_0 x + v_0 y)]$	$\frac{1}{2}[\delta(u - u_0, v - v_0) + \delta(u + u_0, v + v_0)]$
$\text{rect}(x, y)$	$\text{sinc}(u, v)$
$\text{sinc}(x, y)$	$\text{rect}(u, v)$
$\text{comb}(x, y)$	$\text{comb}(u, v)$
$e^{-\pi(x^2 + y^2)}$	$e^{-\pi(u^2 + v^2)}$

- Note the reciprocal symmetry in Fourier transform pairs
  - often 2-D versions can be calculated from 1-D versions by seperability
  - In general: a broad extent in one domain corresponds to a narrow extent in the other domain

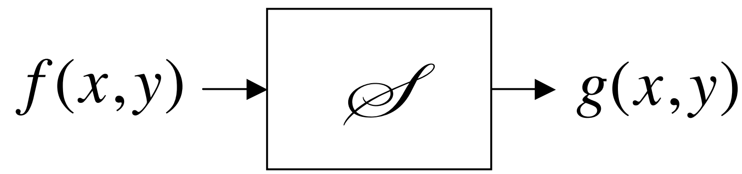
Summary of key properties of the Fourier Transform

Theorem	$f(x,y)$	$F(u,v)$
Similarity	$f(ax,by)$	$\frac{1}{\lvert ab \rvert} F\left(\frac{u}{a},\frac{v}{b}\right)$
Addition	$f(x,y) + g(x,y)$	$F(u,v) + G(u,v)$
Shift	$f(x - a, y - b)$	$e^{-2\pi i(au+bv)}F(u,v)$
Modulation	$f(x,y) \cos \omega x$	$\frac{1}{2} F\left(u + \frac{\omega}{2\pi}, v\right) + \frac{1}{2} F\left(u - \frac{\omega}{2\pi}, v\right)$
Convolution	$f(x,y) * g(x,y)$	$F(u,v)G(u,v)$
Autocorrelation	$f(x,y) * f^*(-x,-y)$	$\lvert F(u,v) \rvert^2$
Rayleigh	$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \lvert f(x,y) \rvert^2 \, dx \, dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \lvert F(u,v) \rvert^2 \, du \, dv$	
Power	$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)g^*(x,y) \, dx \, dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u,v)G^*(u,v) \, du \, dv$	
Parseval	$\int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \lvert f(x,y) \rvert^2 = \sum \sum {a_{mn}}^2,$ where $F(u,v) = \sum \sum a_{mn} [{}^2\delta(u - m, v - n)]$	
Differentiation	$\left(\frac{\partial}{\partial x}\right)^m \left(\frac{\partial}{\partial y}\right)^n f(x,y)$	$(2\pi i u)^m (2\pi i v)^n F(u,v)$

Transfer Functions

Transfer Function for an LSI System

- Recall that for an LSI system



$$g(x,y) = f(x,y) * h(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)\, h(\xi - x, \eta - y)\, d\xi\, d\eta$$

- We can define the <u>Transfer Function</u> as the 2D Fourier transform of the PSF

$$H(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\xi,\eta) e^{j2\pi(u\xi+v\eta)}\, d\xi\, d\eta = \mathscr{F}_{2D}\left\{h(x,y)\right\}$$

- In this case the LSI imaging system can be simply described by:

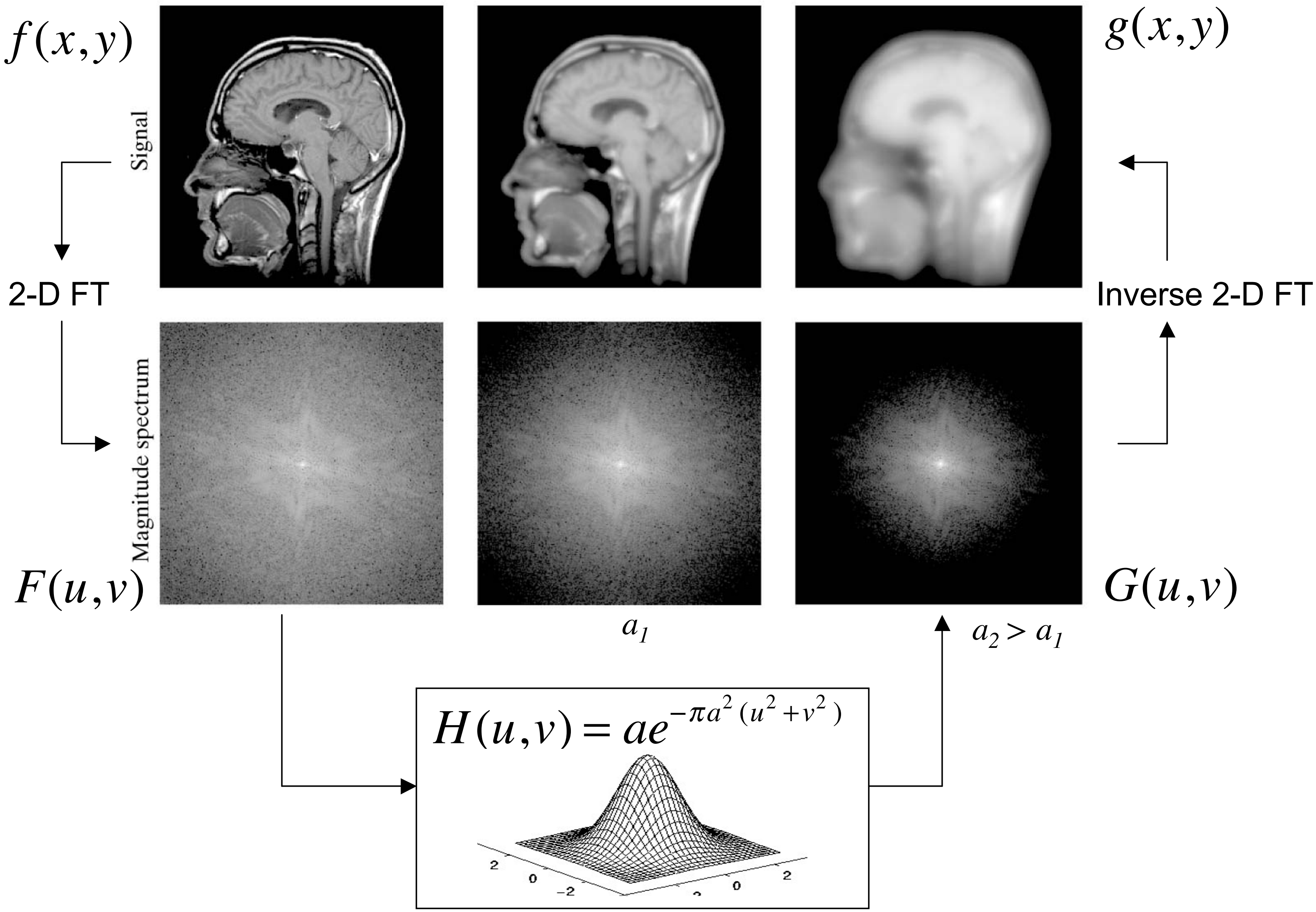
$$g(x,y) = f(x,y) * h(x,y) = \mathscr{F}_{2D}^{-1}\left\{F(u,v)H(u,v)\right\}$$

- or $G(u,v) = F(u,v)H(u,v)$

- which provides a very powerful tool for understanding systems

Illustration of transfer function

$$f(x,y) \rightarrow \boxed{h(x,y)} \rightarrow g(x,y)$$



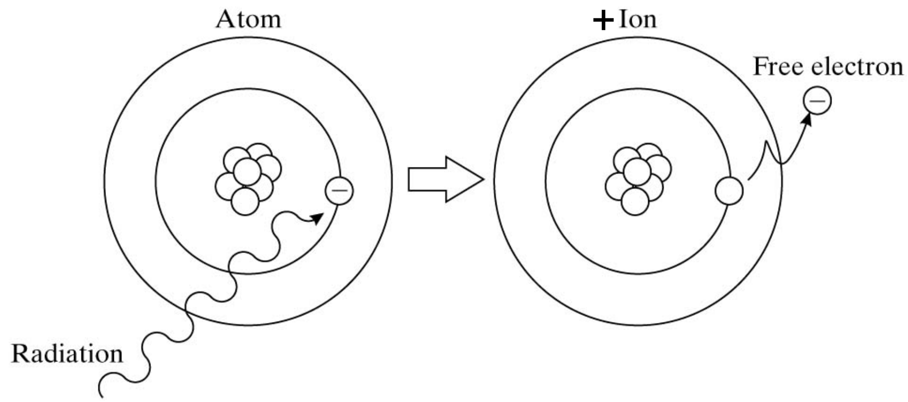
X-ray Radiography

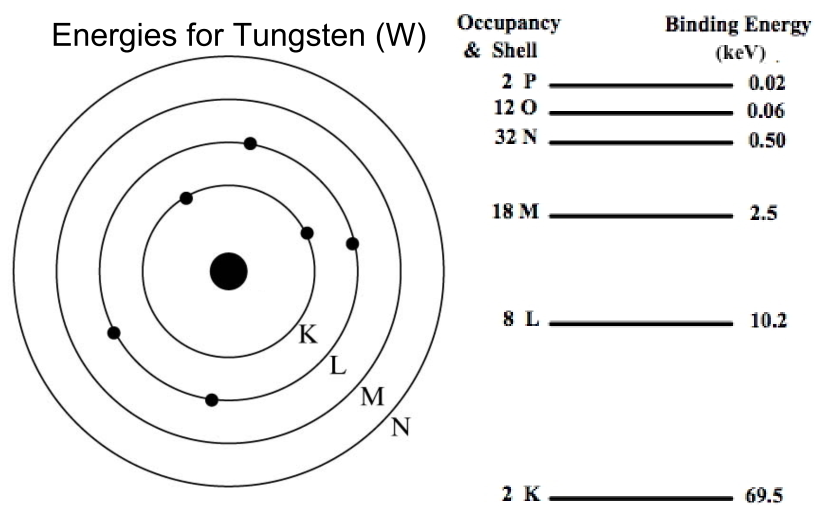
Definitions

- Ion: an atom or molecule in which the total number of electrons is not equal to the total number of protons, giving it a net positive or negative electrical charge
- Radiation: a process in which energetic particles or energetic waves travel through a medium or space

Ionizing Radiation

- Radiation (such as high energy electromagnetic photons behaving like particles) that is capable of ejecting orbital elections from atoms
- Can also be particles (e.g. electrons)
- Ionizing energy required is the binding energy for that electron's shell
- Energy units are electron volts (eV or keV), the energy of an electron accelerated by 1 volt
- For Hydrogen K orbital electrons, E=13.6 eV
- For Tungsten K orbital electrons, E=69.5 keV
- In medical imaging we need photons with enough energy to transmit through tissue so are in range of 25 keV to 511 keV and is thus ionizing



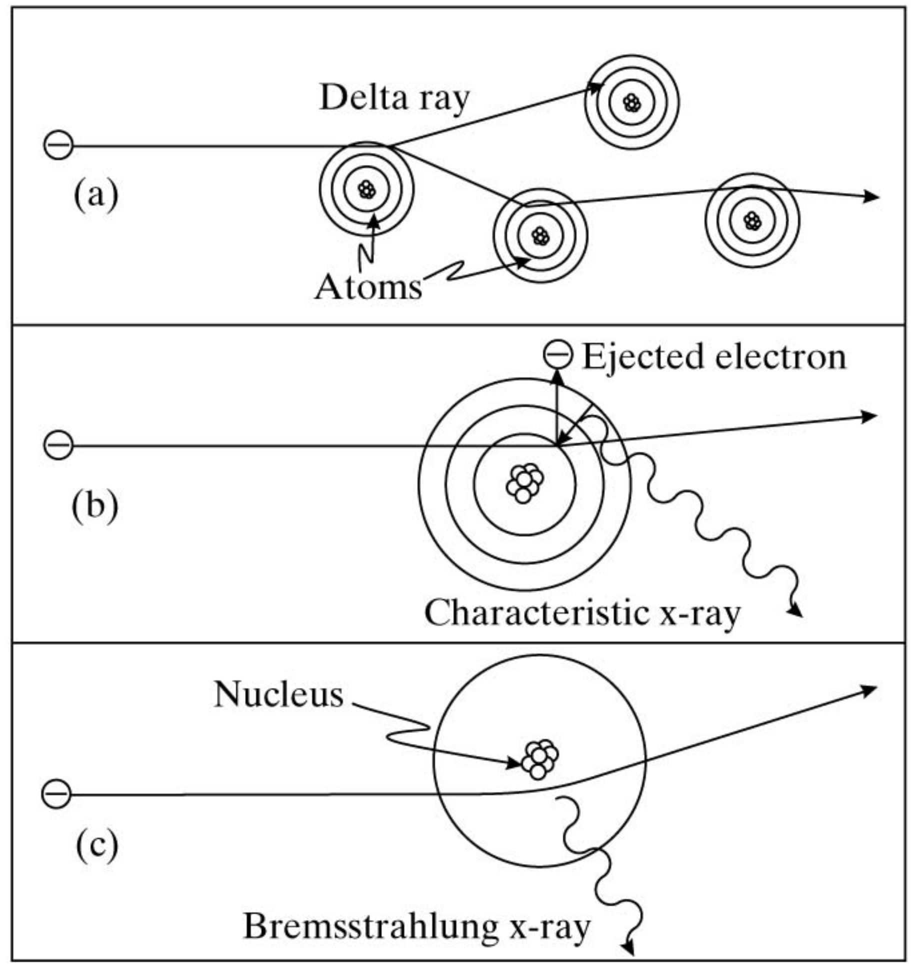


Electrons as Ionizing Radiation

- Electron kinetic energy $E = (mv^2)/2$
- Three main modes of interaction in the energy range we are considering
  a) Collision with other electrons and possible creation of delta-rays (high-energy electrons)
    - This is the most common mode and excited atoms loose energy by IR radiation (heat)

  b) Ejection of an inner orbital electron
    - This orbit is filled by an outer electron and the difference in energy is released as a 'characteristic x-ray'

  c) Bending of trajectory by nucleus
    - Since acceleration of a charged particle causes radiation, this causes 'braking radiation' or *bremsstrahlung*



X-ray Spectrum from Electron Bombardment

When high energy electrons hit tungsten (symbol W), three effects occur

1. Heat (> 99.9% of the energy)
2. Characteristic x-rays
3. Bremsstrahlung x-rays