PHYS 536

R. J. Wilkes

Session 3

Fourier methods

Fourier transforms

TONIGHT ONLY: CLASS STARTS AT 7:30 PM

1/10/2023

Course syllabus and schedule – first part…

See : http://courses.washington.edu/phys536/syllabus.htm

Session	date	Day	Readings:	K=Kinsler, H=Heller	Topic
1	3-Jan	Tue	K ch. 1	H: Ch. 1, 2	Course intro, acoustics topics, overview of wave properties; pulses, transverse and longitudinal waves, overview of sound speeds
2	5-Jan	Thu	K ch. 1	H: Ch. 9, 10	harmonic oscillators: simple, damped, driven; complex exponential solutions, electrical circuit analogy, resonance, Q factor
3	10-Jan	Tue	K ch. 1	H: Ch. 3	Fourier methods: Fourier series, integrals, Fourier transforms, discrete FTs, sampling and aliasing ← Tonight
4	12-Jan	Thu	K. chs 10	H: Ch. 4, 11	Frequencies and aliasing; convolution and correlation; discrete convolution; digital filtering, optimal filters, FIR filters, noise spectra; power spectra. REPORT 1 PROPOSED TOPIC DUE
5	17-Jan	Tue	K. ch. 2, 3, 4	H: Ch. 13, 15	waves in strings, bars and membranes; Acoustic wave equation, speed of sound; Harmonic plane waves, intensity, impedance.
6	19-Jan	Thu	K. Ch. 5, 6	H: Ch. 1	Spherical waves; transmission and reflection at interfaces
7	24-Jan	Tue	K. Ch. 8	H: Ch. 7	Radiation from small sources; Baffled simple source, piston radiation, pulsating sphere;
8	26-Jan	Thu	K: Ch. 10	H: Chs. 13-15	Near field, far field; Radiation impedance; resonators, filters
9	31-Jan	Tue	K. Ch. 9-10	H: Chs. 16-19	Musical instruments: wind, string, percussion
10	2-Feb	Thu	K. Ch 14		Transducers for use in air: Microphones and loudspeakers
11	7-Feb	Tue	K. Ch 11	H: Chs. 21-22	The ear, hearing and detection
12	9-Feb	Thu	K. Chs 5,11		Decibels, sound level, dB examples, acoustic intensity; noise, detection thresholds. REPORT 1 PAPER DUE by 7 PM; REPORT 2 PROPOSED TOPIC DUE

Announcements

We now have a TA to help you with problems and papers:

- Yiyun Dong <yiyund@uw.edu>

Her main job is to grade papers, but Ms Dong can help you if you get stuck on the homework problems

Contact her by email if you want to make an appointment for phone or zoom meetings

Announcements

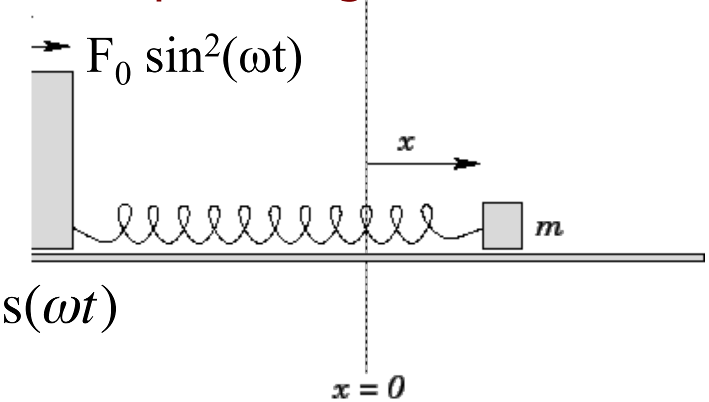
- REMINDER: term paper #1 proposals are due Thursday!
  - Remember: only 5 pages – NARROW your scope!
  - Please send me a brief email with
    - Topic chosen
    - Resources to be used in your study (books, journal articles, etc)
    - Format chosen: term paper or website
      - You can submit a 5p paper, or build a website with the same amount of content
      - For info on how to create a website @uw, see

      https://sites.uw.edu/your-first-site/

Driven mechanical oscillator example

- Driven (undamped) oscillator has mass m, spring constant s, and is driven by $F(t) = F_0 \sin^2(\omega t)$. At t=0 the mass is at x = 0.
  - What is x(t) given the above initial conditions?
  - In terms of m and ω, what value of k produces resonance?
    - First, let's solve a less complicated problem:

    let the driver be just $F(t) = F_0 \cos(\omega t)$ – this simplifies algebra



$$m\frac{d^2x}{dt^2} + sx = F_0\cos(\omega t) \rightarrow \frac{d^2x}{dt^2} + \left(\frac{s}{m}\right)x = \frac{F_0}{m}\cos(\omega t)$$

$$\frac{d^2x}{dt^2} + \omega_0^2 x = \frac{F_0}{m}\cos(\omega t);\quad \text{assume solution } x(t) = A\cos(\omega t)$$

$$-\omega^2 A\cos(\omega t) + \omega_0^2 A\cos(\omega t) = \frac{F_0}{m}\cos(\omega t)$$

$$\text{So must have } A = \frac{F_0}{m\left(\omega_0^2 - \omega^2\right)} \rightarrow \boxed{x(t) = \frac{F_0}{m\left(\omega_0^2 - \omega^2\right)}\cos(\omega t)}$$

This can't be whole solution - there are no free parameters for initial conditions!

Driven mechanical oscillator example

Full solution is $x(t) = x_{DRIVER}(t) + x_0(t) = \frac{F_0}{m\left(\omega_0^2 - \omega^2\right)}\cos(\omega t) + B\cos(\omega_0 t) + C\sin(\omega_0 t)$

Need to include x(t) of free oscillator

Initial conditions $x(0) = 0, \quad \frac{dx(0)}{dt} = 0 \rightarrow C = 0,$

$$x(t) = \frac{F_0}{m\left(\omega_0^2 - \omega^2\right)}\left(\cos(\omega t) - \cos(\omega_0 t)\right)$$

That solves the case of the simple driver $F_0\cos(\omega t)$

For the driver $F(t) = F_0 \sin^2(\omega t)$, use identity $2\sin^2(x) = 1 - \cos(2x)$

$$\rightarrow F_0 \sin^2(\omega t) = \frac{F_0}{2m}\left(1 - \cos(2\omega t)\right) = \frac{d^2x}{dt^2} + \omega_0^2 x$$

Now it gets a bit messy – patience...

$$Let \quad X = x - \frac{F_0}{2m\omega_0^2} \rightarrow \frac{d^2X}{dt^2} + \omega_0^2 X = -\cos(2\omega t)\frac{F_0}{2m}$$

Start with the same approach again: insert $X(t) = A\cos(2\omega t)$

$$\frac{d^2X}{dt^2} + \omega_0^2 X = -4\omega^2 A\cos(\omega t) + \omega_0^2 A\cos(\omega t) = -\cos(2\omega t)\frac{F_0}{2m} \rightarrow A = -\frac{F_0}{2m\left(\omega_0^2 - 4\omega^2\right)}$$

Driven mechanical oscillator example

Go back to $x = X + \frac{F_0}{2m\omega_0^2}$: Full solution is $x(t) = x_{DRIVER}(t) + x_0(t)$

$$x(t) = -\frac{F_0}{2m\left(\omega_0^2 - 4\omega^2\right)}\cos(2\omega t) + B\cos(\omega_0 t) + C\sin(\omega_0 t) + \frac{F_0}{2m\omega_0^2}$$

Apply initial conditions: $x(0) = 0, \quad \frac{dx(0)}{dt} = 0 \rightarrow C = 0$, and $B = \frac{F_0}{2m}\left(\frac{1}{\left(\omega_0^2 - 4\omega^2\right)} - \frac{1}{\omega_0^2}\right)$

$$\text{So } x(t) = \frac{-F_0\cos(2\omega t)}{2m\left(\omega_0^2 - 4\omega^2\right)} + \frac{F_0\cos(\omega_0 t)}{2m}\left(\frac{1}{\left(\omega_0^2 - 4\omega^2\right)} - \frac{1}{\omega_0^2}\right)$$

$$= \frac{-F_0}{2m\left(\omega_0^2 - 4\omega^2\right)}\left(\cos(\omega_0 t) - \cos(2\omega t)\right) + \frac{F_0}{2m\omega_0^2}\left(1 - \cos(\omega_0 t)\right)$$

$$\left(1 - \cos(\omega_0 t)\right) = 2\sin^2(\omega_0 t) \rightarrow x(t) = \frac{-F_0\cos(2\omega t)}{2m\left(\omega_0^2 - 4\omega^2\right)}\left(\cos(\omega_0 t) - \cos(2\omega t)\right) + \frac{F_0\sin^2(\omega_0 t)}{m\omega_0^2}$$

In terms of m and driver ω, what should spring constant be to get maximum amplitude?

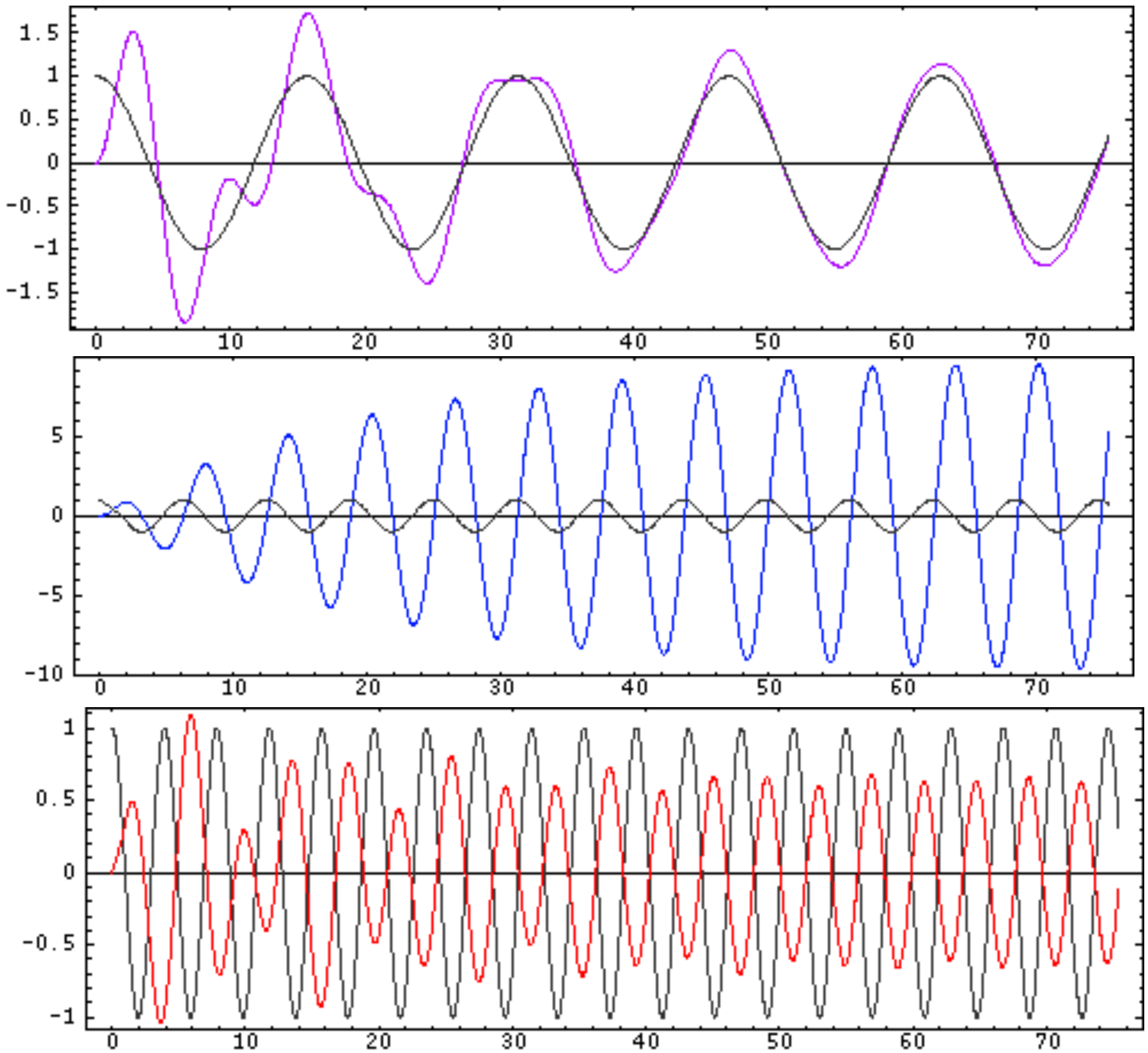
amplitude of oscillation is maximized when $\left(\omega_0^2 - 4\omega^2\right) = 0$

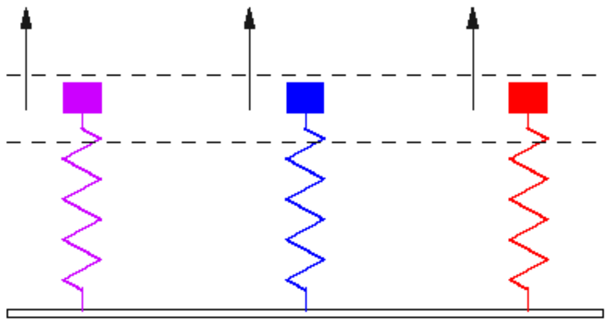
$\omega_0^2 = 4\omega^2, \quad s = m\omega_0{,}^2$ so $s_{MAX-A} = 4m\omega^2 \quad (\text{then } \omega_0 = 2\omega)$

From last time

Resonance

– Time plots for oscillators (matching colors) f1=0.4, f2=1.01, f3=1.6





Fourier methods

- "Fourier" = generic term for any numerical method using harmonic functions (sin/cos/exp(ix)) as an *orthonormal basis set**
  - Much of the following can be applied to other basis sets (e.g., wavelets)

1. Fourier interpolation (trigonometric interpolation)

Given N *equally spaced* points $\{x_k\}_N$ on $[0,2\pi]$ (N *even*)

$$x_k = 2\pi \frac{k}{N}; \quad \text{interpolating function is}$$

$$f(x) = a_0 + \sum_{j=1}^{N/2} \left(a_j \cos(jx) + b_j \sin(jx) \right)$$

$$j = 1 \text{ has 1 cycle in } x = [0, 2\pi]$$

$$j = N/2 \text{ has } (N/2) \text{ cycles in } x = [0, 2\pi]$$

** orthonormal basis:* Set of vectors that
- Are normalized ($|V_j| = 1$)
- Are orthogonal ($V_j * V_k = 0$ if i≠j)
- Form a basis to represent desired fns

Why stop at N/2 instead of N?
We'll see later

- Theorem: For truncated interpolation ($j_{MAX}<N/2$) this gives the *best fit* in the sense of least squares of any trig-fn interpolant with the same number of terms - optimal

Fourier methods

- Since exp(ix) = cos(x) + ***i*** sin(x), we can write

engineers pls note: this is a physics class so $\sqrt{-1} = i, \ \text{not}\ j$

(Here j is just a running index)

$$f(x) = \sum_{j=-N/2}^{+N/2} c_j \exp(i\,jx)$$

$$c_0 = a_0, \quad c_{+j} = \frac{a_j - ib_j}{2}, \quad c_{-j} = \frac{a_j + ib_j}{2}$$

"Negative frequencies" are just a book-keeping artifact to get neat notation, as we'll see later

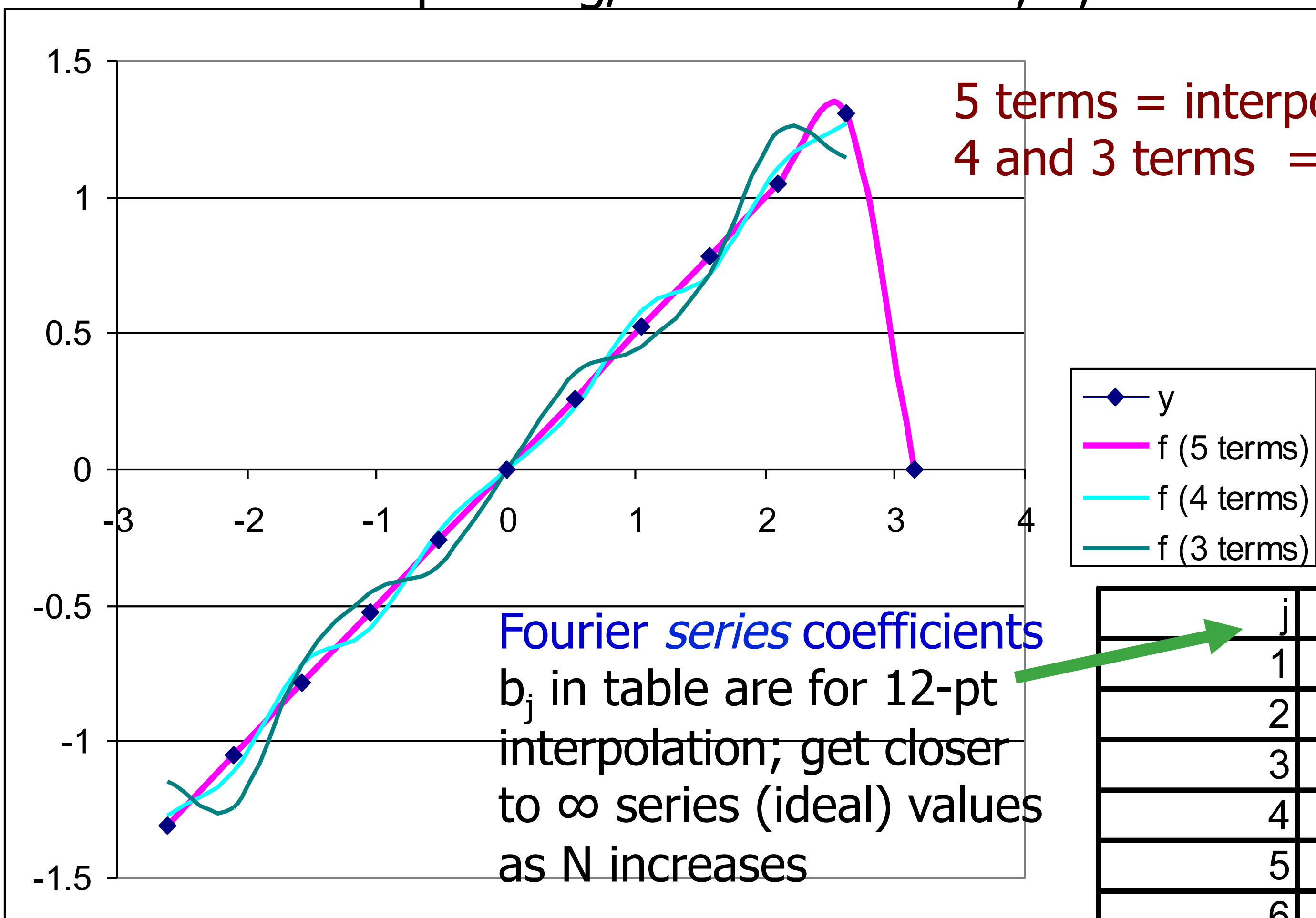
  - Get coefficients of the interpolant from

$$a_j = \frac{1}{2N} \sum_{k=0}^{N} f(x_k)\cos(jx), \quad b_j = \frac{1}{2N} \sum_{k=0}^{N} f(x_k)\sin(jx)$$

or $$c_j = \frac{1}{N} \sum_{k=0}^{N} f(x_k)\exp(-i\,jx)$$

Fourier interpolation: Example

- Data: straight line, y=x/2, at 12 points in x from $-\pi$ to $+\pi$
  - So N=12, number of coefficients = 6, a_j = 0 (y=0 at x=0 → odd fn)
- Set y = 0 at x = $\pm\pi$, to make it "periodic"
- Results: interpolating/fit function for 3, 4, 5 terms



5 terms = interpolation,
4 and 3 terms = least squares fits

Fourier *series* coefficients
b_j in table are for 12-pt interpolation; get closer to ∞ series (ideal) values as N increases

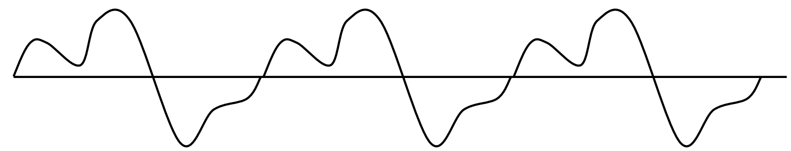
j	b_j	b(infinite)
1	0.977049	1
2	-0.45345	-0.5
3	0.261799	0.333333
4	-0.15115	-0.25
5	0.070149	0.2
6	0	0

Fourier analysis

- ***Fourier series*** for a *periodic* (but *not necessarily harmonic*) function:

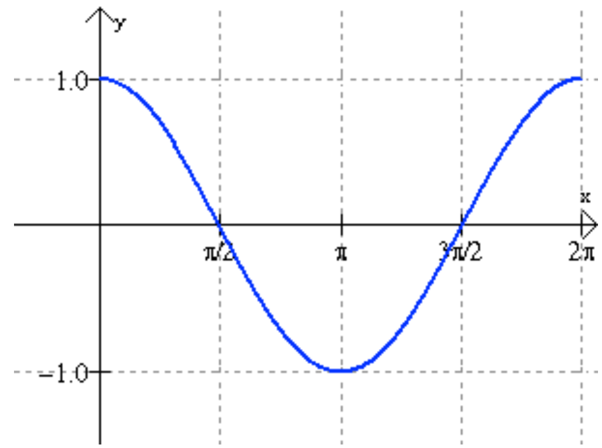
("Harmonic" = sin/cos only)

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(nkx) + \sum_{n=1}^{\infty} B_n \sin(nkx)$$



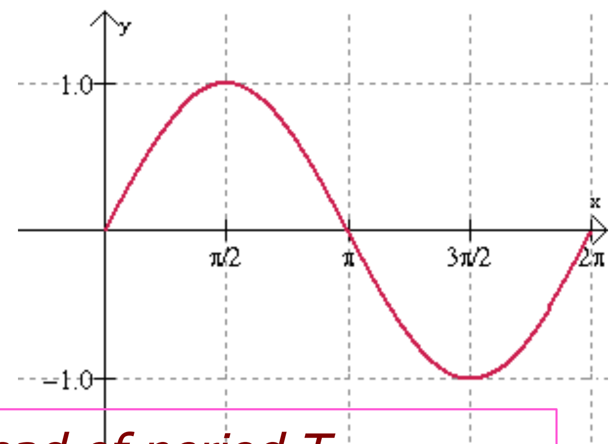
where $k = \dfrac{2\pi}{\lambda}$ (frequency)

- cos = even function: f(-x)=f(x)
  - Symmetrical about x=0, π
- sin = odd function: f(-x)=-f(x)
  - Antisymmetrical about x=0, π
- Find A, B coefficients from:



$$A_n = \frac{2}{\lambda}\int_0^{\lambda} f(x)\cos(nkx)dx \qquad \lambda = \frac{2\pi}{k}$$

$$B_n = \frac{2}{\lambda}\int_0^{\lambda} f(x)\sin(nkx)dx$$



Using f(x) rather than f(t) → uses wavelength λ instead of period T
But f(t) is exactly the same, just replace k with ω = 2π/T

Fourier series – complex exponential representation

Periodic functions: f(x+L)=f(x)   period = L

– *orthogonality:*

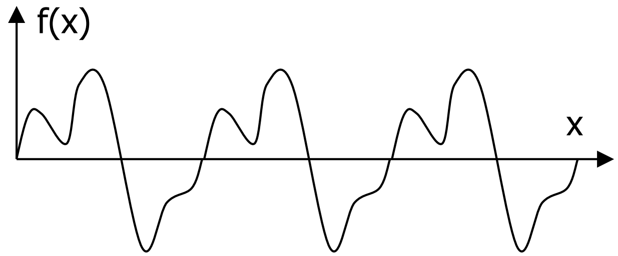
$$\text{for } \phi_j = \exp(ijx), \quad \int_{-\pi}^{+\pi} \phi_j \phi_k^* \, dx = 0 \text{ if } j \neq k \; ; \quad = 2\pi \text{ if } j = k$$

(* = *complex conjugation*)

We can always scale period L $\rightarrow 2\pi$:

$$x = [a,b] \rightarrow x' = \frac{2\pi(x-a)}{b-a} = [0,\ 2\pi]$$

Then $f(x) = \frac{a_0}{2} + \sum_{j=1}^{\infty} a_j \cos(jx) + b_j \sin(jx) = \sum_{j=-\infty}^{+\infty} c_j \exp(ijx)$



Fourier series - sum = infinite series for exact representation

[Recall: truncated sum ($j_{\max} = m$) gives best (least squares) approximation for trig polynomial of m terms]

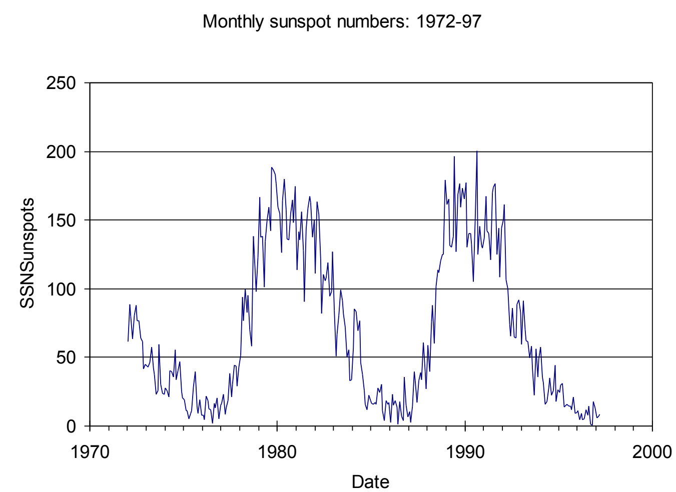
Find coefficients from

$$a_j = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(jx)\,dx; \quad b_j = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(jx)\,dx$$

$$\text{or} \quad c_j = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\exp(ijx)\,dx$$

If $f(-x) = f(x)$ (even function) then $b_j = 0$ (cos series only)

If $f(-x) = -f(x)$ (odd function) then $a_j = 0$ (sin series only)



Fourier synthesis examples

- We can synthesize a periodic waveform using Fourier series sums

Can generate any periodic waveform by choice of amplitudes and phase offsets in sum of harmonics: (phase shift term ϕ makes sin → cos as needed)

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \sin(n\omega t + \phi_n), \quad \text{where} \quad \omega = 2\pi f_0 = \frac{2\pi}{T_0}, \quad \phi_n = \text{phase offset of nth harmonic}$$

$$\rightarrow with \quad A_0 = \frac{a_0}{2}, \quad A_n = a_n \sin(\phi_n), \quad B_n = b_n \cos(\phi_n)$$

$$f(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(n\omega t) + \sum_{n=1}^{\infty} B_n \sin(n\omega t),$$

Examples:

Amplitude	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}	A_{11}	A_{12}	A_{13}	A_{14}	A_{15}	A_{16}
Sine	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Triangle	1	0	1/9	0	1/25	0	1/49	0	1/81	0	1/121	0	1/169	0	1/225	0
Square	1	0	1/3	0	1/5	0	1/7	0	1/9	0	1/11	0	1/13	0	1/15	0
Ramp	1	1/2	1/3	1/4	1/5	1/6	1/7	1/8	1/9	1/10	1/11	1/12	1/13	1/14	1/15	1/16
Pulse Train	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

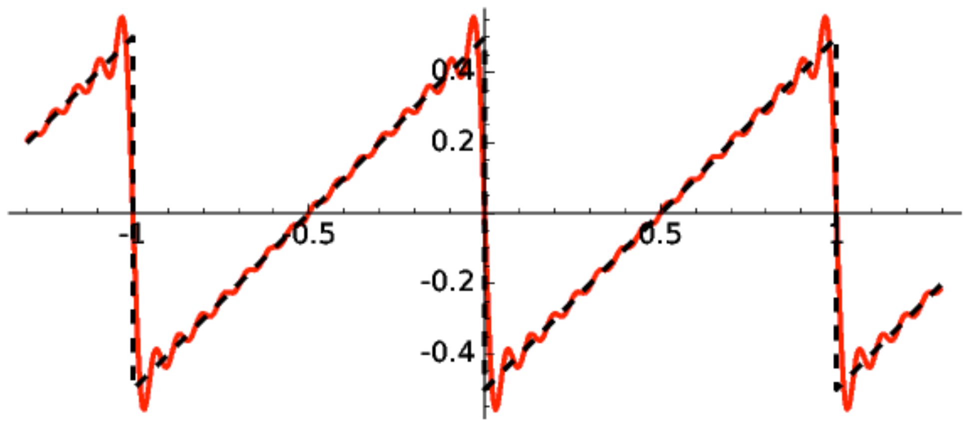
Phase	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	ϕ_7	ϕ_8	ϕ_9	ϕ_{10}	ϕ_{11}	ϕ_{12}	ϕ_{13}	ϕ_{14}	ϕ_{15}	ϕ_{16}
Sine	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Triangle	0	0	180	0	0	0	180	0	0	0	180	0	0	0	180	0
Square	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Ramp	180	180	180	180	180	180	180	180	180	180	180	180	180	180	180	180
Pulse Train	90	90	90	90	90	90	90	90	90	90	90	90	90	90	90	90

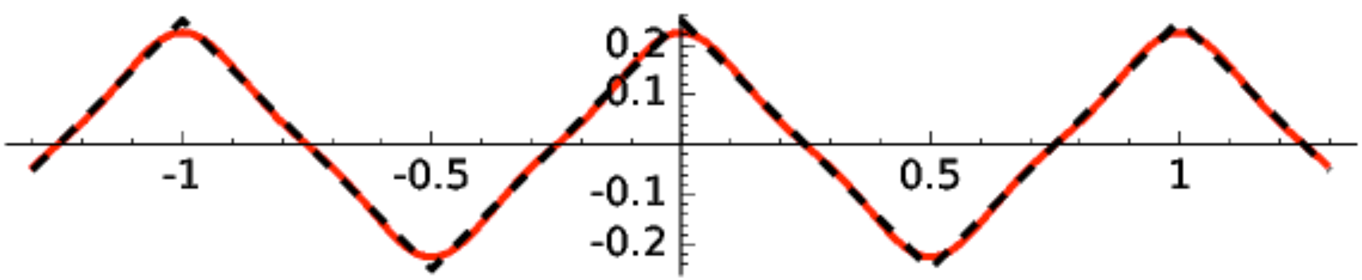
Fourier synthesizer demonstration

- http://www.mjtruiz.com/ped/fourier/

From Michael Ruiz, U. NC/Asheville

Sawtooth function using 16 terms





Triangle function on ly needs 3 terms for decent approximation

Fourier analysis

- **Example: square wave**

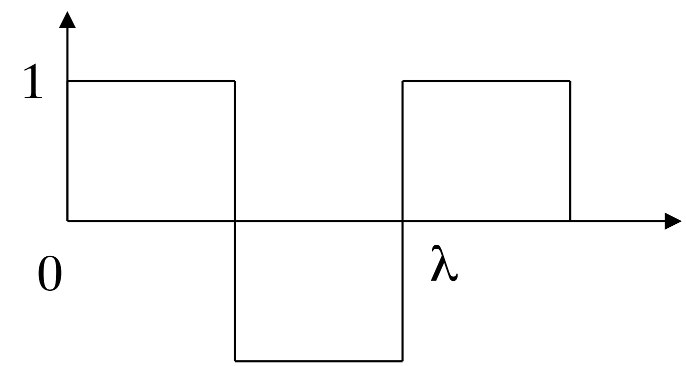
(with leading edge at x=0)

- *odd* function, so all $A_n = 0$

$$B_n = \frac{2}{\lambda}\int_0^{\lambda/2}(+1)\sin(nkx)\,dx + \int_{\lambda/2}^{\lambda}(-1)\sin(nkx)\,dx$$

$$\int \sin ax = \frac{-\cos ax}{a}$$

$$B_n = \frac{2}{n\pi}(1-\cos(n\pi))$$



- so $B_n=0$ for n=2,4,6..., $B_n=(4/n\pi)$ for n=1,3,5...

$$f(x) = \frac{4}{\pi}\left(\sin(kx) + \frac{1}{3}\sin(3kx) + \frac{1}{5}\sin(5kx) + \cdots\right)$$

- **So *any* periodic fn can be represented as a sum of sin/cos fns**
  - a_0 (or c_0) = "DC component" (vertical offset)

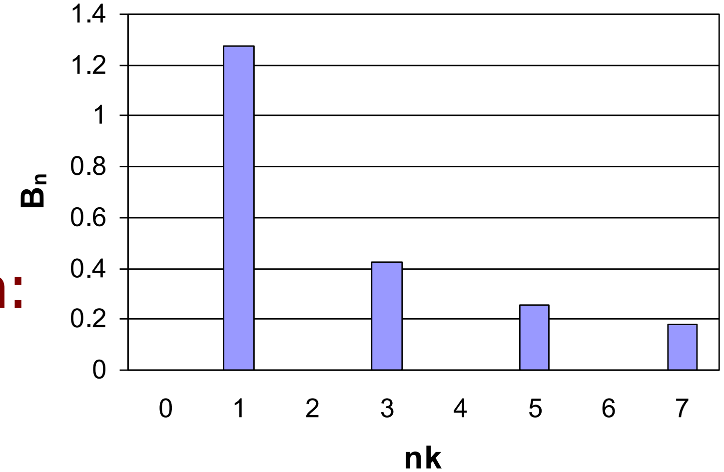
Notice *harmonics* are equally spaced in frequency

- lowest frequency (longest wavelength) corresponds to 1 cycle within period L
- *j* th term has *j* cycles in L

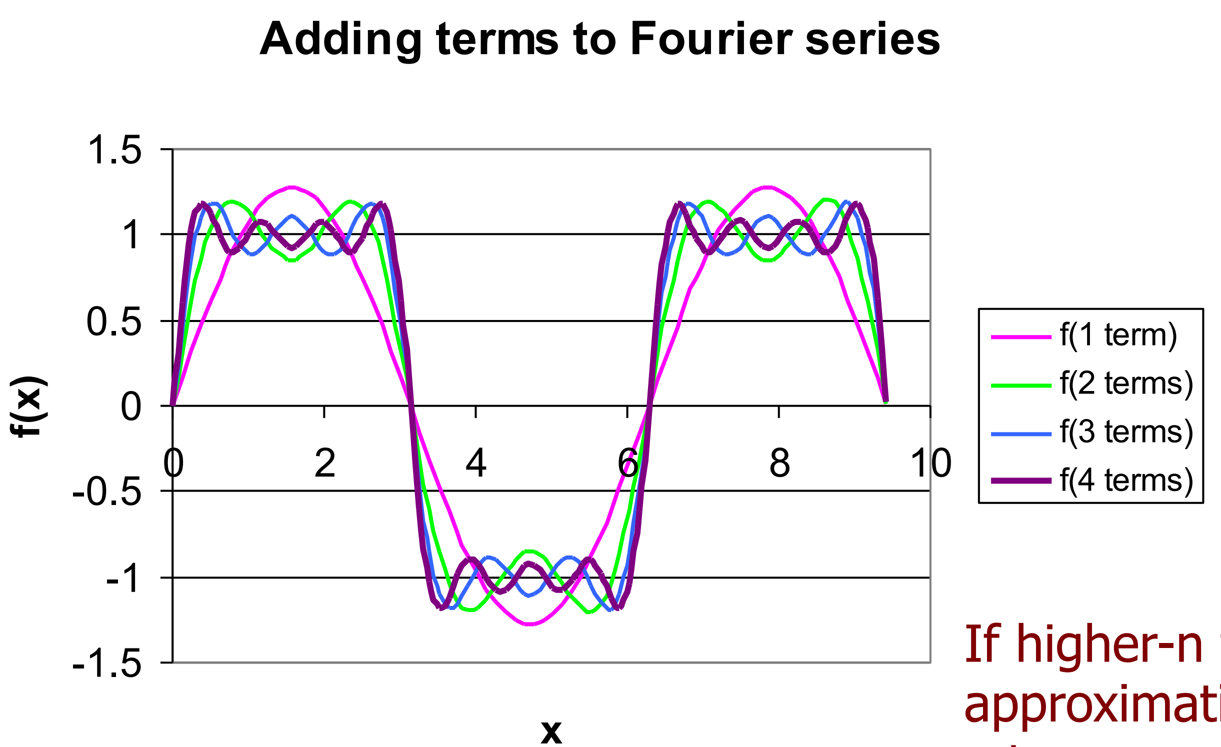
Fourier series

- Coefficients in the series give a *discrete spectrum* for square wave

$$f(x) = \frac{4}{\pi}\left(\sin(kx) + \frac{1}{3}\sin(3kx) + \frac{1}{5}\sin(5kx) + \cdots\right)$$



Only *odd* terms, amplitude drops as 1/n:

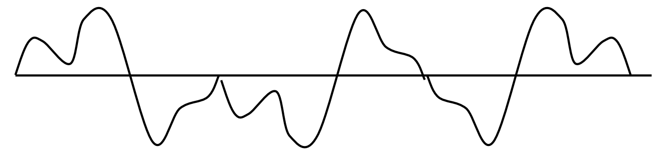


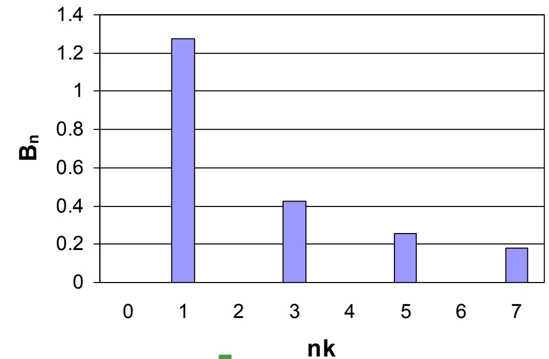
Adding more terms (higher frequencies *nk*) gives better approximation to f(x): faster rise, flatter tops

If higher-n terms are missing, approximation is poor: tops have ripples, edges are curved
(limited *bandwidth* : high frequencies lost)

Fourier analysis for non-periodic function

- Let λ→∞: then *anything* can be "periodic"
  - λ→∞ implies k=2π/λ → dk
    - frequencies are "infinitesimally" spaced
  - Fourier *series* → Fourier *integral*





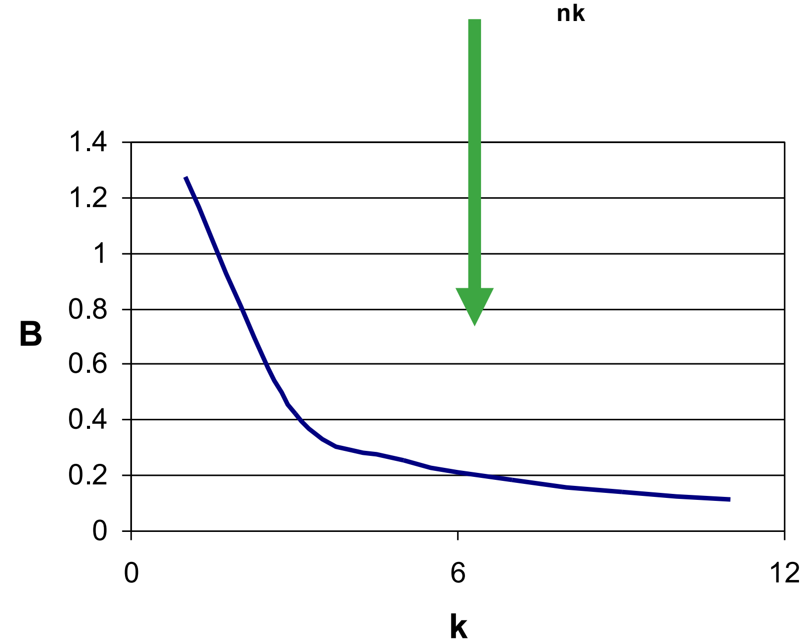
$$f(x) = \frac{1}{\pi}\int_0^\infty A(k)\cos(kx)\,dx + \frac{1}{\pi}\int_0^\infty B(k)\sin(kx)\,dx$$

where $A(k) = \int_{-\infty}^{\infty} f(x')\cos(kx')\,dx'$

(*cosine transform* of f)

and $B(k) = \int_{-\infty}^{\infty} f(x')\sin(kx')\,dx'$

(*sine transform* of f)



A(k), B(k) become *continuous* spectra

Fourier transforms

- Combine the sine and cosine integrals into the complex exponential form:

$$F(k) = \int_{x'=-\infty}^{\infty} f(x')e^{ikx'}\,dx'$$

F(k) = Fourier Transform of *f(x)*

where $F(k) = A(k) + iB(k)$

- Inverse transform: $f(x) = F^{-1}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)e^{-ikx}\,dk$
  - Note: some books interchange +/- in exponentials, or have normalization ($1/\sqrt{2\pi}$) on both transform and inverse – read carefully

- Signals vs *x* or *t*, and spectra vs *k* or *f* are dual spaces
  - Fourier transform connects them: linear functions of each other

Fourier transforms

Handy properties of FTs:

- If $g(t) = f_1(t) + f_2(t) + f_3(t) + \dots$ then $G(k) = F_1(k) + F_2(k) + F_3(k) + \dots$

  and FT($a\, f(t)$) = $a\, F(k)$

  - Parity: $$g(t)\ even \leftrightarrow G(f)\ even \text{ (ditto for odd)}$$
  - Time scaling: $$g(at) \leftrightarrow \frac{1}{|a|} G\left(\frac{f}{|a|}\right)$$
  - Time translation $$g(t - t_0) \leftrightarrow G(f) \exp\left(-i\, 2\pi f\, t_0\right) \qquad (2\pi f = \omega)$$
  - Frequency shift $$g(t) \exp\left(i\, 2\pi f_0\, t\right) \leftrightarrow G(f - f_0)$$

Famous Fourier transforms:

– Single square pulse centered on 0

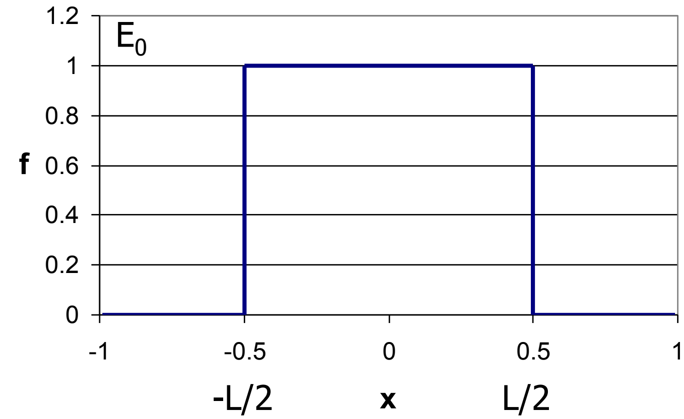
- f(x)=even, so B(k)=0

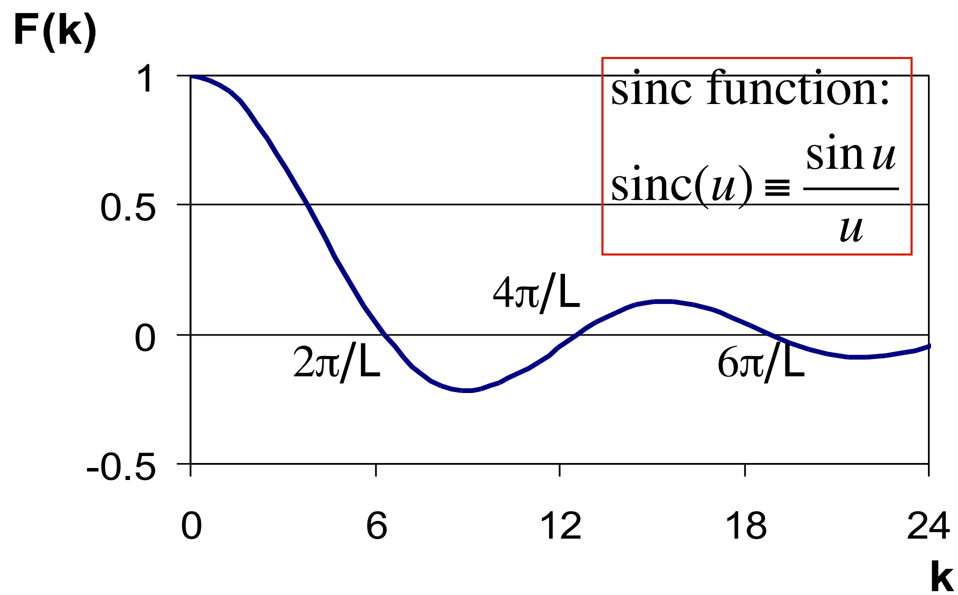
$$A(k) = \int_{-L/2}^{L/2} E_0 \cos(kx)\,dx$$

$$= \frac{2E_0}{k}\sin\left(\frac{kL}{2}\right)$$

$$= E_0 L \frac{\sin u}{u} \quad \text{where} \quad u = \frac{kL}{2}$$

$$= E_0 L \,\mathrm{sinc}\left(\frac{kL}{2}\right)$$

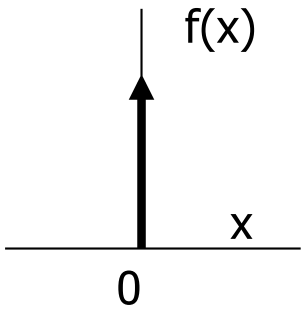




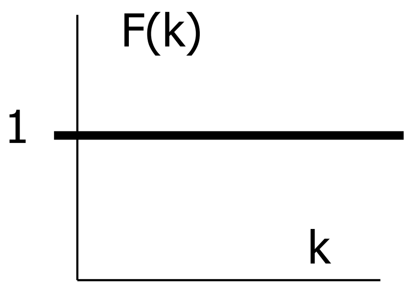
- Note that
  - As u→0, sinc u→1
  - sinc(u) is an *even* function
  - f(x) is very *localized*, but F(k) has *infinite extent*

Famous Fourier transforms

- Let width of a square pulse →0 while keeping area=const.
  - e.g., let L→0 while E_0=1/L
  - So f(x)→∞ for x=0, f(x)=0 everywhere else, and f(x)→δ(x) **Dirac *delta function***
    (=Heaviside *unit impulse fn*)



$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \qquad F(k) = \lim_{L\to 0}\left[E_0 L \,\mathrm{sinc}(kL/2)\right] = 1$$



f(x) is totally local, F(k) is totally un-localized!

– Properties of delta function:

$$\delta(x) = \delta(-x)$$

$$\delta(ax) = \frac{1}{|a|}\delta(x) \qquad (a \neq 0)$$

$$f(x)\delta(x-a) = f(a)\delta(x-a)$$

Famous Fourier transforms

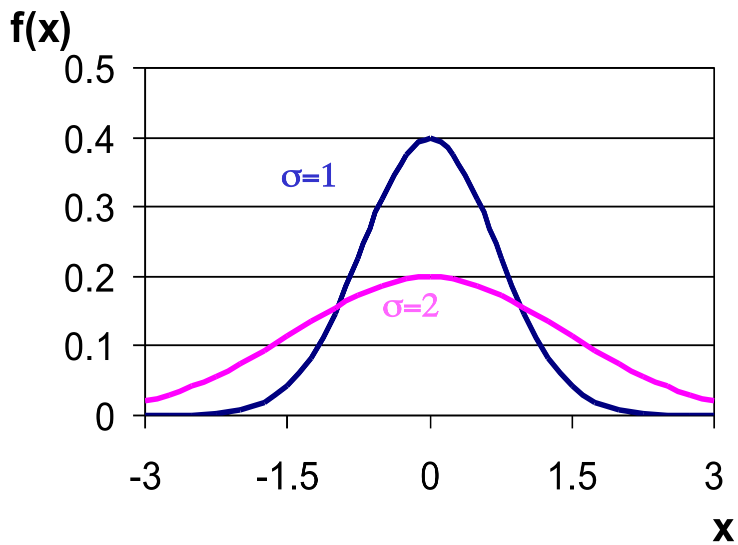
- Gaussian ("normal distribution")

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2}$$

This form has area = 1

Full width (at f = $\frac{1}{e}$) = $\sqrt{2\sigma^2}$

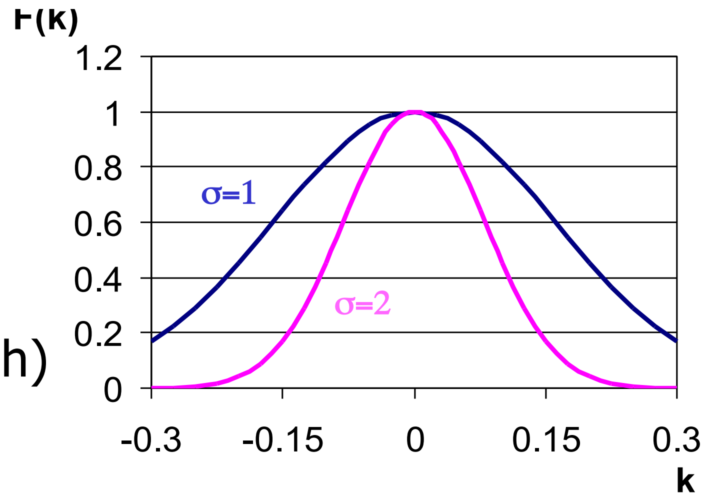
Height (at $x = 0$) = $\frac{1}{\sqrt{2\pi\sigma^2}}$



- FT: $F(k) = e^{-\pi^2(2\sigma^2)k^2}$ (another Gaussian)

Full width = $\frac{1}{\pi\sqrt{2\sigma^2}}$ (~ inverse of f(x) width)

Height (at $k = 0$) = 1 (independent of σ)



- So: narrower f(x)=broader F(k) and vice versa
- Both f(x) and F(k) are *semi-localized:* degree of localization depends on σ

Fourier integrals: Review of terminology

- Time/frequency domains (signal processing)
  - period T (sec)
  - Frequency f (cycles/sec=Hz)
  - "angular frequency" $\omega=2\pi f$ (radians/s)
- Space/Spatial frequency domains (wave motion, optics, image analysis)
  - Spatial frequency f (cycles/meter)
  - Period L (meters)
  - Wavelength λ (meters)
  - Wave number $k=2\pi/\lambda$ (meters^{-1})

- Fourier series → integral :: discrete f's → continuous spectrum

$$f(t) = \sum_{j=-\infty}^{+\infty} c_j \exp(i\,jt) \rightarrow f(t) = \int_{-\infty}^{+\infty} F(\omega)\exp(i\omega t)\,d\omega$$

$F(\omega)$ gives the relative weight of each frequency

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(t)\exp(-i\omega t)\,dt$$

Note: There are other conventions, for example in *Numerical Recipes*

$$f(t) = \int_{-\infty}^{+\infty} F(\omega)\exp(-i\omega t)\,d\omega;$$

$$F(f) = \int_{-\infty}^{+\infty} f(t)\exp(-i\omega t)\,dt$$

$F(\omega)$ is the *Fourier Transform* of $f(t)$ (and vice-versa)

frequency domain ("spectrum") — time domain (signal)

Sampling signals

- Common case: signal is *sampled* N times, at *equally-spaced* intervals:
  - Data set = $\{f(t_k\}_N$ N samples of f(t)
  - Sampling interval $\Delta = t_{k+1} - t_k$ for all k ➔ $t_k = t_0 + k\Delta$
  - We need *2 coefficients* for each freq (a_j and b_j, or $c_{\pm j}$)
  - So we can Fourier-interpolate N *intervals* with N/2 harmonics:

(Here, k=index, not freq.)

$t_0 \rightarrow t_{N-1}$ corresponds to phase $0 \rightarrow 2\pi$ for lowest f

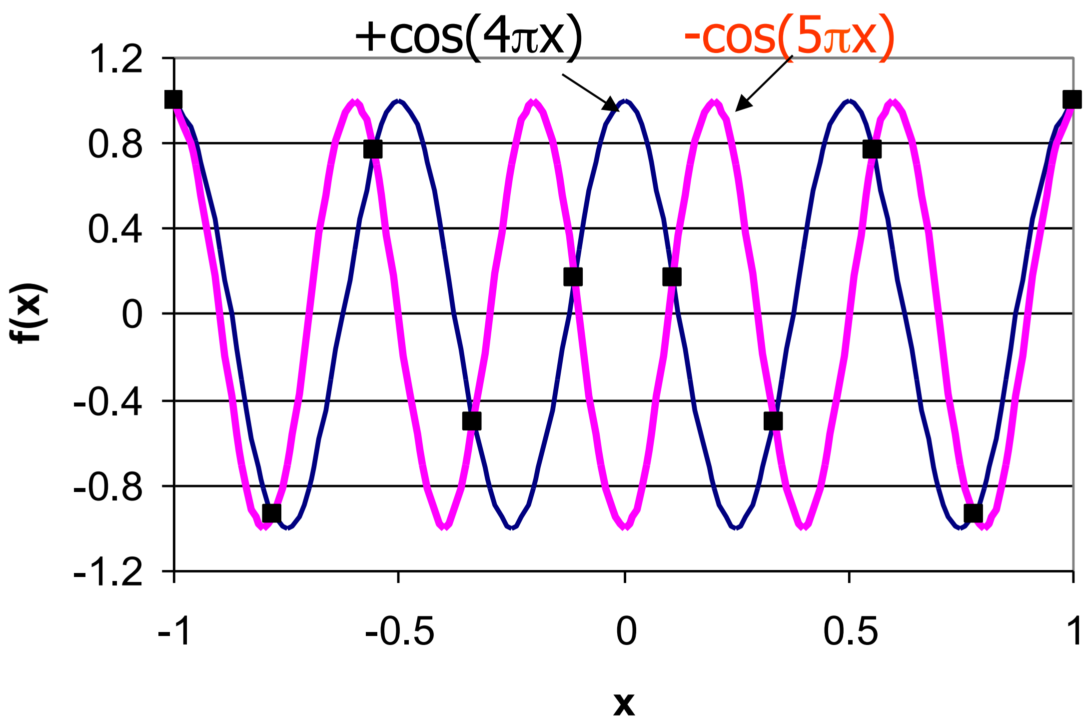
Cycle length for max $f \rightarrow \lambda_{MIN} = \dfrac{2\pi}{N/2} = 2\left(\dfrac{2\pi}{N}\right) = 2\Delta$

$\therefore f_{MAX} = \dfrac{1}{2\Delta} =$ *Nyquist frequency*

- Cannot get any meaningful information by trying to include higher frequencies

Sampling and aliasing

- So, to get max frequency (*bandwidth*) = *f*, we must sample at frequency ***2f***
  - Sample at least 2 pts/cycle of the highest *f* component
- If we try to use harmonics j > N/2 ($f > f_{Nyq}$), we get *aliasing* (phony matching of sampled points):



Example:
- cos(5πx) and +cos(4πx) have the *same values* at 10 equally-spaced points on [-1,1]
(10 pts = 9 intervals, so only 9/2= 4 harmonics are valid)

Discrete Fourier transform

- For function sampled at N equally spaced t values,

$h_k(t) = h(k\Delta)$ $\quad \Delta$ = sampling interval, $\quad k = 0 \ldots N, \quad N$ *even*

Nyquist : $f_c = \dfrac{1}{2\Delta} \rightarrow$ cannot find $H(f)$ for $|f| > f_c$

- With N points, we can only find N values of the Fourier transform: *discrete* H_n, not *continuous* H(f)

$$f_n = \frac{n}{N\Delta}, \quad n = -\frac{N}{2} \ldots 0 \ldots + \frac{N}{2}$$

(looks like $N+1$ f's, but $f_{-N/2} = f_{+N/2}$)

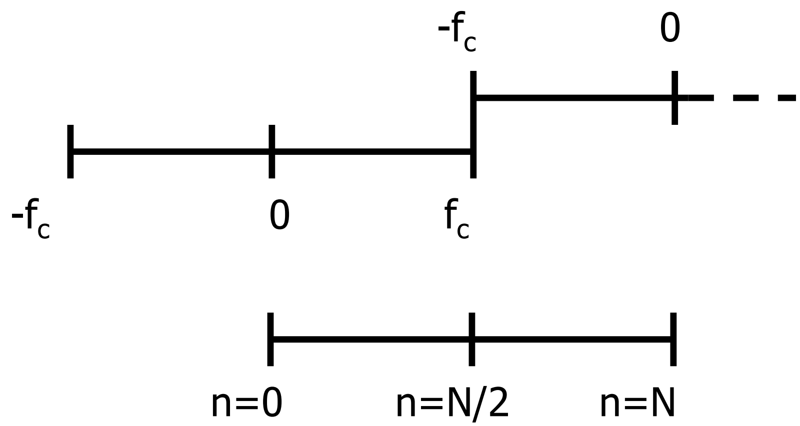
$$H(f) = \int_{-\infty}^{+\infty} h(t) \exp(i2\pi f t)\, dt \rightarrow \sum_{k=0}^{N-1} h_k \exp(i2\pi kn / N)\Delta$$

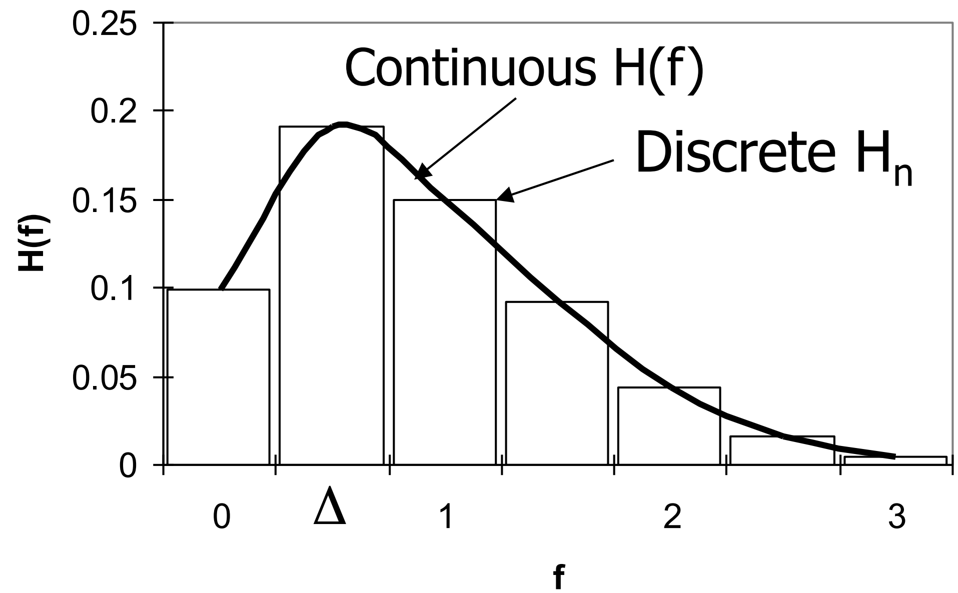
$$H_n = \sum_{k=0}^{N-1} h_k \exp\left(i\, 2\pi k \frac{n}{N}\right) \approx \frac{H(f_n)}{\Delta}$$

Discrete Fourier transform

- H_n is periodic in n
  - Period = N
  - So $H_{N-n} = H_n$

$$H_n = \sum_{k=0}^{N-1} h_k \exp\left(i\,2\pi k \frac{n}{N}\right) \approx \frac{H(f \sim f_n)}{\Delta}$$





  - So we can let n=0,1…N-1 (same index range as for h_k)

Transform pairs

- The discrete *inverse* transform is thus:

$$h_k = \frac{1}{N}\sum_{n=0}^{N-1} H_n \exp\left(-i\,2\pi k\frac{n}{N}\right)$$

- Parseval's theorem says "energy" is conserved between time and frequency domains:

$$\sum_k |h_k(t)|^2 = \frac{1}{N}\sum_n |H_n(f)|^2$$

Recall from your E&M class:
wave *amplitude* E(t) ➔ *power* ~ | E |^2

  – Parseval's theorem (= "energy conservation")
    - Total power in signal: $$P = \int_{-\infty}^{+\infty} |h(t)|^2\,dt = \int_{-\infty}^{+\infty} |H(f)|^2\,df$$
    - Power at frequency f: $$dP(f) = \int_{|f|}^{|f|+df} |H(f)|^2\,df$$
    - Power Spectral Density **(PSD)** $$\frac{dP}{df} = |H(f)|^2 + |H(-f)|^2$$

Discrete FT example

- Fourier interpolation: we can derive f(t) from FT(sampled data):
  - Example:

  Step fn: y={1, 1, 1, 0, 0, 0}, equally spaced on t=[0,2π)

  N = 6, k = 0,1…5, and $\{x_k\} = k(2\pi/6)$

  = {0, 1.05, 2.09, 3.14, 4.19, 5.24}

Trigonometric interpolating function is

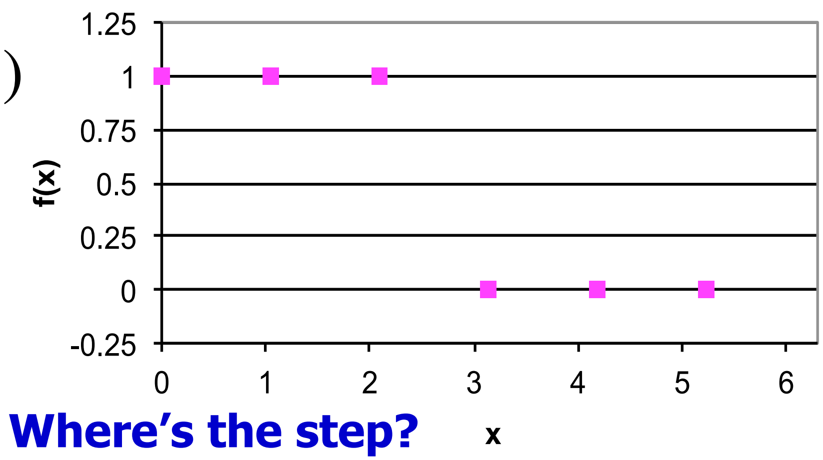
$$f_N(t) = \frac{a_0}{2} + \sum_{j=1}^{m-1}\left(a_j \cos(jt) + b_j \sin(jt)\right) + \frac{a_m}{2}\cos(mt)$$

Where $m = N/2 \rightarrow N = 2m$ (assumes even number of pts)

$$a_j = \frac{2}{N}\sum_{k=0}^{N-1} y_k \cos(jt), \quad b_j = \frac{2}{N}\sum_{k=0}^{N-1} y_k \sin(jt)$$

or $$c_j = \frac{2}{N}\sum_{k=0}^{N-1} y_k \exp(i\,jt_k)$$

$$\rightarrow a_j = \mathrm{Re}(c_j), \quad b_j = \mathrm{Im}(c_j)$$



Where's the step?

Do we know?

Discrete FT example

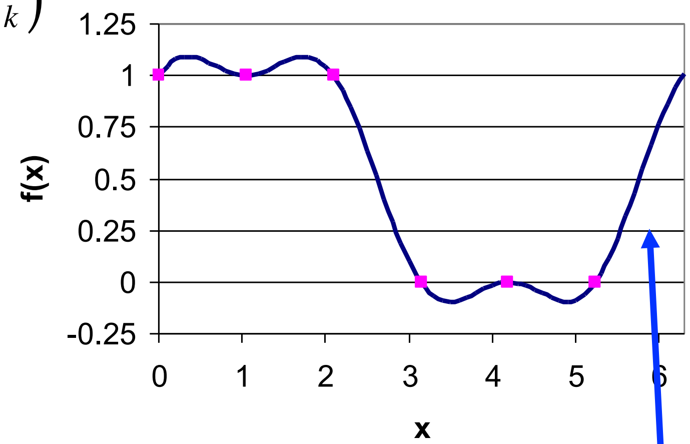
$$f_N(t) = \frac{a_0}{2} + \sum_{j=1}^{m-1}\left(a_j \cos(jt) + b_j \sin(jt)\right) + \frac{a_m}{2}\cos(mt), \qquad m = N/2 = 3$$

- Connection between FT coeffs and f(t) coeffs:

$$Y_j = \sum_{k=0}^{N-1} y_k \exp\left(i\,2\pi k \frac{j}{N}\right) = \sum_{k=0}^{N-1} y_k \exp\left(i\, jt_k\right)$$

$$c_j = \frac{2}{N}\sum_{k=0}^{N-1} y_k \exp(i\,jt_k) \rightarrow c_j = \frac{2}{N}Y_j$$

$$a_j = \frac{2}{N}\mathrm{Re}(Y_j), \quad b_j = \frac{2}{N}\mathrm{Im}(Y_j)$$



- Run FT on these data, results are:

  Y_j={3.0, 1.0+1.73i, 0, 1.0, 0, 1.0-1.73i}

  c_j=(1/3) Y_j ={1.0, 1.0+1.73i, 0, 1.0, 0, 1.0-1.73i}

- So coefficients of interpolating function are:

$a_0/2$=Re(Y_0/3)/2=0.5, a_1=Re(Y_1/3)=0.333, a_2=0, a_3=Re(Y_3/3)/2=0.167; b_1=Im(Y_1/3)=0.577, b_2=0

$$f_{N=6}(x) = 0.5 + 0.33\cos(t) + 0.167\cos(3t) + 0.577\sin(t)$$

Summary of FT properties

- Note: for real f(t): F(-f) = F*(f) (* = conjugate)
  - So, providing F(f) does not contain any δ-functions (i.e. discrete sinusoids) $\left|F(-f)\right|^2 = \left|F(f)\right|^2 \rightarrow PSD = 2\left|F(f)\right|^2$
  - Parseval's thm ➔ RMS f(t) = area under PSD
- Summary:

$\{f(t_k)\}, \quad k = 0,\ 1 \ldots N-1$ N samples of signal at intervals Δ

$\{F_n\}, \quad n = -\frac{N}{2} \ldots 0 \ldots \frac{N}{2}$ Discrete Fourier transform of *h*

F_n = *Amplitudes* in frequency domain (spectrum of *h*):

$$F(f_n) \approx F_n \Delta, \quad \text{with frequencies } f_n = \frac{n}{N\Delta}, \quad -f_c \le f_n \le f_c, \quad f_c = \frac{1}{2N}$$

$$F(f) = \int_{-\infty}^{+\infty} f(t) \exp(i 2\pi f t)\, dt \rightarrow F_n = \sum_{k=0}^{N-1} f_k \exp\left(i\, 2\pi k \frac{n}{N}\right)$$

Summary of FT properties

Limited information: we have

N numbers for $f(t_k) \rightarrow N$ numbers for $F_n \quad \left(F_{-N/2} = F_{N/2}\right)$

Amplitude F_n exists from $f = -f_c$ to f_c , but

spectrum only has meaning for $f = 0$ to f_c :

$$P_n(f_n) = \left|F(f_n)\right|^2 + \left|F(-f_n)\right|^2$$

For {h} real, {H} will be all reals (if h is odd, cos series), or all imaginaries (if h is even, sin series)

H_n is explicitly periodic with period N, so $F_{-n}=F_{N-n}$

$$\left.\begin{array}{l} n = 0 \ldots \frac{N}{2} \quad \rightarrow \quad f = 0 \ldots f_c \\ n = \frac{N}{2} \ldots (N-1) \rightarrow f = -f_c \ldots 0 \end{array}\right\}$$

$\Big\{$ $Note: \pm f_c = f_{N/2}$ so we can have both k and n run from $0 \ldots (N\text{-}1)$