PHYS 536

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Session 4

Discrete Fourier transforms

Sampling theorem

Convolution and correlation

Digital filtering

1/12/2023

Course syllabus and schedule – first part…

See : http://courses.washington.edu/phys536/syllabus.htm

Session	date	Day	Readings:	K=Kinsler, H=Heller	Topic
1	3-Jan	Tue	K ch. 1	H: Ch. 1, 2	Course intro, acoustics topics, overview of wave properties; pulses, transverse and longitudinal waves, overview of sound speeds
2	5-Jan	Thu	K ch. 1	H: Ch. 9, 10	harmonic oscillators: simple, damped, driven; complex exponential solutions, electrical circuit analogy, resonance, Q factor
3	10-Jan	Tue	K ch. 1	H: Ch. 3	Fourier methods: Fourier series, integrals, Fourier transforms, discrete FTs, sampling and aliasing
4	12-Jan	Thu	K. chs 10	H: Ch. 4, 11	Frequencies and aliasing; convolution and correlation; discrete convolution; digital filtering, optimal filters, FIR filters, noise spectra; power spectra. REPORT 1 PROPOSED TOPI Tonight
5	17-Jan	Tue	K. ch. 2, 3, 4	H: Ch. 13, 15	waves in strings, bars and membranes; Acoustic wave equation; speed of sound; Harmonic plane waves, intensity, impedance.
6	19-Jan	Thu	K. Ch. 5, 6	H: Ch. 1	Spherical waves; transmission and reflection at interfaces
7	24-Jan	Tue	K. Ch. 8	H: Ch. 7	Radiation from small sources; Baffled simple source, piston radiation, pulsating sphere;
8	26-Jan	Thu	K: Ch. 10	H: Chs. 13-15	Near field, far field; Radiation impedance; resonators, filters
9	31-Jan	Tue	K. Ch. 9-10	H: Chs. 16-19	Musical instruments: wind, string, percussion
10	2-Feb	Thu	K. Ch 14		Transducers for use in air: Microphones and loudspeakers
11	7-Feb	Tue	K. Ch 11	H: Chs. 21-22	The ear, hearing and detection
12	9-Feb	Thu	K. Chs 5,11		Decibels, sound level, dB examples, acoustic intensity; noise, detection thresholds. REPORT 1 PAPER DUE by 7 PM; REPORT 2 PROPOSED TOPIC DUE

Announcements

- REMINDER: term paper #1 proposals are due TODAY!
  - Still haven't heard from a few students
  - Remember: only 5 pages – NARROW your scope!
  - Please send me a brief email with
    - Topic chosen
    - Resources to be used in your study (books, journal articles, etc)
    - Format chosen: term paper or website
      - You can submit a 5p paper, or build a website with the same amount of content
      - For info on how to create a website @uw, see

https://sites.uw.edu/your-first-site/

From last time

Discrete Fourier transform

- For function sampled at N equally spaced t values,

$h_k(t) = h(k\Delta)$ $\quad \Delta$ = sampling interval, $\quad k = 0 \ldots N, \quad N$ *even*

Nyquist : $f_c = \dfrac{1}{2\Delta} \rightarrow$ cannot find $H(f)$ for $|f| > f_c$

- With N points, we can only find N values of the Fourier transform: *discrete* H_n, not *continuous* H(f)

$$f_n = \frac{n}{N\Delta}, \quad n = -\frac{N}{2} \ldots 0 \ldots + \frac{N}{2}$$

(looks like $N+1$ f's, but $f_{-N/2} = f_{+N/2}$)

$$H(f) = \int_{-\infty}^{+\infty} h(t) \exp(i2\pi f t)\, dt \rightarrow \sum_{k=0}^{N-1} h_k \exp(i2\pi kn/N)\Delta$$

$$H_n = \sum_{k=0}^{N-1} h_k \exp\left(i\, 2\pi k \frac{n}{N}\right) \approx \frac{H(f_n)}{\Delta}$$

From last time

Transform pairs

- The discrete *inverse* transform is thus:

$$h_k = \frac{1}{N} \sum_{n=0}^{N-1} H_n \exp\left(- i\, 2\pi k \frac{n}{N} \right)$$

- Parseval's theorem says "energy" is conserved between time and frequency domains:

$$\sum_k |h_k(t)|^2 = \frac{1}{N} \sum_n |H_n(f)|^2$$

Recall from your E&M class:
wave *amplitude* E(t) ➔ *power* ~ | E |^2

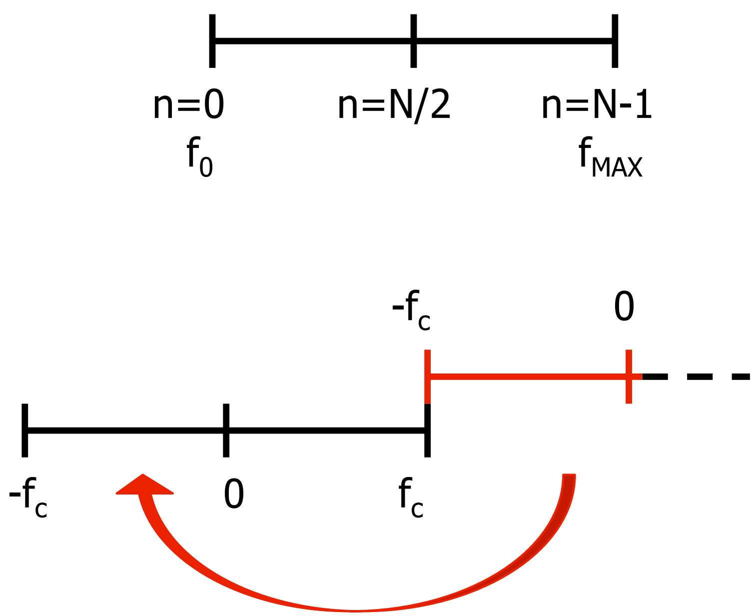
  – Parseval's theorem (= "energy conservation")
    - Total power in signal: $P = \int_{-\infty}^{+\infty} |h(t)|^2 \, dt = \int_{-\infty}^{+\infty} |H(f)|^2 \, df$
    - Power at frequency f: $dP(f) = \int_{|f|}^{|f|+df} |H(f)|^2 \, df$
    - Power Spectral Density **(PSD)** $\frac{dP}{df} = |H(f)|^2 + |H(-f)|^2$

DFT: negative f's and Nyquist frequency

- Given N data samples h(t), with $\{t_n\}$, n=0…N-1
  - Discrete FT produces N values of H(f), k=0…N-1, $f_{MAX}=1/\Delta$

  (Note: Discrete FT implicitly assumes h(t) is periodic)
  - But Nyquist limit allows only N/2 frequencies: max $f_c = 1/2\Delta$
  - Solution:

  Treat these FT components as representing $f_{-N/2} .. f_{+N/2}$ where $f_{-N/2} = f_{+N/2}$



Negative and positive frequencies

- Mathematically, the continuous FT and inverse are defined *symmetrically*:

$$H(f) = \int_{-\infty}^{+\infty} h(t)\exp(i2\pi f t)dt \leftrightarrow h(t) = \int_{-\infty}^{+\infty} H(f)\exp(-i2\pi f t)df$$

  - So negative f's are handled naturally
- But discrete transform sums are *periodic* in n:

$$F_n = \sum_{k=0}^{N-1} f(t_k)\exp(i2\pi kn/N)$$

where k indexes N signal samples, $k = 0 \ldots N-1$,

and n indexes N frequencies, $n = (-N/2)\ldots 0 \ldots (+N/2)$

$\exp(i2\pi kn/N)$ is periodic in n, with period = N, so $F_{-n} = F_{N-n}$

  - Periodicity means $f_{-N/2} = f_{+N/2}$, so only get N distinct f's

- Note: "Fast Fourier Transform" (FFT) = clever algorithm to minimize CPU time required for DFT-ing large sample sets
  - Requires N to be a power of 2 - otherwise, just a DFT

Negative and positive frequencies

- We can let n=0…N/2 instead (symmetry wrt $f(t_k)$ indexing)
- Then
  - n=0 ➔ f=0
  - n=1 ➔ $f=f_1$
  - n=N/2 ➔ $f=\pm f_c = f_1 N/2$
  - n=(N/2)+1 ➔ negative freq $f= -f_{c+1}$
  - n=(N-1) ➔ $-f_1$

example: say N=6, and Δ=1 ms: then $f_c=1/2\Delta=0.5$ kHz

$f_{-3}=0.5$kHz, $f_{-2}=-0.33$kHz, $f_{-1}=0.16$kHz, $f_0=0$ (DC), $f_1=0.16$kHz, $f_2=0.33$kHz, $f_3=0.5$kHz

- Notice that negative f' s (or n>N/2) contain no new information, but must be taken into account when computing intensity ("power"):

$$P_n(f_n) = |H(f_n)|^2 + |H(-f_n)|^2, \quad n = 0 \ldots N/2$$

DFT: Sampling theorem and Nyquist frequency

- Sampling theorem:

"If a continuous function *h(t),* sampled at an intervals, Δ, is ***bandwidth limited*** to frequencies smaller in magnitude than f_c, so $H(f) = 0$ for $|f| \geq f_c$

Then h(t) is completely determined by its samples for $\Delta < 1/(2f_c)$

– So: If the signal is *known* not to contain harmonics > f_c then the Fourier interpolation is an exact representation (and vice-versa!)

  - Often we have a signal that is bandwidth-limited (by amplifier or cable limitations)

    – sampling theorem tells us that the entire information content of the signal can be recorded by sampling it at a rate $f_s = 2 f_c$

  - Remarkable because a continuous function seems to have infinitely more "information content" than the series

– But: if h(t) is not bandwidth limited to $f \leq f_c$, all the spectral power density outside of the frequency range $-f_c \leq f \leq f_c$ is (falsely) pushed into that range: aliasing

(more on this later)

From last time

Discrete FT example

- Fourier interpolation: we can derive f(t) from FT(sampled data):
  - Example:

Step fn: y={1, 1, 1, 0, 0, 0}, equally spaced on t=[0,2π)

N = 6, k = 0,1…5, and $\{x_k\} = k(2\pi/6)$

= {0, 1.05, 2.09, 3.14, 4.19, 5.24}

Trigonometric interpolating function is

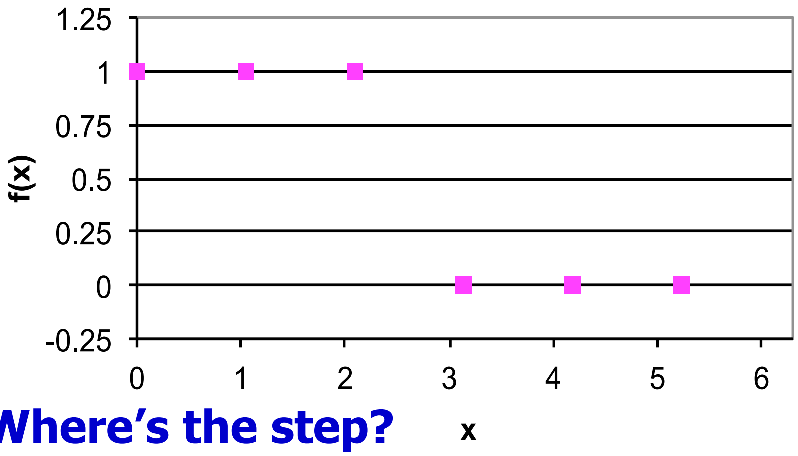
$$f_N(t) = \frac{a_0}{2} + \sum_{j=1}^{m-1}\left(a_j \cos(jt) + b_j \sin(jt)\right) + \frac{a_m}{2}\cos(mt)$$

Where $m = N/2 \rightarrow N = 2m$ (assumes even number of pts)

$$a_j = \frac{2}{N}\sum_{k=0}^{N-1} y_k \cos(jt), \quad b_j = \frac{2}{N}\sum_{k=0}^{N-1} y_k \sin(jt)$$

or $c_j = \frac{2}{N}\sum_{k=0}^{N-1} y_k \exp(i\,jt_k)$

$$\rightarrow a_j = \mathrm{Re}(c_j), \quad b_j = \mathrm{Im}(c_j)$$



Where's the step?

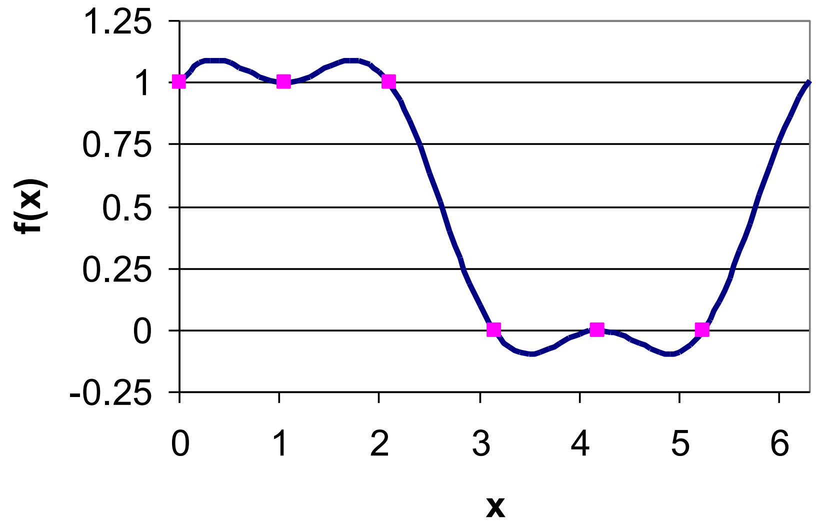
Do we know?

Example of frequency issues

- Let's revisit the earlier example of FT on a sampled square wave:
  - Data: y={1, 1, 1, 0, 0, 0}, equally spaced on x=[0,2π)

  $N = 6$, $k = 0,1\ldots5$, $\{x_k\} = k(2\pi/6)$

  $= \{0, 1.05, 2.09, 3.14, 4.19, 5.24\}$
  - Notice that data are *assumed to be periodic* (basis of discrete FT), so "x_7"=2π ➔ "y_7"=y_1 … (repeat)



  - Run DFT on these data, results are:

    $Y_0=3.0$ (constant term, baseline)

    $Y_1=1.0+ 1.73\,i$

    $Y_2=0$

    $Y_3=1.0$

    $Y_4=0$ $(=Y_{-2})$

    $Y_5=1.0 - 1.73\,i$ $(=Y_{-1})$
  - Here $\Delta= 2\pi/6$, $f_n=n/(N\Delta)=n/2\pi$, $f_c=1/(2\Delta)$,

Example of frequency issues

– We can identify the frequencies in the discrete spectrum as
$f_0=0$, $f_1=1/2\pi$, $f_2=2/2\pi$, $f_3=3/2\pi=f_c$ (Nyquist f in this example)

But we are entitled to 6 H(k) components for 6 h(t) samples, so we get 2 more, corresponding to *negative* frequencies:

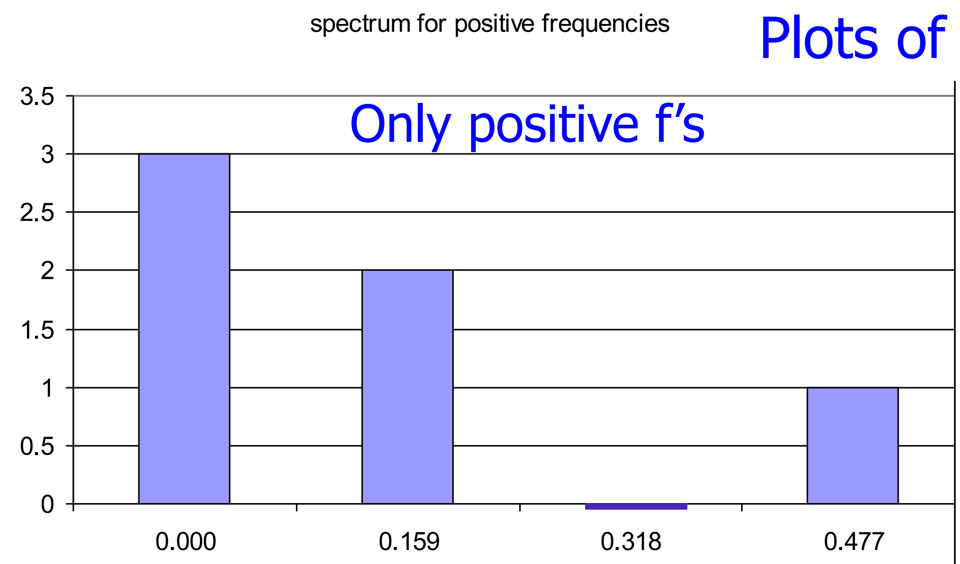
$f_4 = f_{-2} = - 2/2\pi$, $f_5 = f_{-1} = - 1/2\pi$

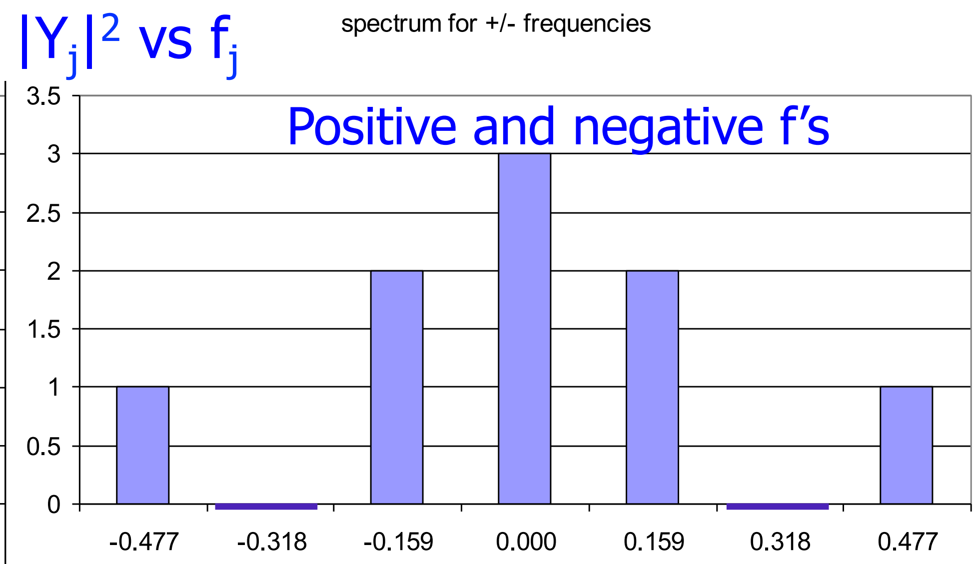
Notice that $|Y_4| = |Y_2|$ and $|Y_5| = |Y_1|$

If we use the indexing -N/2…+N/2, we get

$f_{-3}=3/2\pi$, $f_{-2}= -2/2\pi$, $f_{-1}= -1/2\pi$, $f_0=0$,
$f_1 = 1/2\pi$, $f_2 = 2/2\pi$, $f_3=3/2\pi$ (7 n's but 6 f's)

Plots of $|Y_j|^2$ vs f_j





Summary of FT properties

- Note: for real signal f(t): F(-f) = F*(f) (* = conjugate)
  - So, if F(f) does not contain any δ-functions (i.e. discrete sinusoids)

$$\left|F(-f)\right|^2 = \left|F(f)\right|^2 \rightarrow PSD = 2\left|F(f)\right|^2$$

  - Parseval's thm ➔ RMS f(t) = area under PSD
- Summary:

$\{f(t_k)\}, \quad k = 0,\ 1 \ldots N-1$ N samples of signal at intervals Δ

$\{F_n\}, \quad n = -\frac{N}{2} \ldots 0 \ldots \frac{N}{2}$ Discrete Fourier transform of h

F_n = *Amplitudes* in frequency domain ("spectrum" of f):

$$F(f_n) \approx F_n \Delta, \quad \text{with frequencies } f_n = \frac{n}{N\Delta}, \quad -f_c \le f_n \le f_c, \quad f_c = \frac{1}{2N}$$

$$F(f) = \int_{-\infty}^{+\infty} f(t) \exp(i2\pi f t)\, dt \rightarrow F_n = \sum_{k=0}^{N-1} f_k \exp\left(i\, 2\pi k \frac{n}{N}\right)$$

Summary of DFT properties

Limited information: we have only

N numbers for $f(t_k) \rightarrow N$ numbers for $F_n \quad \left(F_{-N/2} = F_{N/2}\right)$

Amplitude F_n exists from $f = -f_c$ to f_c , but

spectrum only has meaning for $f = 0$ to f_c :

$$P_n(f_n) = \left|F(f_n)\right|^2 + \left|F(-f_n)\right|^2$$

With $\{f_n\}$ real, $\{F_n\}$ will be all reals (eg, if f(t) is odd, cos series), or all imaginaries (if f is even, sin series)

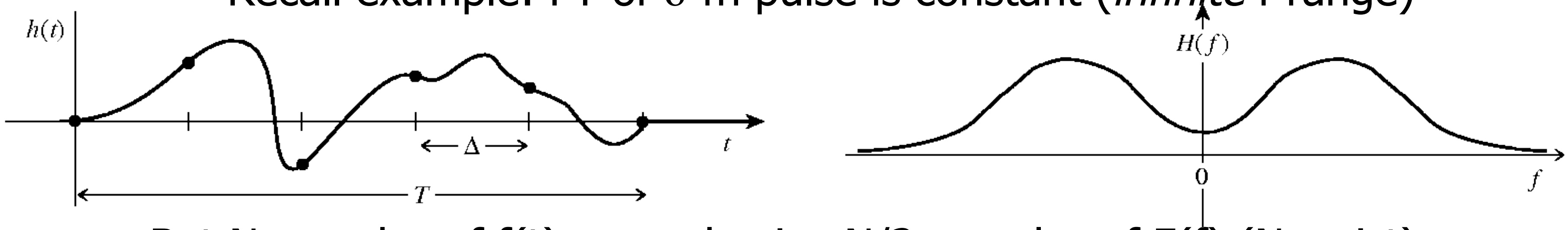
F_n is assumed to be periodic, with period N, so $F_{-n}=F_{N-n}$

$$\left.\begin{array}{l} n = 0 \ldots \dfrac{N}{2} \rightarrow f = 0 \ldots f_c \\ n = \dfrac{N}{2} \ldots (N-1) \rightarrow f = -f_c \ldots 0 \end{array}\right\}$$

$Note: \pm f_c = f_{N/2}$ so we can have both k and n run from $0 \ldots (N\text{-}1)$

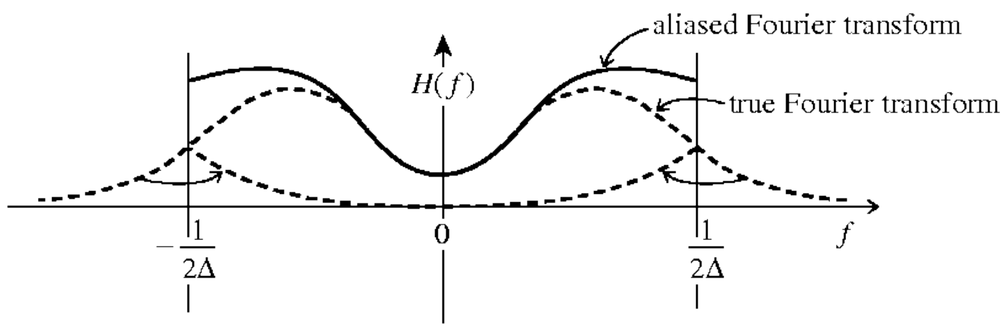
Understanding frequencies and aliasing

- Let's review the meaning of frequencies in FTs
  - *Signal* occupies limited range of t: finite sampling
  - We know FT connects *limited* t range to *broad* freq range
    - Recall example: FT of δ-fn pulse is constant (*infinite* f range)



  - But N samples of f(t) can only give N/2 samples of F(f) (Nyquist)
    - f-range limited to $\pm f_c$
      - f_c defined by range T ("period") and N: ➔$f_1=1/T$, $f_c = (N/2)f_1$
    - True spectrum must have broader tails
      - "Power" (area under tails of true f spectrum) will get *aliased* into the limited f range of the discrete spectrum

Figs. from *Numerical Recipes*



Applying Fourier methods: Convolution and correlation

- Given 2 signals, g(t) and h(t)

$$g * h = \int_{-\infty}^{+\infty} g(t')h(t-t')dt' \quad = \textit{convolution of } g \text{ and } h$$

Note: minus sign

multiply g by *shifted* h and integrate (notice $g * h = f(t)$)

Then $g * h = h * g,$ and

$$FT(g * h) = \int_{-\infty}^{+\infty} \exp(i2\pi f t)dt \int_{-\infty}^{+\infty} g(t')h(t-t')dt'$$

So $FT(g * h) = G(f)H(f)$ *Convolution theorem*

FT of *convolution* ⇔ simple *product* of FTs in f - domain

Similarly : $\mathrm{corr}(g,h) = \int_{-\infty}^{+\infty} g(t'+t)h(t')dt'$ = *correlation* of g and h

Note: plus sign

$$\mathrm{corr}(g,h) = FT\{\, G(f)H(-f)\,\}$$

For real functions $g, h,\ H(-f) = H^*(f)$

so $\mathrm{corr}(g,h) \underset{FT}{\Leftrightarrow} G(f)H^*(f)$

Common applications

– Convolution: describes effect of a *filter* on a signal
– Correlation: identify and locate a *specified waveform* in noisy signal

Convolution and correlation in signal processing

- Convolution and correlation are mathematically similar but have different interpretations:

convolution $g * h = \int_{-\infty}^{+\infty} g(t')h(t-t')dt' \leftrightarrow G(f)H(f)$

correlation $\mathrm{corr}(g,h) = \int_{-\infty}^{+\infty} g(t+t')h(t')dt' \leftrightarrow G(f)H^*(f)$

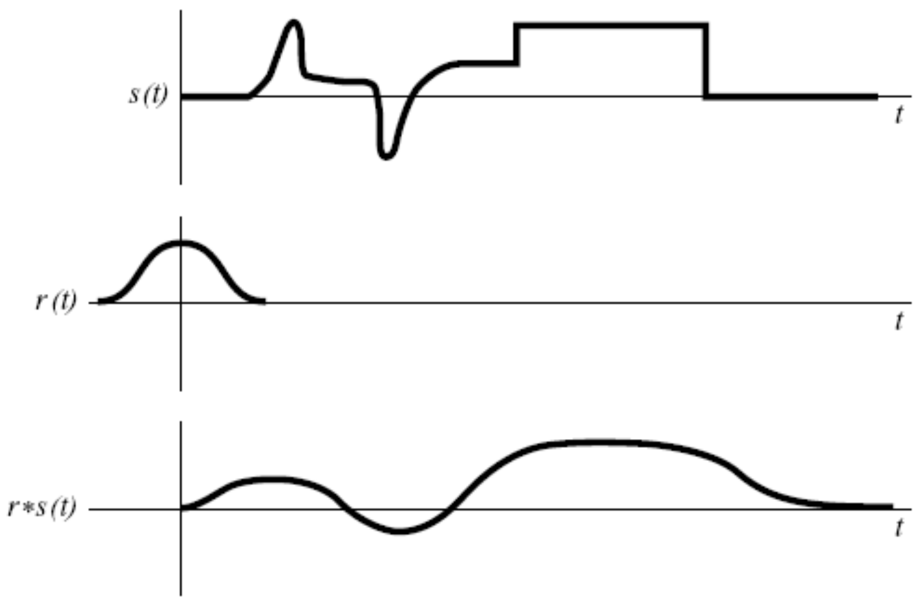
  – Convolution = smearing or smoothing "signal" h(t) with "filter" g(t)
    - Typically g covers *smaller* range than h (shorter time span or fewer samples)
  – Correlation = checking for common features (modulo some unknown *shift* Δx) between 2 signals
    - Typically g and h have ~same sample size
    - *Autocorrelation* = check for cyclic behavior in signal itself
      – Important tool in acoustics (more later)

Convolution

- For each value t' within the selected t range, multiply g(t') by h(t-t') and add up contributions:

Continuous version:

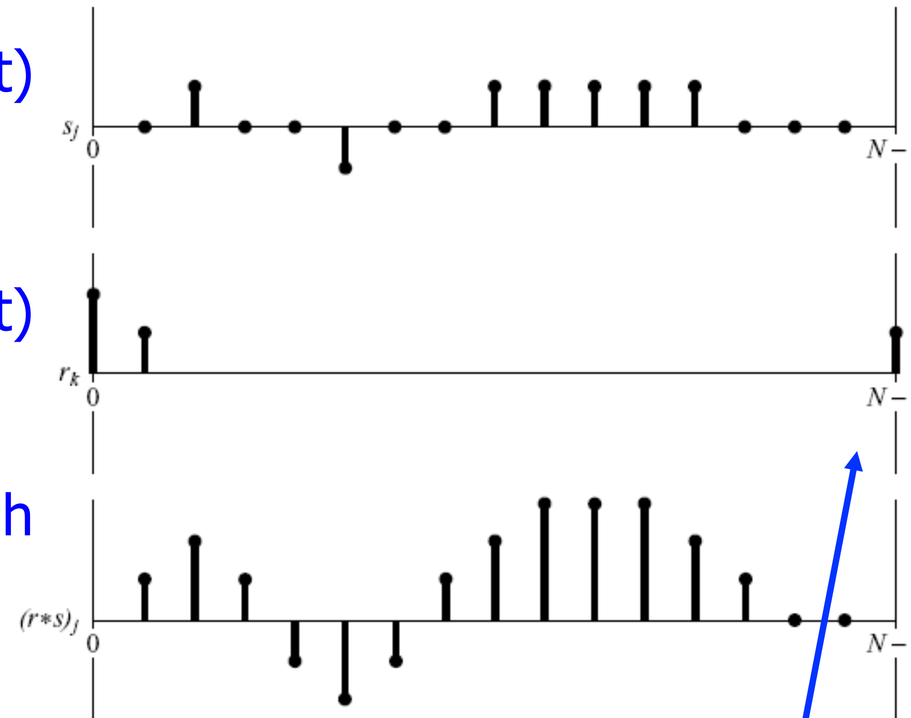
Discrete version



g(t)

h(t)

g*h



note wraparound of g(t)

NR fig. 13.1.1

Convolution

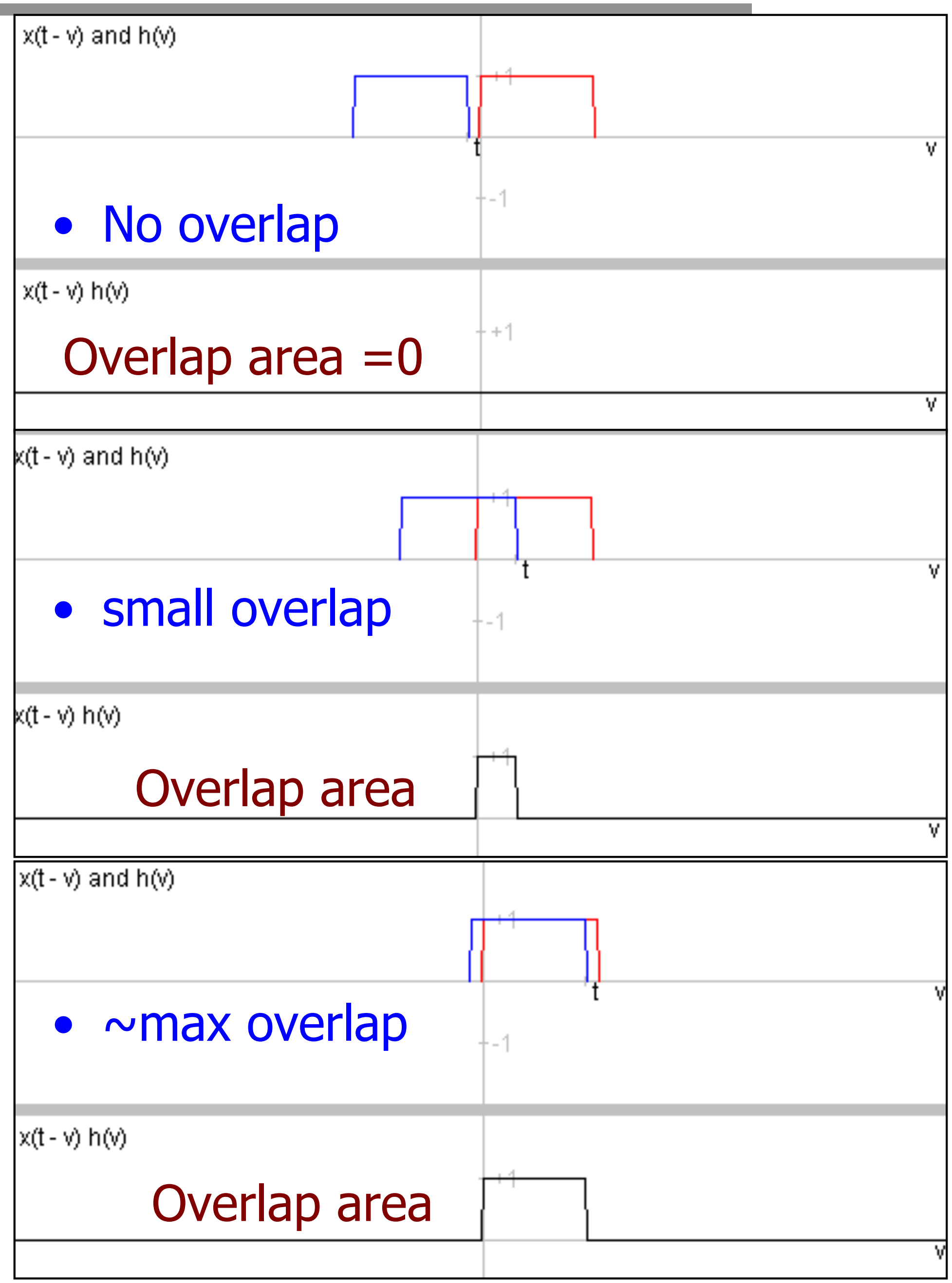
Convolution amounts to calculating overlap vs displacement and integrating:

Example: Convolution of 2 square pulses, x(v) and h(v)

(see http://www.jhu.edu/~signals/convolve/)

In these plots, t=offset of h(v) relative to x(v)

(reversed!)



Convolution (continued)

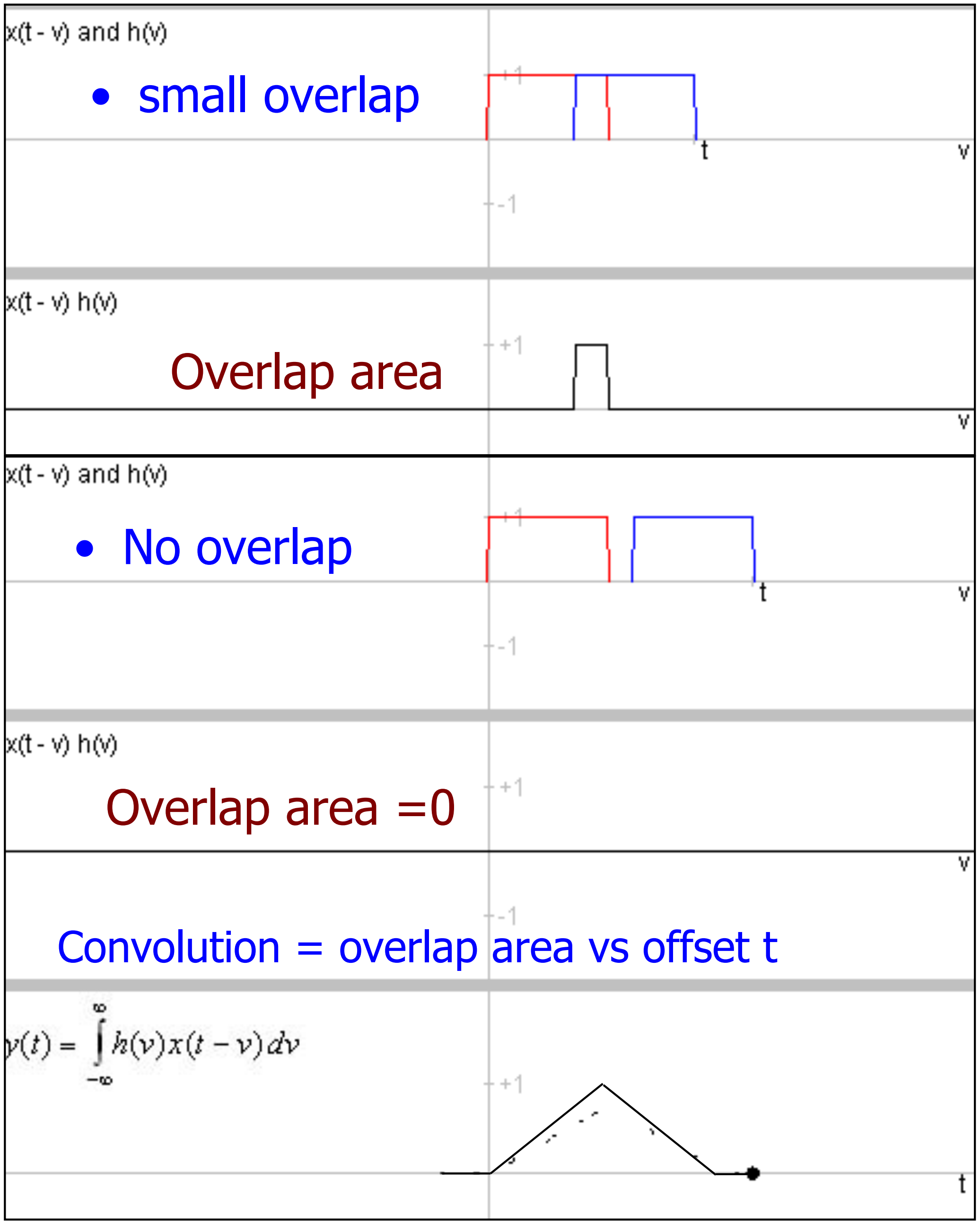
- past max overlap:

small overlap again

No overlap again

Plot of overlap area vs offset t

= Convolution vs t



Discrete convolutions and correlations

Discrete Convolution:

- Application: model effect of a filter (or any process) on a known input signal
  - Electrical signal passed through transmission line
  - Point-source (eg, distant star) imaged in optical system
  - Physical process in detector with systematic error
- For periodic signal g = s(t) (N samples), and
- Response h = r(t) with finite shorter duration ($M \leq N$ samples):
  - *Finite impulse response* = FIR (important case in signal analysis)
  - Calculating (g*h) is really simple! Time shift is just change of index:

$$(g*h)_j = \sum_{k=-M/2+1}^{M/2} s_{j-k} r_k = FT(G_j H_j)$$

- Just sum of products of *shifted* elements of g & h(t)
- Can be implemented on a specialized DSP chip for real-time apps
- Can also be implemented in hardware as a *FIR filter* with only passive components

Discrete convolutions and correlations

Discrete Correlation:

- Application: search for pattern in a data stream
  - Search for specified signal in noise
  - Test for similarity of signals (in time-series sense)
- Very similar to convolution, but typically M=N
  - Discrete corr:

$$\mathrm{corr}(g,h)_j = \sum_{k=0}^{N-1} g_{j+k} h_k = FT(G_k H_k^*)$$

    - Notice index shift has + sign instead of -
- corr(g,h) vs t (="lag") : correlogram
  - corr(g,h) is large when g~h at lag t (location of h in signal stream g)
- Wiener-Khinchin Theorem: autocorrelation is Fourier dual of signal's power spectral density PSD: $corr(h,h) \leftrightarrow |H(f)|^2$

Autocorrelation

- Wiener-Khinchin Theorem: $corr(h,h) \leftrightarrow |H(f)|^2$
  - Meaning (as with all FT pairs): if the autocorrelation is narrow, the PSD will be broad

→ "Uncertainty principle" :

Δt = width of autocorrelation in time, Δf = width of PSD in frequency

Then $\Delta f \, \Delta t \sim$ constant

(analogous to QM uncertainty principle $\Delta x \, \Delta p \sim$ constant, and from the same source: in QM, x and p are dual spaces → FT partners

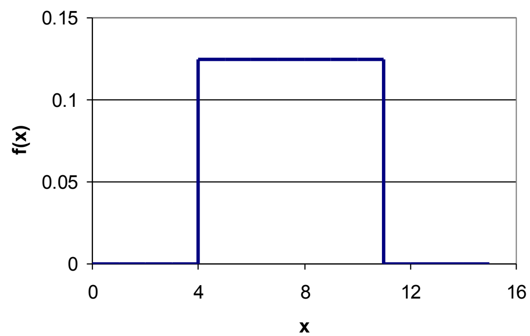
- Chorus effect, and fast echoes:
  1. If members of a chorus could all sing a given note without vibrato, we would hear it as one voice (with greater amplitude)
     - Real chorus sounds pleasantly complex, but we recognize the note
     - Variations between voices broaden the signal's autocorrelation, so PSD is relatively narrow
  2. If a sound is repeated after a very short delay (few msec) we cannot register it as separate, but it "colors" the sound by altering autocorrelation – room echoes do this

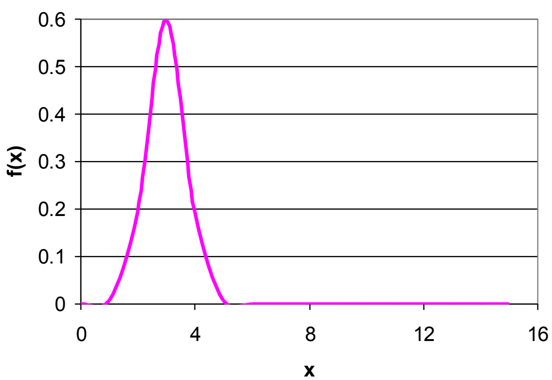
Example of convolution

- Example of convolution (this is from optics, but same idea for f(t)):
  - *Scene* is imaged by lens with limited aperture: clips off higher *spatial frequencies*
  - *Point spread function* = impulse response of lens (image of ideal mathematical point) (acoustic equivalent: FT of a sharp bang)
  - Convolution= apply PSF to each point of input scene and add to get resulting image

original

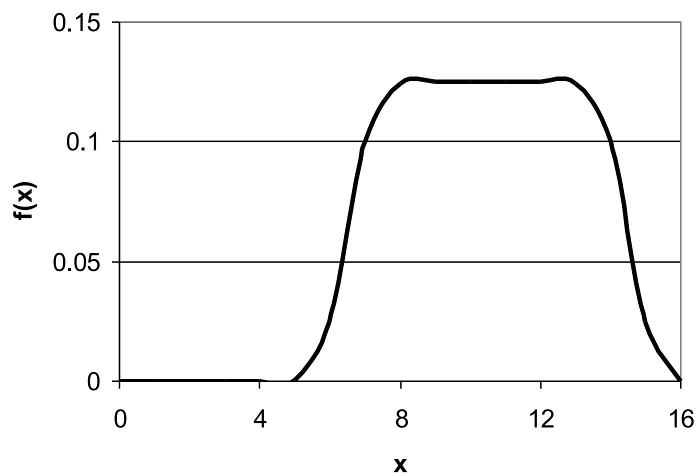
Simplest possible scene
= uniform bright disk
(sharp edge)





Point spread function = Gaussian
(impulse response of lens)

Image = convolution (r*s)
(edges of disk become fuzzy)



Lens output
(convolved with
PSF of lens)

Applications of digital convolution in acoustics

Reverberation: effects of multiple reflections of a sound source arriving at listener's location, in a given room or other enclosure

- Direct sound arrives first, followed by direct reflections, then multipath reflections

Convolution Reverb

- Simulate effect of room acoustics on a digitized sound stream (eg music)
  - Get impulse response of room using sharp noise (like a gunshot)
  - Convolve IR with signal of interest
  - Can predict how music will sound in room

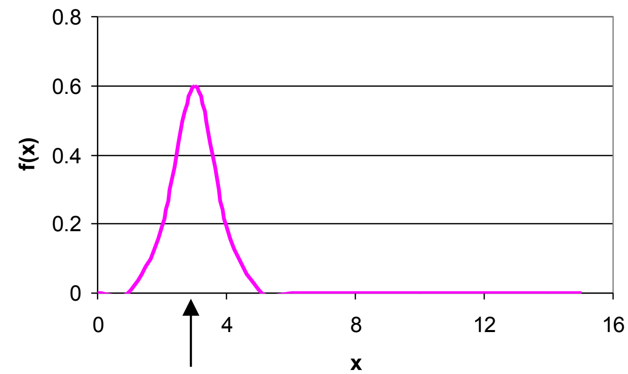
Digital Reverb

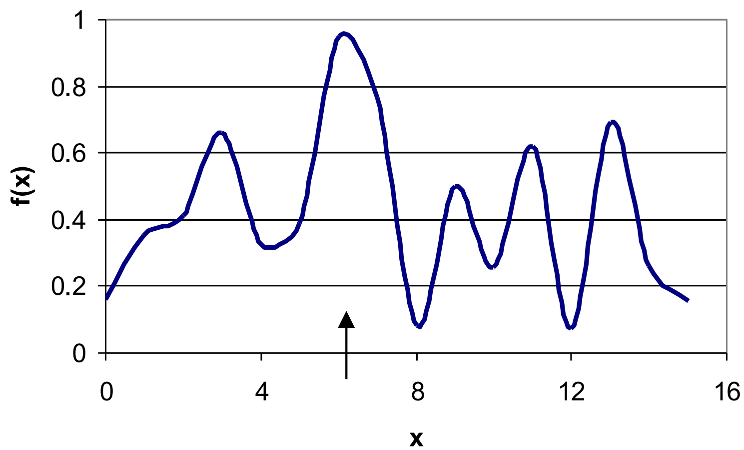
- Apply any desired set of delays and frequency dependent effects to a digital signal stream
  - Use specialized electronics, or computer software, to filter, attenuate and delay multiple copies of original signal
  - at lag t (location of h in signal stream g))

Example of correlation

- Example of correlation from acoustic signal processing
  - Gaussian-shaped sonar pulse is buried in noisy data stream
  - Find arrival time of pulse

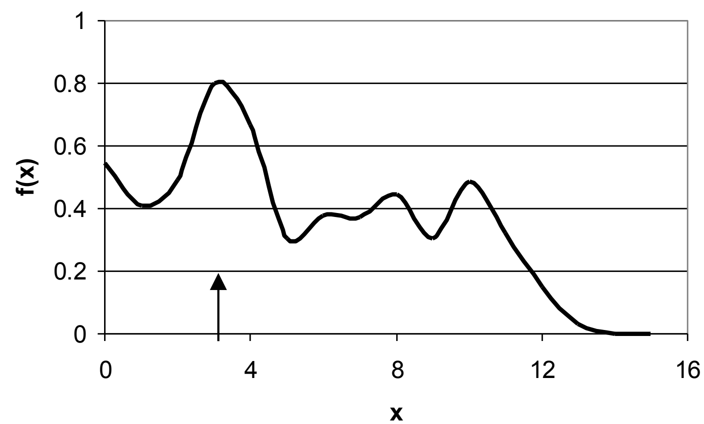
Gaussian-shaped pulse model:
Starts at x=1, centered at x=3





Pulse buried in white noise
(25% signal, 75% noise)
Here, pulse starts at x=4, centered at x=6

Correlogram:
Peak location shows arrival time
of pulse is at x' =3 in correlation
4-1=3 is "lag" relative to model function,
so pulse center is located at x= 3 + 3 = 6
in signal stream (position in model + lag
in correlation



Filters in signal processing

Typically we

Measure signal s(t) (e.g., voltage vs time from microphone)

FTs:

Assume s(t)=u(t)*r(t) → S(f)

u(t)= true underlying signal → U(f)

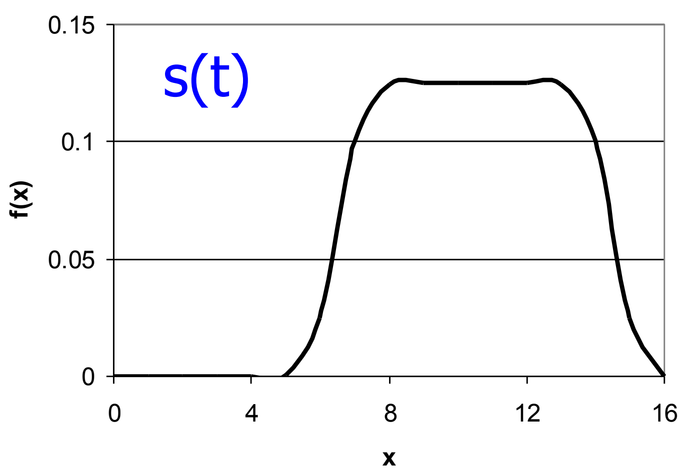
r(t)= measuring system's response fn → R(f)

Then S(f)=U(f)R(f) (simple product of FTs)

So U(f)=S(f)/R(f) (S=FT[s(t)], R=FT[r(t)])

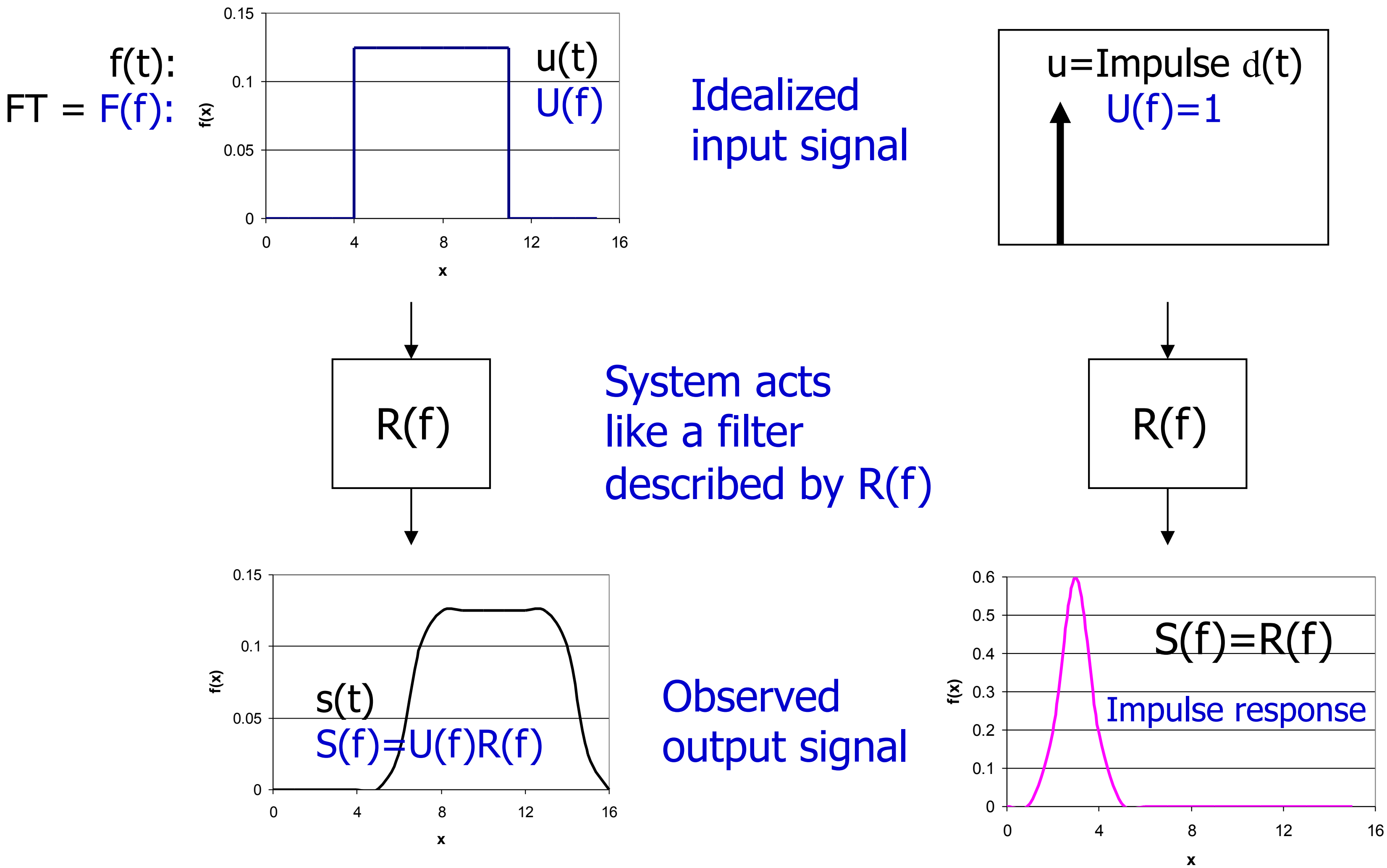
$u(t)=FT^{-1}[U(f)]$ (deconvolve to recover true signal)

This describes the action of a *filter* on a signal



S(f)=U(f)R(f)

signal processing acts like a filter



To find R(f), we input an impulse u(t): U(f) = flat (all f’s present)
Then output spectrum S(f) = filter characteristic R(f)

- R(f) = *Impulse response*

Digital filtering

- So far we have assumed *offline* filtering in f-domain (in a computer)
  - "*Acausal*": we have the full signal history in hand, a priori
- Often must do *realtime* filtering in t-domain (in the field!)
  - "*Causal*": we have only the current and a few recent samples
  - Historically: used analog devices: capacitors, inductors, transistors; or lenses, apertures, filters…
  - Currently: digital filtering using DSP chips or fast devices (GHz rates)
- Linear filter: $$(\{f(t_k)\}_n = \sum_{k=0}^{M} c_k t_{n-k} + \sum_{j=1}^{N} d_j f(t_{n-j})$$
  - Output at t=nΔ is function of
    - Previous M+1 inputs
    - Previous N outputs
  - If N=0 (no feedback), *non-recursive* filter
    - FIR filter: y → 0, after x → 0

    (Finite Impulse Response)
  - If N>0, f= *recursive* filter
    - IIR (infinite impulse response):

    Infinite impulse response possible: feedback → output may howl!

    Sharper filtering, but at cost of potential instability

For FIR filters

$$g(f) = FT(f(t)) = \sum_{k=0}^{M} c_k \exp(-i2\pi fk\Delta)$$

So $FT^{-1}[g]$ gives c_k = fn of $g(f_k)$:

- Get M frequency points with an M-point sample window

Optimal filtering

- Usually system introduces *noise* as well as distortion of signals
  - Measured signal is c(t) = s(t) + n(t)   (where s=u*r)
- We want an *optimal filter* ϕ(f) which removes noise and recovers u(t) via deconvolution of system response R(f)

  $s(t) = FT^{-1}[C(f)*\phi(f)]$

  $U(f) = S(f)/R(f) = C(f)\phi(f)/R(f)$

  - Unlike R(f), we cannot determine noise precisely (noise = *stochastic* process)
    - Cannot find exact ϕ(f) directly, like R(f)
    - Estimate U(f) using (e.g.) least squares (LSQ) criterion:

$$\tilde{U}(f) \approx U_{TRUE} \text{ in sense of LSQ} \rightarrow \quad \text{minimize} \int_{-\infty}^{+\infty} \left| \tilde{U}(f) - U_{TRUE} \right|^2 df$$

$$\left| \tilde{U}(f) - U_{TRUE} \right|^2 = \left| \frac{(S+N)\varphi}{R} - \frac{S}{R} \right|^2$$

Notice that

n(t) and s(t) are *uncorrelated* by definition (else n(t) is not noise!)

Optimal filtering

to minimize $\int_{-\infty}^{+\infty} \left| \tilde{U}(f) - U_{TRUE} \right|^2 df = \int_{-\infty}^{+\infty} \left| \frac{(S+N)\varphi}{R} - \frac{S}{R} \right|^2 df$

$$= \int_{-\infty}^{+\infty} \frac{1}{|R|^2} \left\{ |S|^2 |\varphi - 1|^2 + |N|^2 |\varphi|^2 \right\} df$$

(n(t) and s(t) are uncorrelated, so cross terms integrate to 0)

Minimize integrand: $\frac{\partial}{\partial \varphi} \left\{ |S|^2 |\varphi - 1|^2 + |N|^2 |\varphi|^2 \right\} = 0$

$$\rightarrow \varphi(f) = \frac{|S(f)|^2}{|S(f)|^2 + |N(f)|^2}$$

- Notice ϕ(f) does not depend on R(f)

– Problem: we need S(F) and N(F) but have only the FT of their **sum**, C(f)=FT[s(t)+n(t)]

Noise spectra

- Must make an *hypothesis* about the form of n(t)
  - Typically choose one of:
    - Flat spectrum (white noise, Johnson* noise)
    - Exponential spectrum (pink noise, "1/f noise")
    - Power-law noise (eg, $1/f^2$ = red noise)

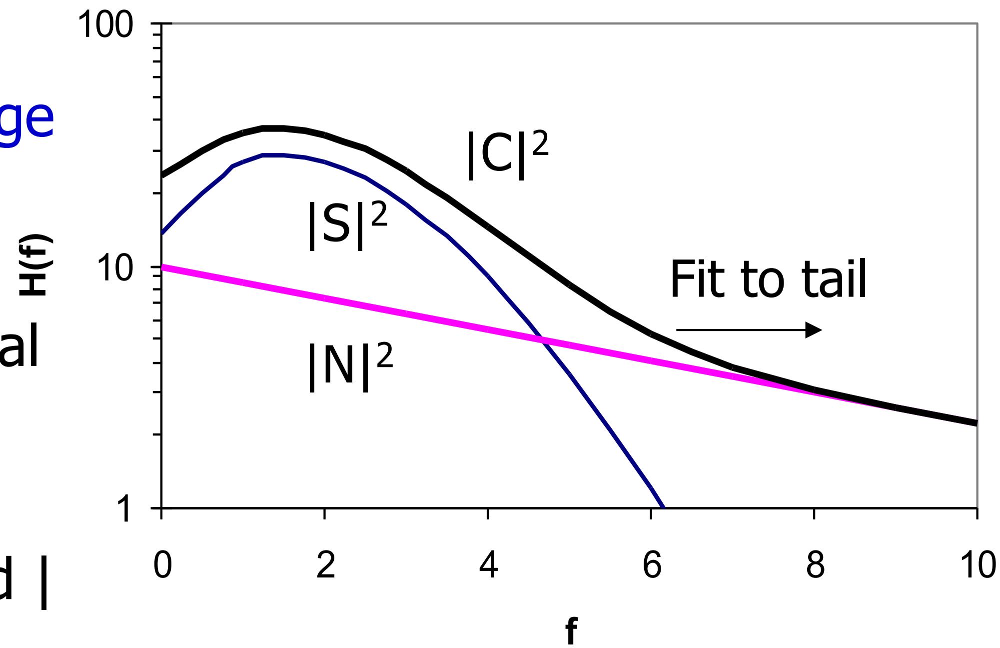
      (Power of f = 0 for white, 1 for pink, 2 for red)

* first observed by J. Johnson in 1926. He described his findings to H. Nyquist, who explained them - both worked at Bell Labs.

- Study the spectrum of raw measurements C(f):

Usually signal S(f) has limited f range

- Fit hypothesis to tail of C, where N dominates
- Extrapolate fit back into signal range to estimate N
- Estimate $|S|^2 = |C|^2 - |N|^2$
- Use fitted $|N|^2$ and estimated $|S|^2$ to find $\phi(f)$



Representing spectra

- Recall that for N samples $\{c_j\}_N$ we get

$$FT[c(t)] = C_k = \sum_{j=0}^{N-1} c_j \exp(i2\pi jk/N)$$

C_k is *complex amplitude* at frequencies

$$f_k = \frac{k}{N\Delta} = 0 \ldots f_{NYQUIST} = \frac{1}{2\Delta}$$

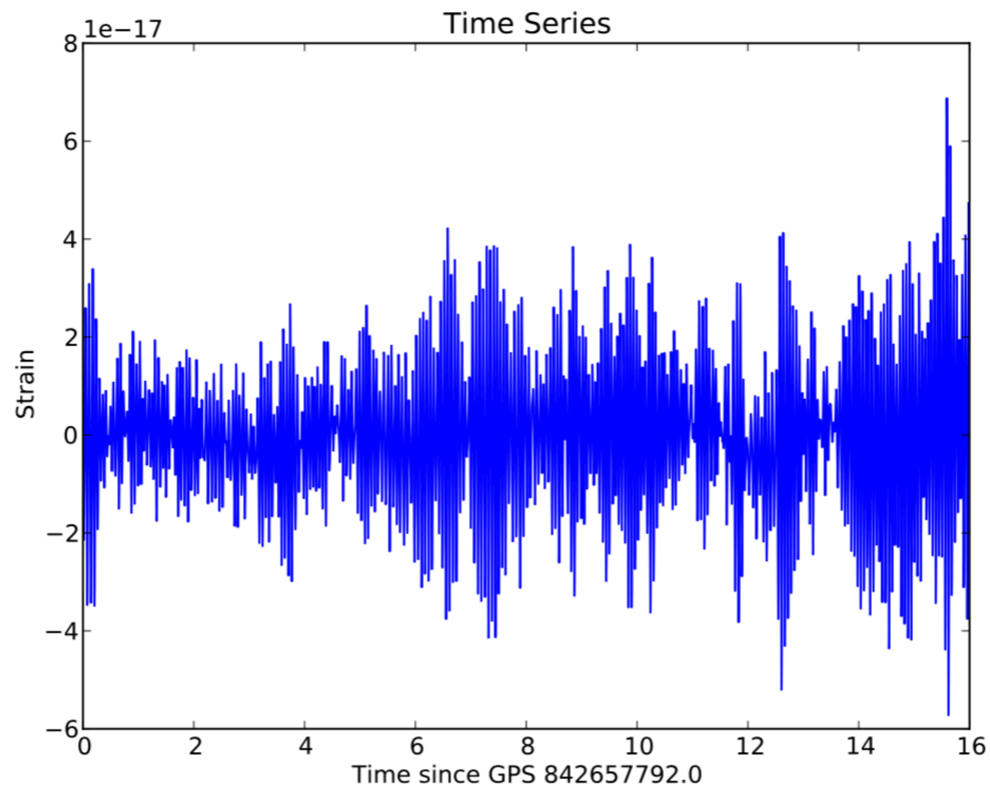
- How to describe the "power" at a given f_k?
  - Simple intensity calculation (plot = "periodogram")

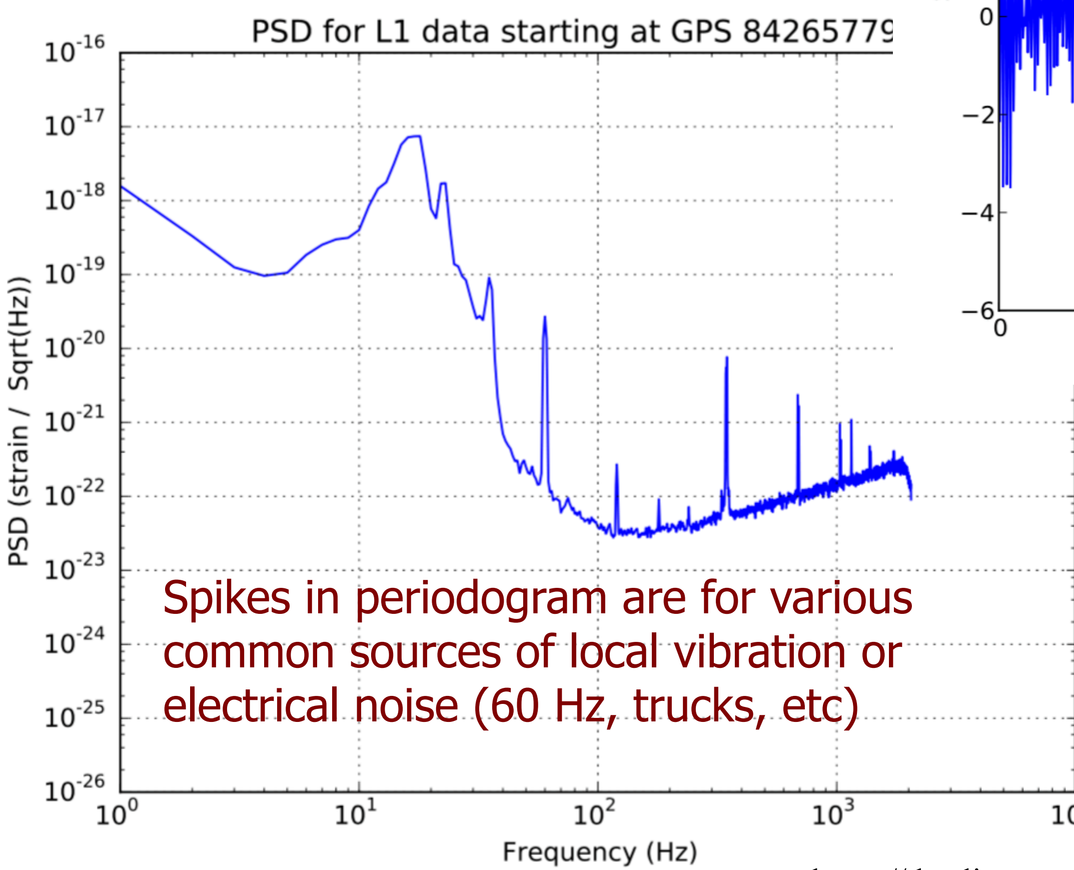
$$P_k(f_k) = \frac{1}{N^2}\left\{\left|C_k\right|^2 + \left|C_{N-k}\right|^2\right\}$$

Note: C_{N-k} term is absent for $k = 0$ and $N/2$

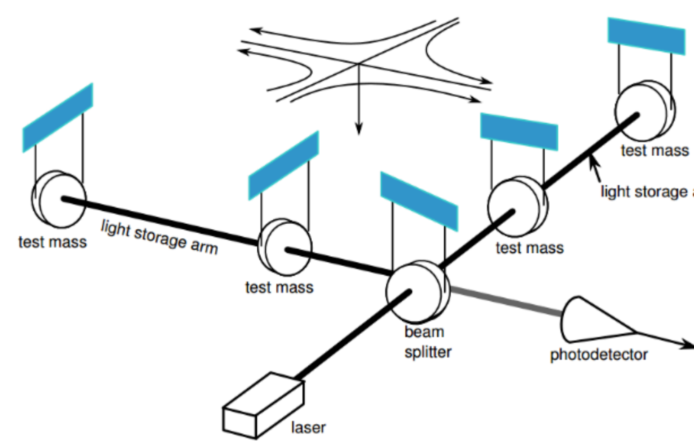
Representing spectra

- Example: random example of the signal stream and corresponding periodogram from LIGO gravitational wave detector at Hanford, WA





Spikes in periodogram are for various common sources of local vibration or electrical noise (60 Hz, trucks, etc)



https://dcc.ligo.org/public/0118/T1500123/001/SURFpaper.pdf

Representing spectra

Parseval's theorem says

$$\sum \left|c_j(t)\right|^2 = \frac{1}{N}\sum \left|C_k(f)\right|^2 \rightarrow N^2 \sum P_k(f_k) = N \sum \left|c_j(t)\right|^2$$

So $\sum P_k(f_k) = \frac{1}{N}\sum \left|c_j(t)\right|^2$ i.e.,

$\sum P_k(f_k) =$ mean squared amplitude of signal in t-domain

$$\rightarrow \sum P_k(f_k) = \int_0^T \left|c(t)\right|^2 dt$$

– Note we get discrete C_k, not a continuous C(f)
  - Sampled data → sampled spectrum
– Each C_k contains "power" (area under C(f)) for a *bin* in frequency
  - Like histogram vs probability density function

Frequency resolution vs bandwidth

- Fourier analysis reminds us of “conservation of information”: you never get something for free
  - f_c (Nyquist f) depends only on sampling *rate:* $f_c=1/2\Delta$
  - Number of frequencies sampled within 0 –> f_c (frequency resolution) depends only on number of samples *N*
  - So:
    - Increasing length of sample does not improve bandwidth
      - For $\Delta = 1\ \mu$s, f_c=0.5 MHz regardless of whether we take 1 sec or 1 year of data
    - Increasing rate of sampling does not improve frequency resolution of the spectrum
      - For 100 samples, we get 100 points on the spectrum, whether we sample at 1 Hz or 1 MHz
    - Neither increasing rate nor sample size improves accuracy of continuous spectrum estimation from discrete spectra

Frequency resolution vs bandwidth

- We can improve accuracy or resolution, but not both
  - "It can be shown" that variance on spectrum estimate is $P^2_k(f_k)$, so σ = 100%, regardless of N or Δ
    - Trick to improve accuracy:

      We can break N samples into K distinct sets of M (so MK=N) and find spectrum for each set, then average over K estimates
    - Breaking set of N samples into K distinct sets gives independent subsamples, so
      - error on mean is ~ sqrt(K)*sigma
      - Improved accuracy, at cost of f resolution