PHYS 536

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Session 5

Autocorrelation

Windowing for limited signal samples

Waves in strings and bars

1/17/2023

Course syllabus and schedule – first part…

See : http://courses.washington.edu/phys536/syllabus.htm

Session	date	Day	Readings:	K=Kinsler, H=Heller	Topic
1	3-Jan	Tue	K ch. 1	H: Ch. 1, 2	Course intro, acoustics topics, overview of wave properties; pulses, transverse and longitudinal waves, overview of sound speeds
2	5-Jan	Thu	K ch. 1	H: Ch. 9, 10	harmonic oscillators: simple, damped, driven; complex exponential solutions, electrical circuit analogy, resonance, Q factor
3	10-Jan	Tue	K ch. 1	H: Ch. 3	Fourier methods: Fourier series, integrals, Fourier transforms, discrete FTs, sampling and aliasing
4	12-Jan	Thu	K. chs 10	H: Ch. 4, 11	Frequencies and aliasing; convolution and correlation; discrete convolution; digital filtering, optimal filters, FIR filters, noise spectra; power spectra. REPORT 1 PROPOSED TOPIC DUE
5	17-Jan	Tue	K. ch. 2, 3, 4	H: Ch. 13, 15	waves in strings, bars and membranes; Acoustic wave equation; speed of sound; Harmonic plane waves, intensity, impedance.
6	19-Jan	Thu	K. Ch. 5, 6	H: Ch. 1	Spherical waves; transmission and reflection at interfaces
7	24-Jan	Tue	K. Ch. 8	H: Ch. 7	Radiation from small sources; Baffled simple source, piston radiation, pulsating sphere;
8	26-Jan	Thu	K: Ch. 10	H: Chs. 13-15	Near field, far field; Radiation impedance; resonators, filters
9	31-Jan	Tue	K. Ch. 9-10	H: Chs. 16-19	Musical instruments: wind, string, percussion
10	2-Feb	Thu	K. Ch 14		Transducers for use in air: Microphones and loudspeakers
11	7-Feb	Tue	K. Ch 11	H: Chs. 21-22	The ear, hearing and detection
12	9-Feb	Thu	K. Chs 5,11		Decibels, sound level, dB examples, acoustic intensity; noise, detection thresholds. REPORT 1 PAPER DUE by 7 PM; REPORT 2 PROPOSED TOPIC DUE

Tonight

Announcements

- You can access most scientific journals, many popular journals, and books online via the UW library – no need to be on campus
  - See http://www.lib.washington.edu/help/connect
  - See also http://www.lib.washington.edu/help/connect/husky-onnet for how to VPN onto campus network

- Revision in posted problem set
  - The version of problem 11 posted is too complicated and difficult (and I won't cover the details needed in class)
  - I've replaced it with:

11. A steel bar of cross section $0.0001m^2$ and 0.25m length is clamped at both ends. a) what is its fundamental frequency for longitudinal vibrations? b) what is the fundamental frequency for the same bar but free at both ends?

Autocorrelation and cosine averaging theorem

- If the signal is a sum of sin/cos functions only, the autocorrelation is easy to compute:

$$\mathrm{corr}(g,g) = \int_{-\infty}^{+\infty} g(t'+t)g(t')\,dt' \quad = \textit{autocorrelation} \text{ of } g$$

$$\text{if} \quad g(t) = \sum_i a_i \cos(\omega_i t + \phi_i), \quad \text{apply cosine averaging theorem:}$$

$$\left\langle \cos(\omega_1 t + \phi_1)\cos(\omega_2 t + \phi_2) \right\rangle = 0 \;\; \text{if} \;\; \omega_1 \neq \omega_2 \quad \langle \text{means average over time} \rangle$$

$$= \frac{1}{2}\cos(\phi_2 - \phi_1) \;\; \text{if} \;\; \omega_1 = \omega_2$$

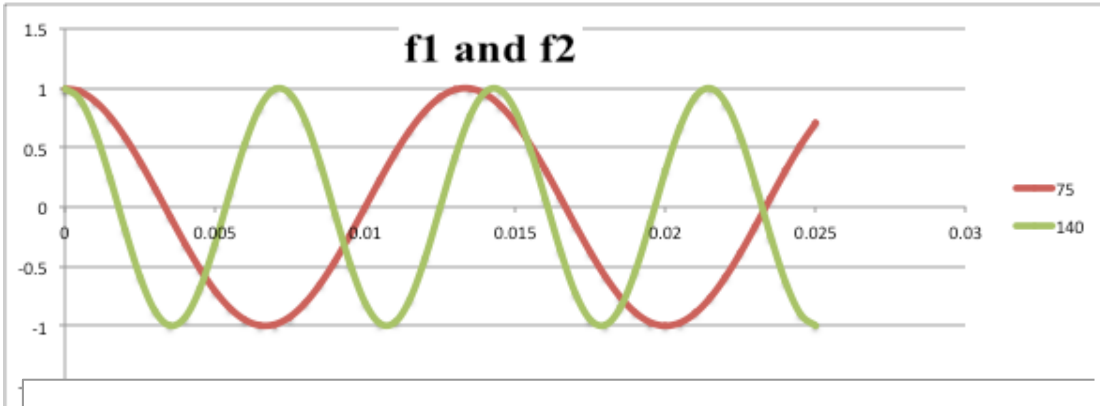
  - Since correlation integral amounts to a time average, "it can be shown" that

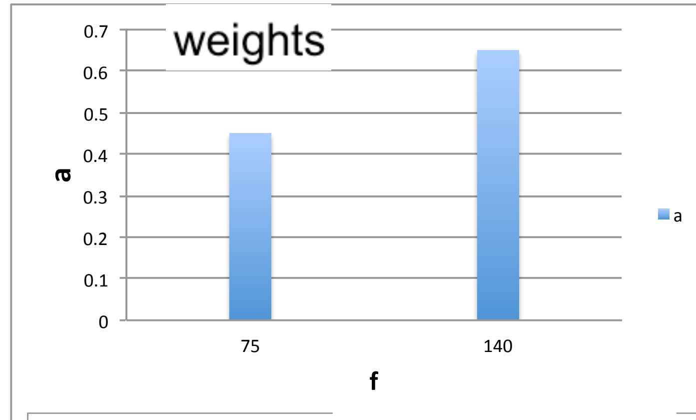
$$\text{for} \;\; g(t) = \sum_i a_i \cos(\omega_i t) \quad \rightarrow \quad \mathrm{corr}(g,g) = \frac{1}{2}\sum_i a_i^2 \cos(\omega_i t)$$

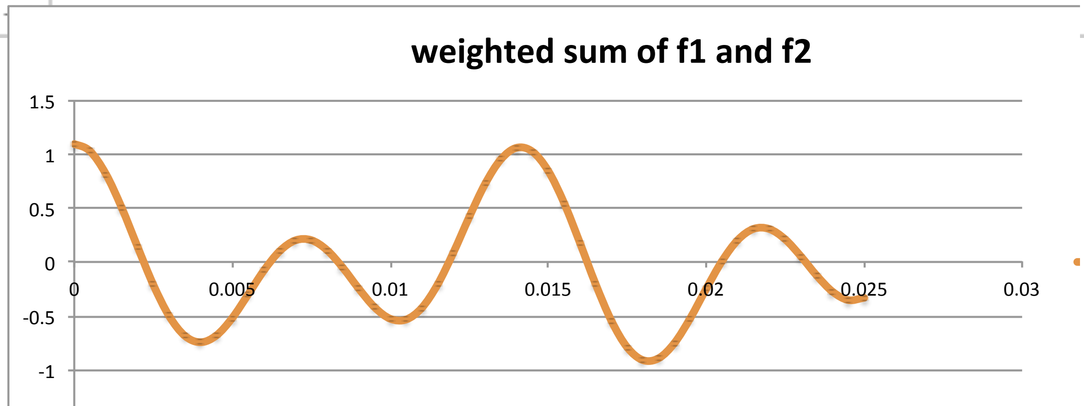
  - So, if signal is a sum of sinusoids of different frequencies, its power spectrum can provide the a_i^2 values (weights) to construct its autocorrelation, or vice versa
    - Can't reconstruct original signal from a_i^2 values – correlation → information loss (sign of a_i)

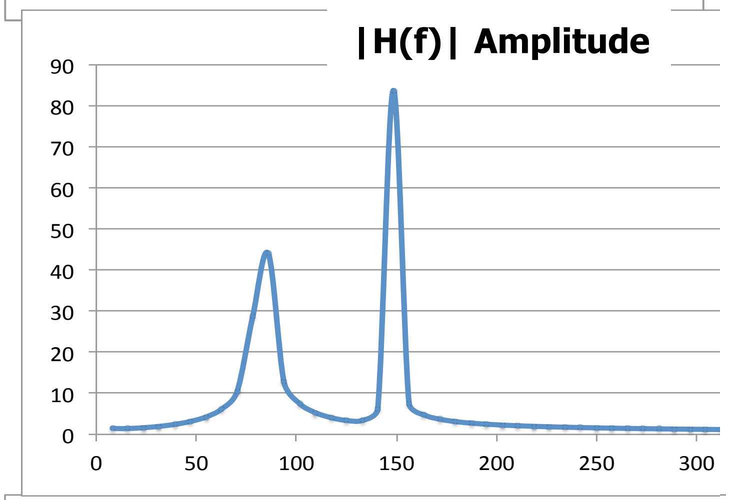
Autocorrelation and cosine averaging theorem

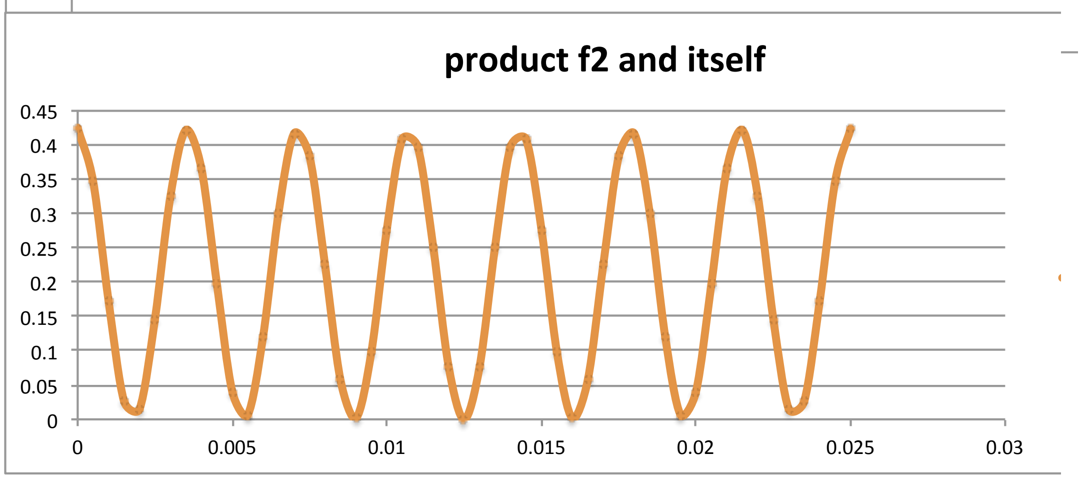
- Example: f=75Hz + 140Hz

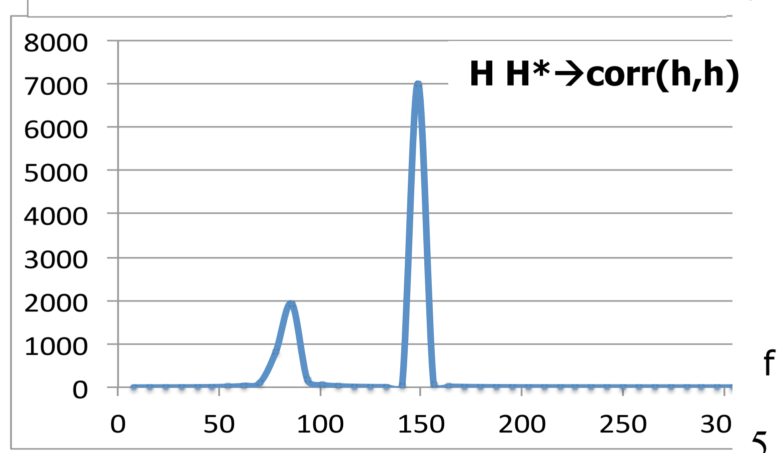








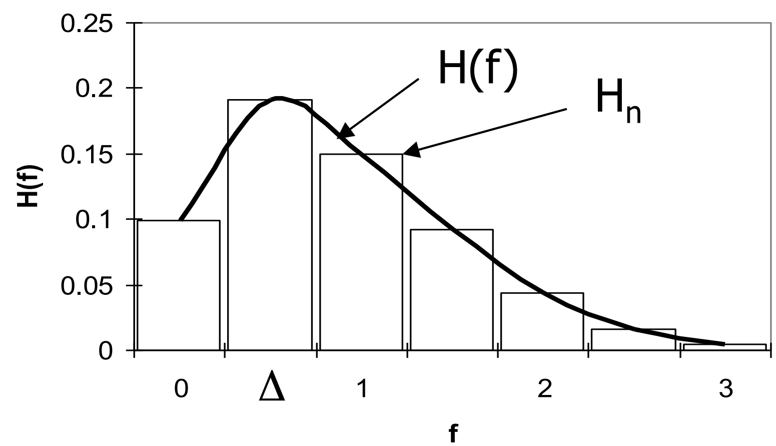




Windows and spectra

- Interpret content of probability histogram bin p_j as *average* of a continuous p(x) over a *uniformly weighted* window Δx

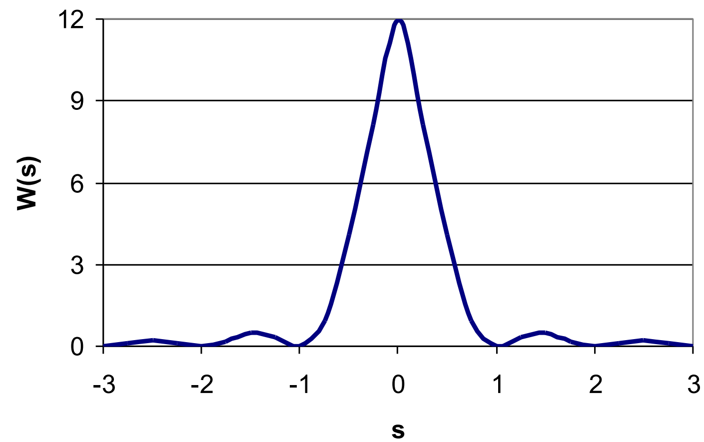
$$p_j = \frac{n_j}{N} \cong \int_{x_j}^{x_j+dx} p(x)dx$$



- Apply same basic idea to spectra:

  P_k=average value of C(f) around f_k

  – But window weight is NOT uniform for spectra:
    - Want uniform weight (constant=1.0) over one full period T in time domain
    - But FT of constant in t-domain= sinc function in f



Windows and spectra

$$Define \quad s = f - f_k$$

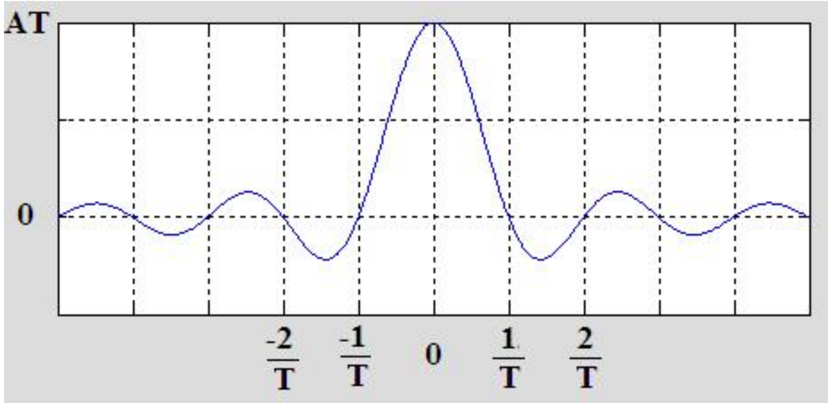
FT of constant weight w(t) in t-domain
→ sinc function W(s) in frequency domain

$$w(t) = 1.0 \rightarrow W(s) = \frac{1}{N^2}\left|\sum \exp(i2\pi ks/N)\right|^2 = \frac{1}{N^2}\left[\frac{\sin(\pi s)}{\sin(\pi s)/N}\right]^2$$

Weighted windows

- Lobes of $sinc^2$ function in W(s) mean
  nearby frequencies outside each bin also contribute to $C_k(f_k)$
- Note: for s=integer ($f=nf_k$), W(s)=0
  - No leakage if spectrum is pure sinusoids (discrete spectrum with fundamental f = sample range)

To minimize "leakage" into adjacent bins, replace uniformly weighted bins (*square window*) with some kind of peaked weighting that minimizes side lobes in the FT



Weighted windows

For discrete FT $C_k = \sum_{j=0}^{N-1} c_j \exp(i2\pi jk/N)$

Let $D_k = \sum_{j=0}^{N-1} c_j w_j \exp(i2\pi jk/N)$,

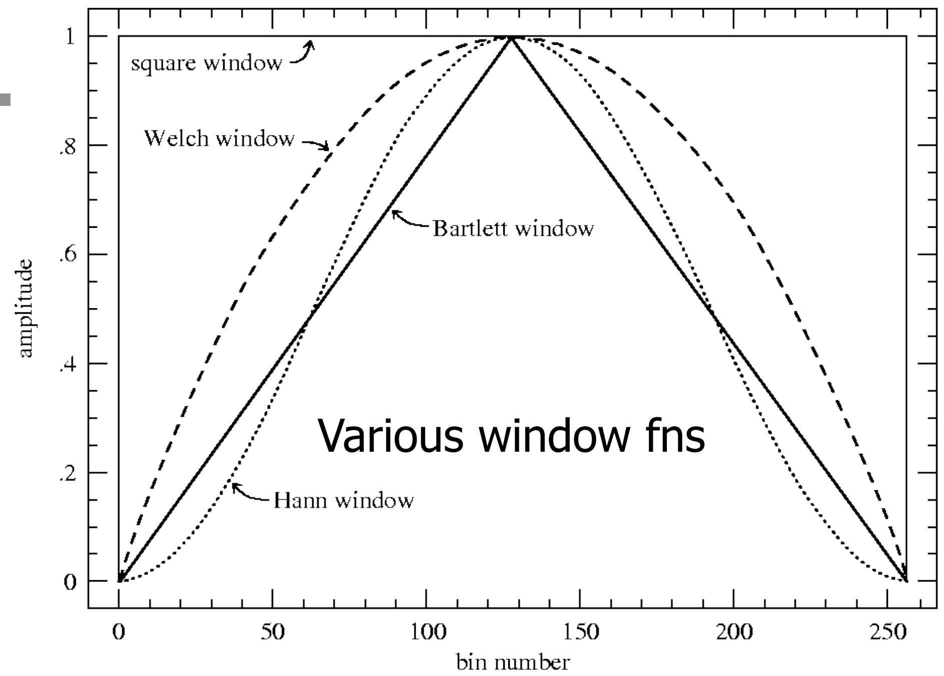
$w = weight$; Then for $W = N\sum_{j=0}^{N-1} w_j^2$

(sum-of-squares window normalization)

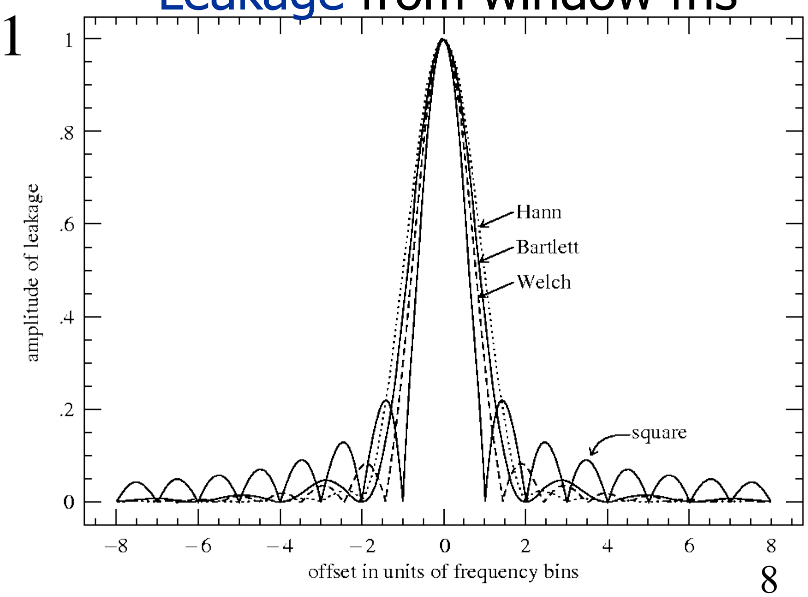
$$P_k(f_k) = \frac{1}{W}\left\{|D_k|^2 + |D_{N-k}|^2\right\} \quad \text{for } k = 1\ldots(N/2)-1$$

$$P_k(f_k) = \frac{1}{W}|C_k|^2 \quad \text{for } k = 0 \text{ and } (N/2)$$

where $f_k = 2f_c\dfrac{k}{N} \quad f_c = \dfrac{1}{2\Delta}$



Leakage from window fns



Windowing

For audio signal analysis

- Almost always have limited sample of a long signal
- Human ear also samples in chunks – properly windowed audio spectrum seems more ‘faithful’
- Side lobes correspond to ‘crosstalk’ between frequencies

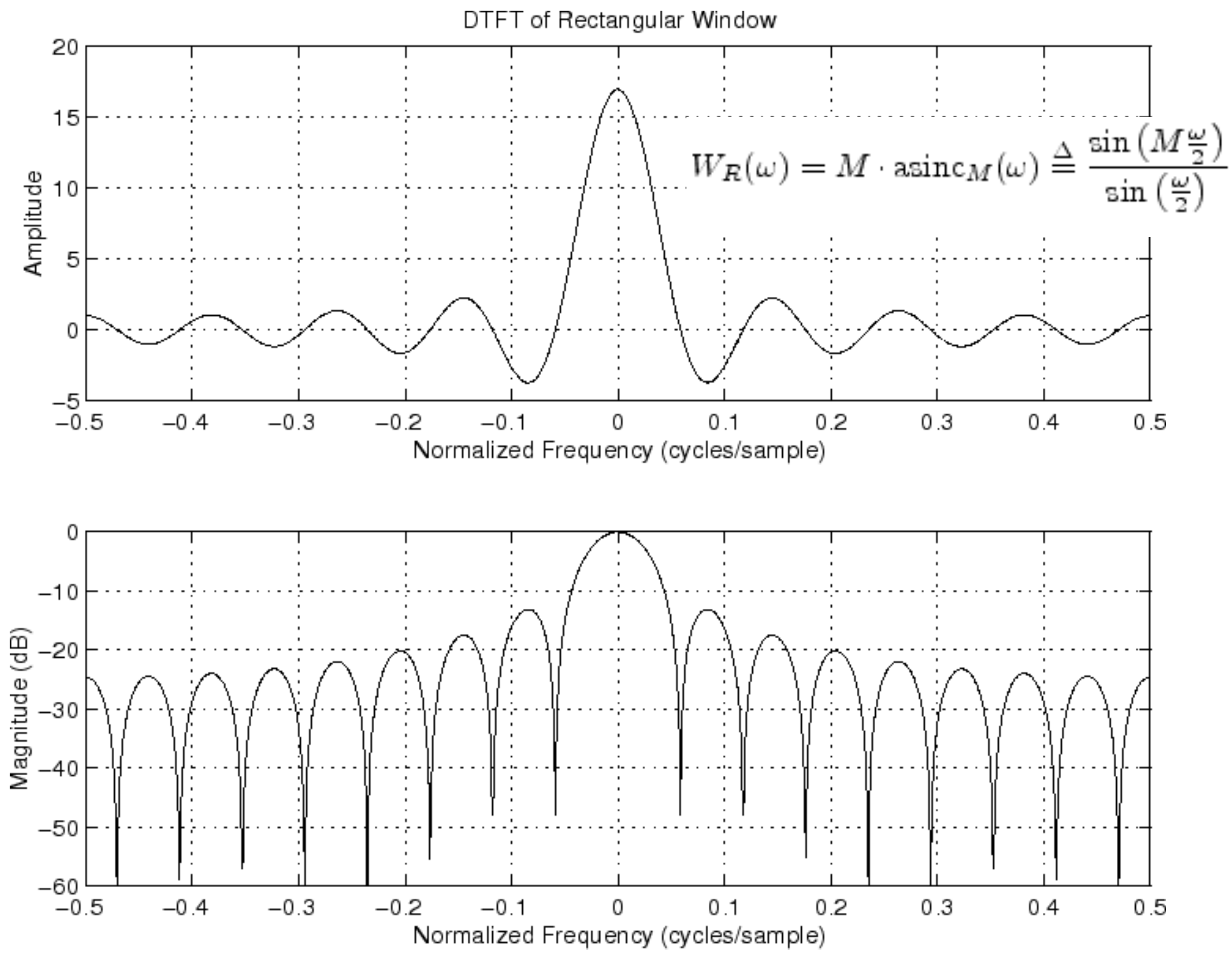
Examples of time-window / frequency domain pairs:

- Rectangular window
- Hamming window
- Gaussian window

Windowing

Rectangular window

- As N increases, the main lobe narrows (better frequency resolution).
- M has *no effect on the height of the side lobes*
- First side lobe only 13 dB down from the main peak.
- Side lobes roll off at approximately 6dB per octave.



In these and following figs, “M” = N

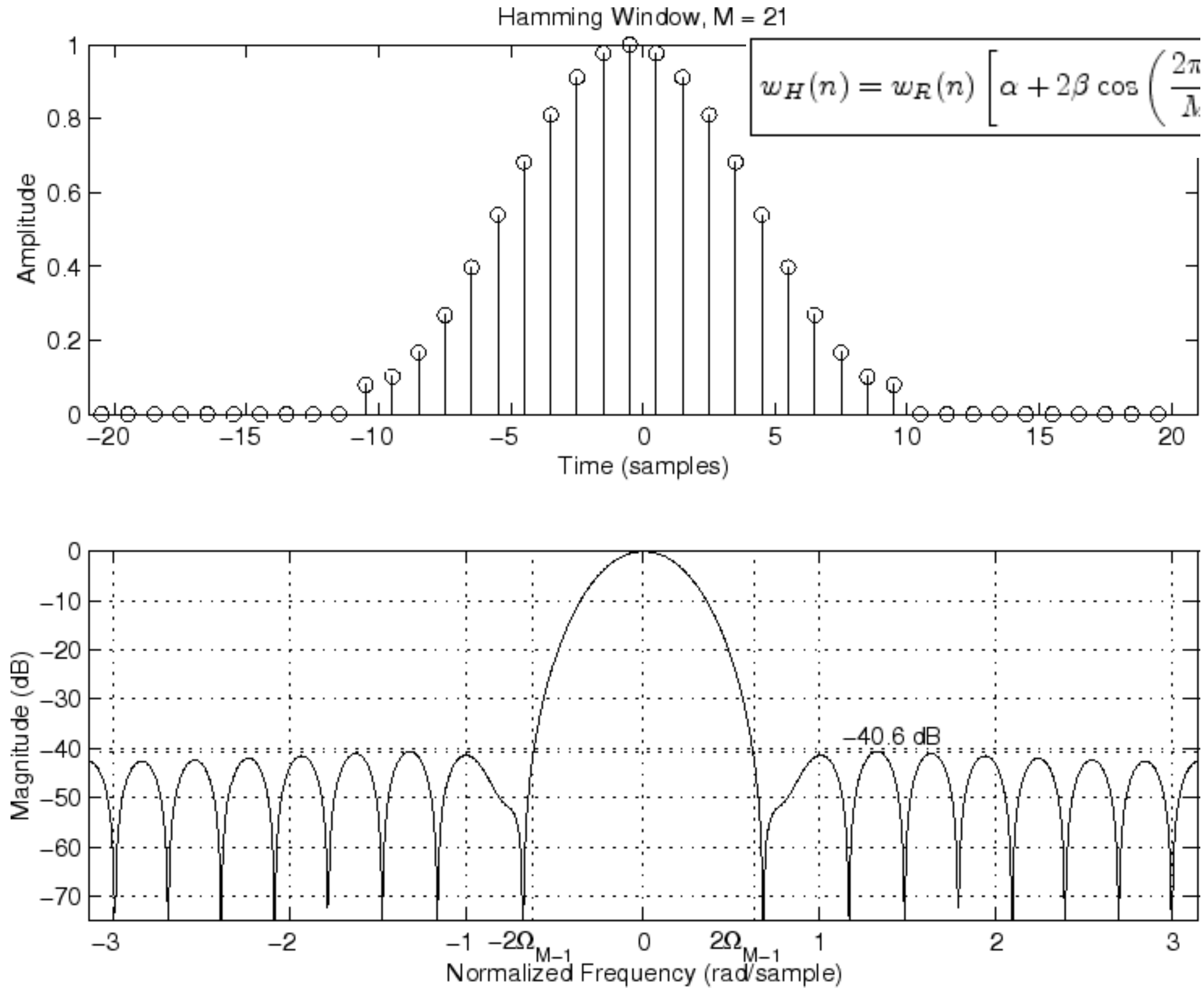
http://ccrma.stanford.edu/~jos/sasp/

More windows

Hamming window

w ~ $\alpha + \beta \cos(2\pi n/M)$

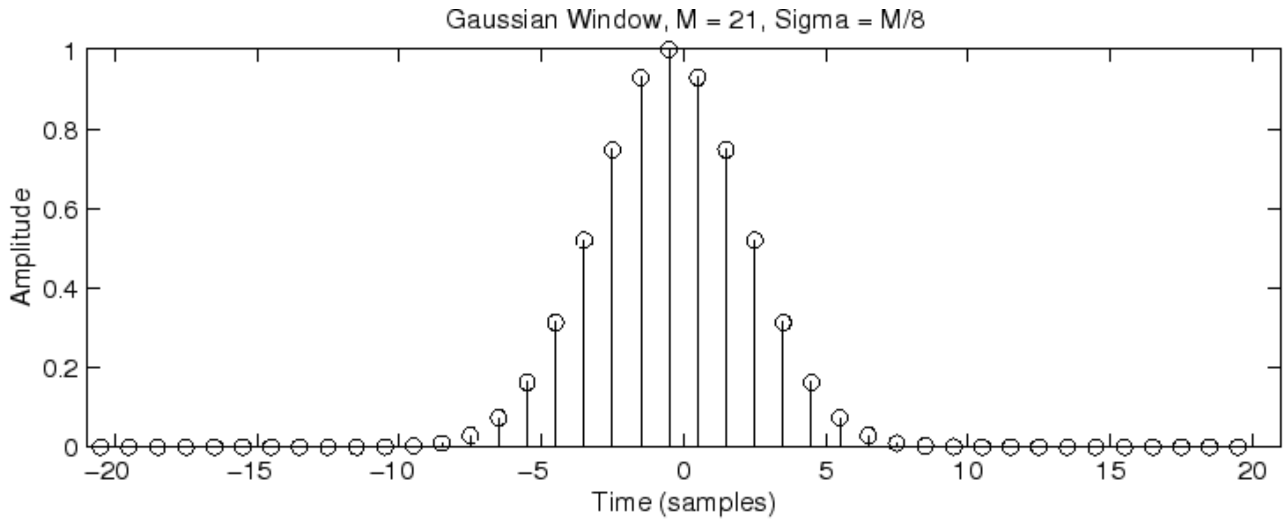
- Choose α=0.54 to cancel largest sidelobe (β=1-α)
- Discontinuous “slam to zero” at endpoints.
- Asymptotic roll-off is approximately -6 dB per octave
- Side lobes are close to “equal ripple”
- First side lobe is 41 dB down!

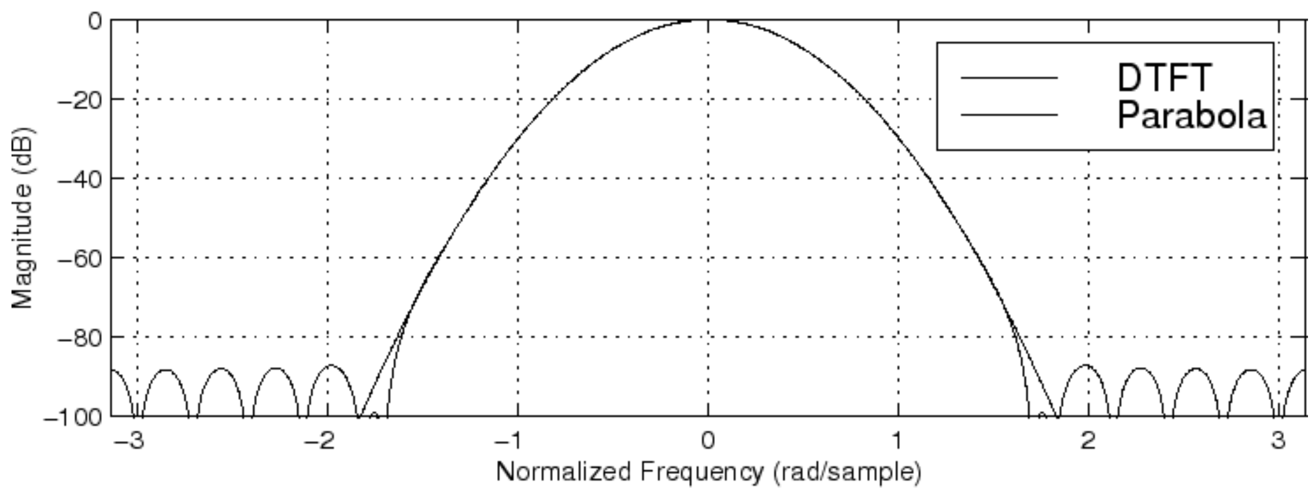


(Hann window = same but with α=1/2, β=1/4 : side lobes roll off gradually)

More windows

- Gaussian window
  - Side lobes way down (80 dB for example, σ=N/8)
  - Main lobe well represented by a simple parabola in f





$$\frac{1}{\sigma\sqrt{2\pi}}e^{-t^2/2\sigma^2} \leftrightarrow e^{-\omega^2/2(1/\sigma)^2}$$

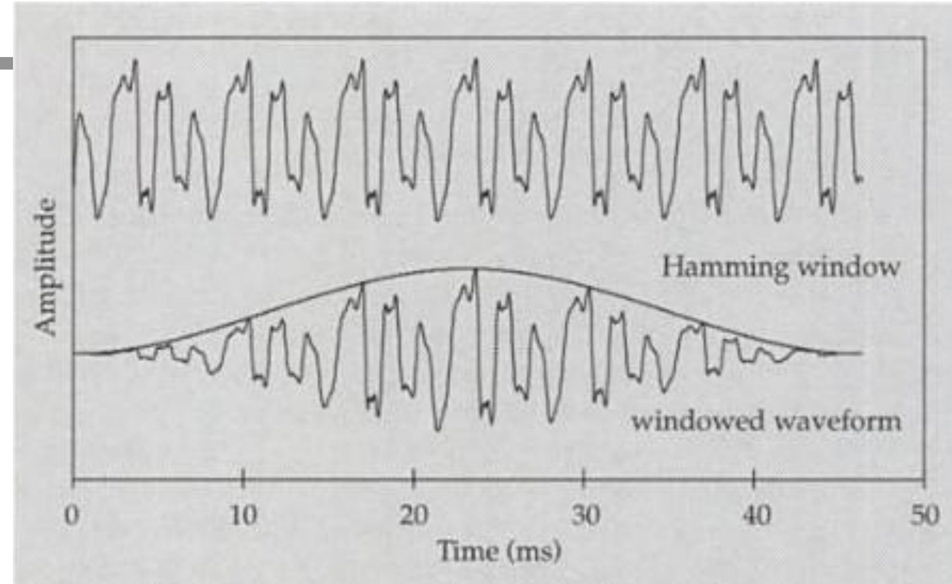
Gaussian FT[Gaussian]

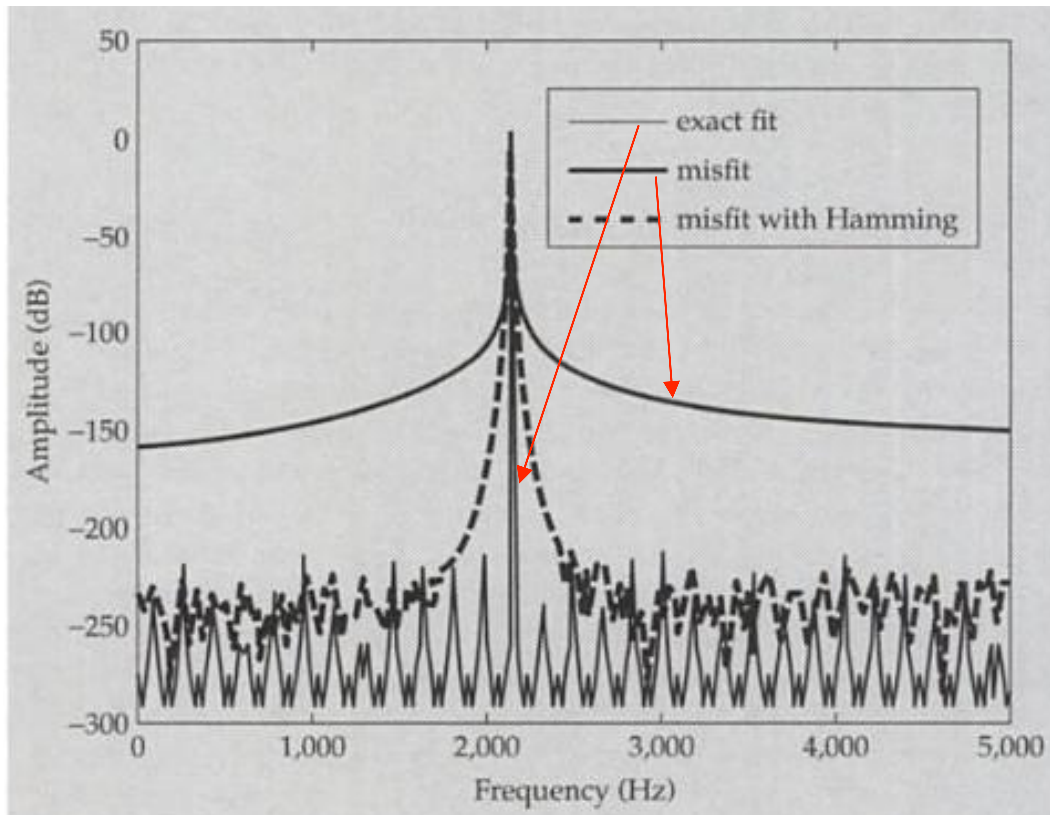
Examples (from *Acoustic and Auditory Phonetics* K. Johnson, Wiley-Blackwell, 2005)

Top: Upper= raw signal; lower= Hamming-weighted signal

Bottom: Discrete sampled power spectrum (signal consists of pure sinusoids)

1. Exact fit: sampling window length = integer multiple of signal period T
2. Misfit: sample window is slightly shorter than nT: mismatch
3. Hamming: same signal as 2 showing improved results from windowing – peak is wider, but S/N is about the same as for exact fit





Waves in strings, in more detail

- Transverse waves on a string
  - Mechanics of tension

$$df_y = T(x+dx)\sin\theta - T(x)\sin\theta$$

apply Taylor expansion: $\rightarrow f(x+dx) = f(x) + \dfrac{\partial f(x)}{\partial x}dx + \dfrac{\partial^2 f(x)}{\partial x^2}dx^2 + \cdots$

$$df_y = \left(T(x)\sin\theta + \frac{\partial T(x)\sin\theta}{\partial x}dx + \cdots\right) - T(x)\sin\theta$$

for small θ, $\sin\theta \sim \tan\theta = \dfrac{\partial y}{\partial x} \rightarrow df_y = \dfrac{\partial}{\partial x}\left(T\dfrac{\partial y}{\partial x}\right)dx = T\dfrac{\partial^2 y}{\partial x^2}dx$

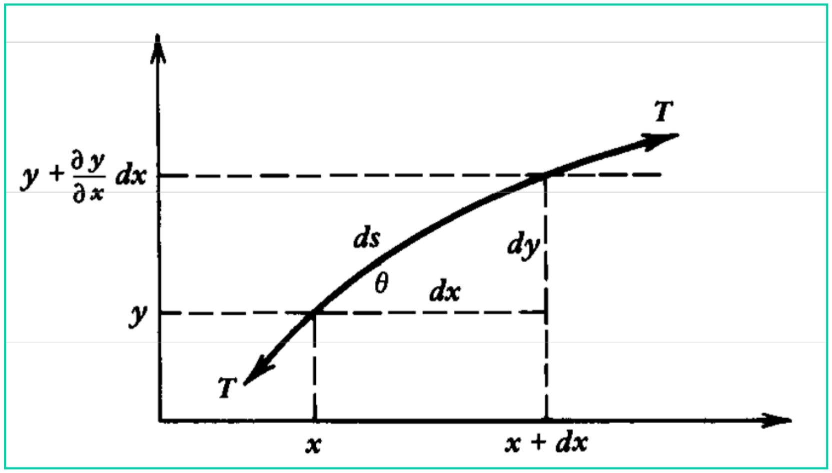
mass density of string $=\rho_L \rightarrow m = \rho_L dx$

$$F = ma \rightarrow df_y = \rho_L dx\left(\frac{\partial^2 y}{\partial t^2}\right) \rightarrow T\frac{\partial^2 y}{\partial x^2} = \rho_L\left(\frac{\partial^2 y}{\partial t^2}\right)$$

so $df_y = a\ dm \rightarrow \dfrac{\partial^2 y}{\partial x^2} = \dfrac{1}{c^2}\left(\dfrac{\partial^2 y}{\partial t^2}\right)$

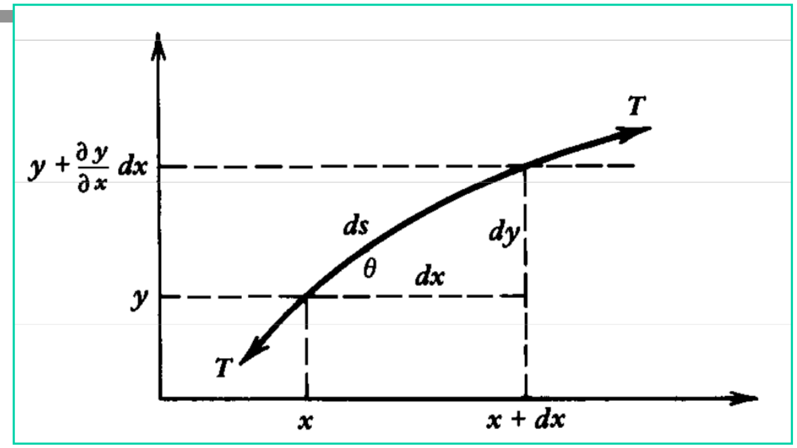
where $c = \sqrt{T/\rho_L}$

Wave eqn for a string



Waves in strings, in more detail

- Transverse waves on a string
  - Solving the wave equation



$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\left(\frac{\partial^2 y}{\partial t^2}\right) \quad \text{with} \quad c^2 = T/\rho_L$$

try $y(x,t) = f(ct \pm x)$: $\quad y(x,t) = y_1(ct - x) + y_2(ct + x)$

for the first term y_1 only, $\dfrac{\partial y_1}{\partial t} = \dfrac{\partial y_1}{\partial (ct - x)} \dfrac{\partial (ct - x)}{\partial t} = c \dfrac{\partial y_1}{\partial (ct - x)}$

let $w = (ct - x)$; apply $\dfrac{\partial}{\partial t}$ again: $\left(\dfrac{\partial^2 y_1}{\partial t^2}\right) = c^2 \dfrac{\partial^2 y_1}{\partial w^2}$

similarly for $\dfrac{\partial y_1}{\partial x}$, we get $\left(\dfrac{\partial^2 y_1}{\partial x^2}\right) = \dfrac{\partial^2 y_1}{\partial w^2}$

so $\dfrac{\partial^2 y_1}{\partial x^2} = \dfrac{1}{c^2} \dfrac{\partial^2 y_1}{\partial t^2}$ -- check! So *any* fn of $(ct - x)$ is a solution of the wave eqn

Repeat for $y_2(ct + x)$: same deal, any fn of $(ct + x)$ is also a solution

Waves in strings: reflections

Reflections at ends: 2 cases, determined by end conditions

- String is rigidly held at x=0 (clamped end)
  - Then for any time t, at x=0

$$y(x,t) = y_1(ct - x) + y_2(ct + x)$$
$$y(0,t) = 0 = y_1(ct) + y_2(ct) \to y_2(ct) = -y_1(ct)$$
$$\text{so } \; y(x,t) = y_1(ct - x) - y_1(ct + x)$$

  - This is the original y_1 plus an inverted duplicate moving in the opposite direction: a reverse-polarity reflected wave
- String is unconstrained in y at x=0 (free end)
  - Then for any time t, at x=0

$$F_y = 0 \to T(x)\sin\theta = \frac{\partial y(0)}{\partial x} = 0 \to \frac{\partial y_1(0)}{\partial x} + \frac{\partial y_2(0)}{\partial x} = 0$$
$$\frac{\partial y_1}{\partial x} = -\frac{\partial y_1}{\partial (ct - x)}, \quad \frac{\partial y_2}{\partial x} = +\frac{\partial y_2}{\partial (ct + x)} \to -\frac{\partial y_1(0)}{\partial (ct)} + \frac{\partial y_2(0)}{\partial (ct)} = 0$$
$$\int_0^t \partial y_1(0)\; d(ct) = \int_0^t \partial y_2(0)\; d(ct) \to y_1(ct) = y_2(ct)$$
$$\text{So } \; y(x,t) = y_1(ct - x) + y_1(ct + x)$$

  - This is the original y_1 plus a reflected wave of the same polarity

Forced waves in strings: first, infinite

Infinite string means no reflections to deal with – simplest case

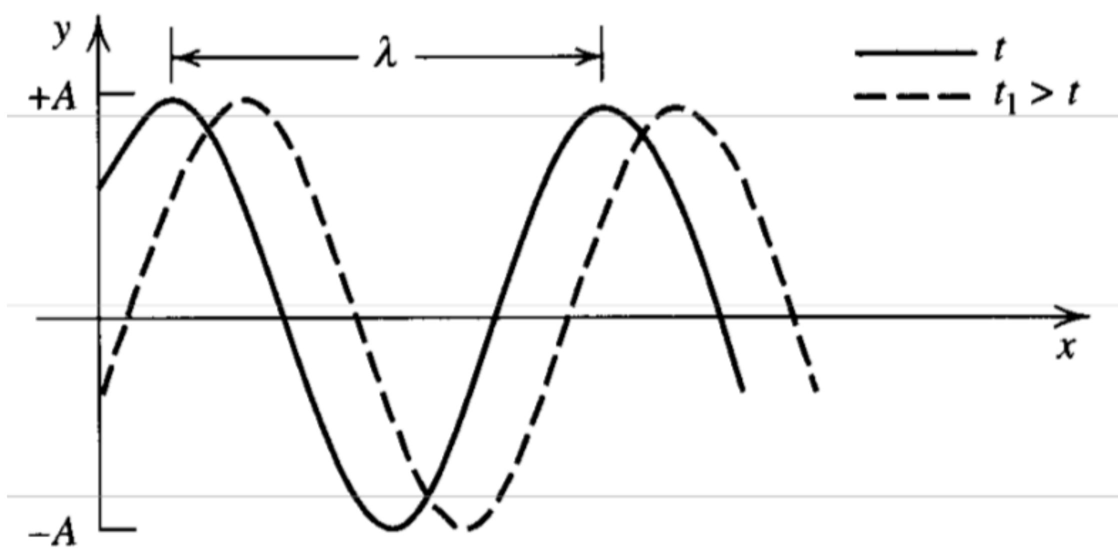
- Solution can only include waves moving in +x direction

$$y(x,t) = y_1(ct - x), \quad \text{with driving force } F_y(t) = Fe^{i\omega t}$$

$$y(0,t) = Ae^{i\omega t}; \quad y_1(0,t) = Ae^{ik(ct)} \quad (\text{wave number } k = \omega / c)$$

$$\text{so for all x, } y(x,t) = y_1(ct - x) = Ae^{ik(ct-x)} = Ae^{i(\omega t - kx)}$$

  - At each x, string oscillates in SHM with f=ω/2π and T=1/f
  - At any time, shape is sinusoidal with amplitude A, and λ=2π/ k
  - Waveform moves in +x direction with (phase) speed $c = \sqrt{T / \rho_L}$



Forced waves in infinite string

- Driving force must balance tension (there is no lump m at x=0)
  - Waveform moves in +x direction with (phase) speed

$$Fe^{i\omega t} = -T\frac{\partial y(0)}{\partial x} = -ikTAe^{i(\omega t-kx)} \rightarrow A = \frac{F}{ikT}; \quad y(x,t) = \frac{F}{ikT}e^{i(\omega t-kx)}$$

$$\text{transverse speed } u(x,t) = \frac{\partial y}{\partial t} = \frac{i\omega F}{ikT}e^{i(\omega t-kx)} = \frac{cF}{T}e^{i(\omega t-kx)}$$

$$c = \sqrt{T/\rho_L} \rightarrow u(x,t) = \frac{F}{\rho_L c}e^{i(\omega t-kx)}$$

- Recall: mechanical impedance = *F/u* so at *x=0*, impedance is

$$Z_{m(0)} = \frac{F(t)}{u(0,t)} = \frac{Fe^{i\omega t}}{\frac{F}{\rho_L c}e^{i(\omega t)}} = \rho_L c$$

Characteristic mechanical impedance of infinite string

  - Instantaneous and average power into string is

$$P(t) = \mathrm{Re}(Fu) = F\cos\omega t\left(\frac{F}{\rho_L c}\right)\cos\omega t;$$

$$\langle P \rangle_{RMS} = \frac{1}{T}\int_0^T P\,dt = \frac{F^2}{2\rho_L c} = \frac{1}{2}\rho_L c U(0), \quad U(0) = |u(0,t)|$$

Forced waves in a finite string

- More complicated – now must deal with *reflected* waves

$$y(x,t) = Ae^{i(\omega t - kx)} + Be^{i(\omega t + kx)}$$

Boundary conditions: at driven end, tension must balance driving force

so as before, $Fe^{i\omega t} + T\dfrac{\partial y(0)}{\partial x} = 0$ for all t,

insert solution: $F + T(-ikA + ikB) = 0.$

At fixed end x=L, must have $y(L,t) = 0$ for all t,

so $Ae^{-ikL} + Be^{+ikL} = 0$

solve these 2 eqns for A and B:

$$\mathrm{A} = \frac{F}{ikT}\frac{e^{ikL}}{e^{ikL} + e^{-ikL}} = \frac{Fe^{ikL}}{2ikT\cos(kL)}; \text{ and } B = \frac{Fe^{-ikL}}{-2ikT\cos(kL)}$$

$$y(x,t) = \frac{Fe^{ikL}}{2ikT\cos(kL)}\left(e^{i[\omega t + k(L-x)]} - e^{i[\omega t - k(L-x)]}\right) = \frac{F\sin[k(L-x)]}{kT\cos(kL)}e^{i\omega t}$$

The 2 versions of y(x,t) describe different pictures:

Two waves *moving* in opposite directions	or	*Stationary* envelope, oscillating in place: ***standing wave***

Forced waves in a finite string

- Standing-wave solution shows locations where y=0 for all t

$$y(x,t)=\left(\frac{F}{kT\cos(kL)}\right)\sin[k(L-x)]e^{i\omega t}, \quad k=\omega/c, \quad F=\text{driver amplitude}$$

$$y=0 \text{ when } k(L-x)=n\pi \rightarrow x_n=L-\frac{n}{2}\lambda, \quad n=0,1,2\cdots 2L/\lambda$$

$$\text{driver is at a node if } L=\frac{n}{2}\lambda, \quad \text{driver is antinode if } L=\frac{m}{4}\lambda, \ m= \text{ odd integer}$$

Amplitude blows up (resonance) when

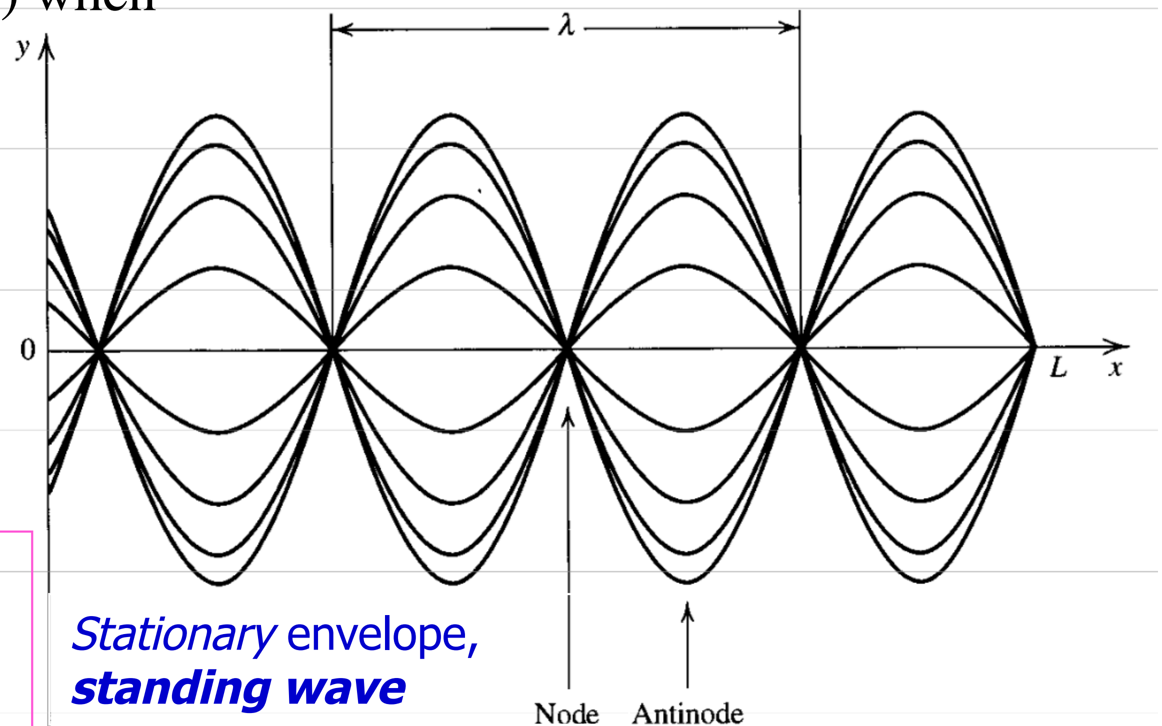
$$\cos(kL)=0 \rightarrow kL=\frac{2n-1}{2}\pi,$$

$$\omega/k=c \rightarrow f_{res}=\frac{2n-1}{4}\frac{c}{L}$$

Amplitude is minimal when

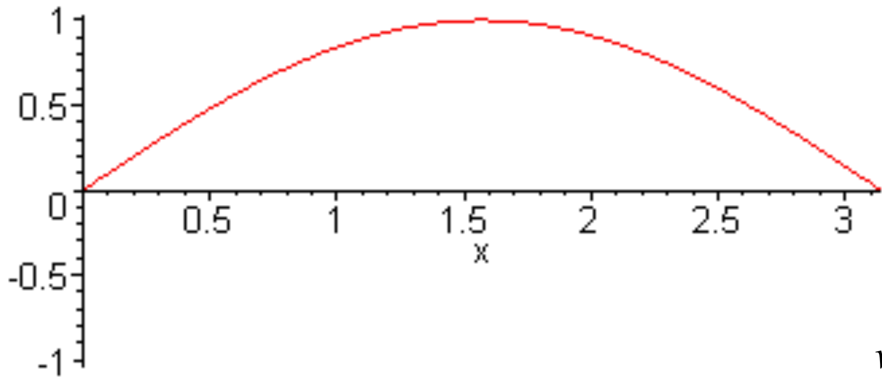
$$kL=n\pi \rightarrow f_{\min}=\frac{n}{2}\frac{c}{L}$$

Resonance amplitude is limited because when y gets too large, small-θ assumption fails



Stationary envelope, ***standing wave***

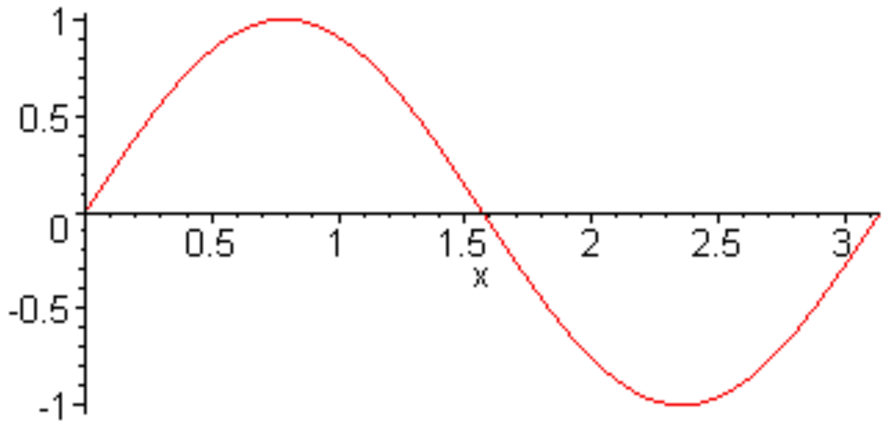
Standing waves on a string



1st harmonic (fundamental)

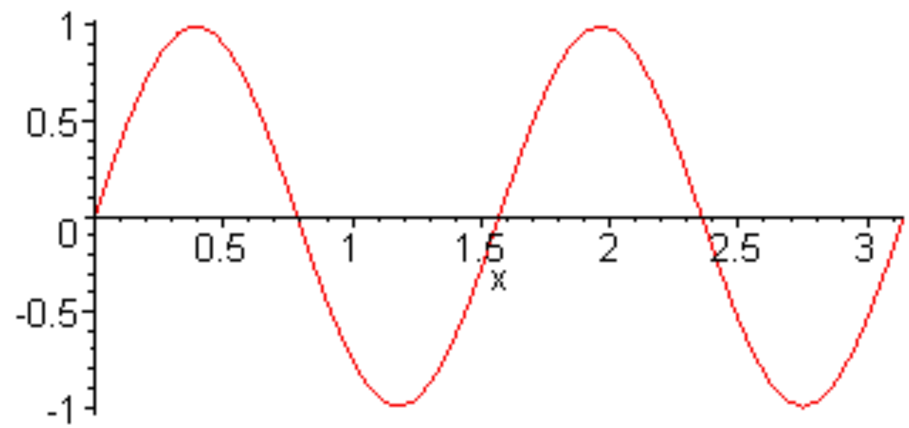
$L = \lambda/2 = \pi$ $f1 = c / (2 \cdot L)$

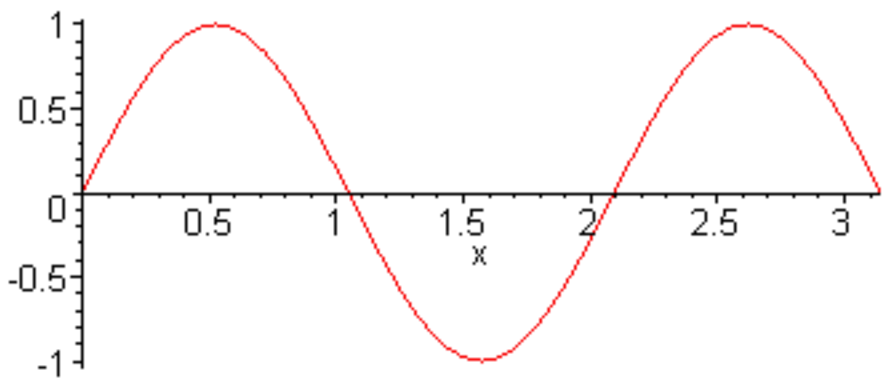
www.sengpielaudio.com/StandingWaves.htm



2nd harm. $L = \lambda$ $f2 = 2 \cdot f1 = c / L$

4th harm. $L = 2\lambda$, $f4 = 4 \cdot f1 = 4 \cdot c / (2 \cdot L)$





3rd harm. $L = 3\lambda / 2$ $f3 = 3 \cdot f1 = 3 \cdot c / (2 \cdot L)$

Impedance in a forced finite fixed string

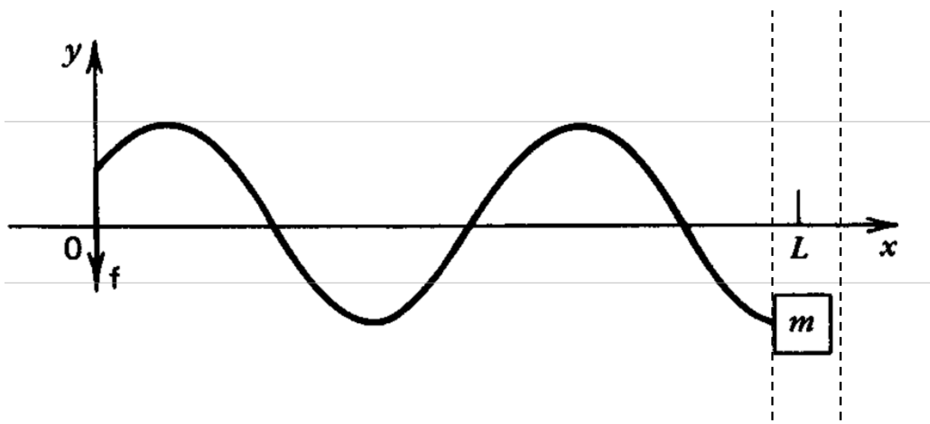
- For fixed finite string, mechanical impedance at the driver is

$$Z_{m(0)} = \frac{F(t)}{u(0,t)} = \frac{Fe^{i\omega t}}{\dfrac{i\omega F \tan(kL)}{kT} e^{i\omega t}} = \frac{kT}{i\omega \tan(kL)} = \frac{-i\,\rho_L c}{\tan(kL)}$$

- For small ω, tan(kL)~kL → $Z_{m(0)} = \dfrac{-i\,\rho_L c}{kL} = -i\left(\dfrac{T}{L}\right)\dfrac{1}{\omega}$

  (same as for spring with s=T/L)
- Notice Z is imaginary (pure reactance): rigid fixed ends

  → string has no way to lose energy, at least ideally
- Things we won't cover in lecture: (see Kinsler)
  - Other driven strings: forced mass-loaded or resistance-loaded



Mass-loaded: m is constrained to move transversely at x=L

Resistance-loaded: same picture except damper instead of m

Normal modes in a fixed-end finite string

- For fixed finite string without driver, when plucked or struck:

$$y(x,t) = Ae^{i(\omega t - kx)} + Be^{i(\omega t + kx)}, \quad k = \omega / c$$

Boundary conditions are $y(0,t) = 0$ and $y(L,t) = 0$, for all t

So, $A + B = 0 \;\rightarrow B = -A$, and

$$Ae^{-ikL} + Be^{+ikL} = 0 \rightarrow 2i\sin(kL) = 0 \rightarrow \sin(kL) = 0 \rightarrow kL = n\pi, \quad n = 1, 2...$$

- So only discrete values of k=ω/c are allowed:

$$k_n = n\pi / L \,; \;\; k = 2\pi f / c \;\rightarrow\; f_n = nc / 2L$$

- For the nth frequency,

$$y_{\mathrm{n}}(x,t) = \mathbf{A}_{\mathrm{n}} \sin(k_n x) e^{i\omega_n t}, \quad where \quad \mathbf{A}_{\mathrm{n}} = A_n + iB_n$$
$$= \left(A_n \cos(\omega_n t) + iB_n \sin(\omega_n t)\right) \sin(k_n x)$$

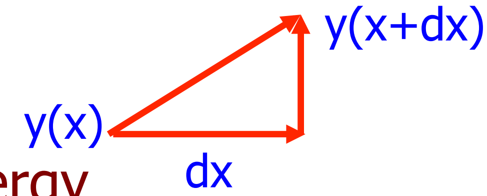
Where A and B will be determined by the initial conditions

- These are the normal modes or eigenfrequencies of the string
    - Fundamental = f_1 =c/2L
    - Harmonics = n f_1 (n=2 → second harmonic, 3=3rd, etc)
    - Overtones = n f_1 for n=2,3… f_1 (n=2 → first overtone, etc)
    - "Partial" = any single frequency component of a sound

Energy of vibration for a fixed-end finite string

- Piece of string between *x* and *x+dx* has kinetic energy *½ mu*²

$$dE_K = \frac{1}{2}\rho_L c^2 \left(\frac{\partial y}{\partial t}\right)^2 dx$$



- Gets stretched by an amount δL → potential energy

$$y(x+dx) = y(x) + \frac{\partial y}{\partial x}dx \rightarrow \delta L = \sqrt{dx^2 + \left(\frac{\partial y}{\partial x}dx\right)^2} - dx = \left(\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2} - 1\right)dx$$

$$\sqrt{1+\varepsilon} \approx 1 + \varepsilon/2 \rightarrow \delta L = \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2 dx$$

- Potential energy due to stretching:

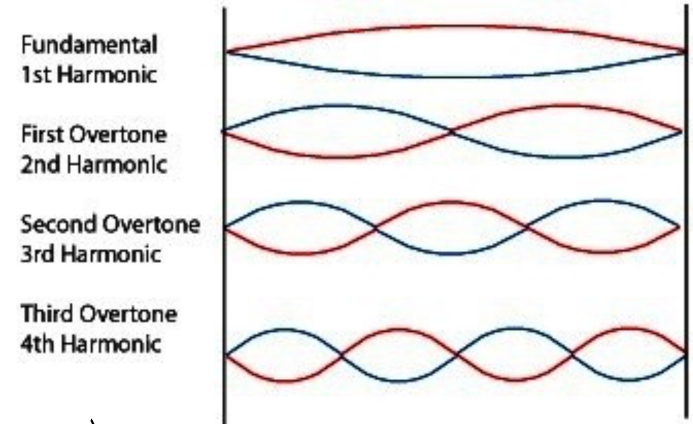
$$dE_P = \frac{1}{2}\rho_L c^2 \left(\frac{\partial y}{\partial x}\right)^2 dx; \text{ the total energy per unit length is}$$

$$\frac{dE}{dx} = \frac{dE_K}{dx} + \frac{dE_P}{dx} = \frac{1}{2}\rho_L c^2 \left[\left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{1}{c}\frac{\partial y}{\partial t}\right)^2\right]$$

$$\rightarrow \text{total energy} = \text{integral over length } L\text{: } E = \int_L \frac{1}{2}\rho_L c^2 \left[\left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{1}{c}\frac{\partial y}{\partial t}\right)^2\right] dx$$

Energy of vibration for a fixed-end finite string

- Example: string of length L is vibrating in its *nth* mode:



$$y_n(x,t) = \mathbf{A}_n \sin(k_n x) e^{i\omega_n t} \rightarrow$$

$$\frac{\partial y}{\partial x} = k_n \left(A_n \cos(\omega_n t) + B_n \sin(\omega_n t) \right) \cos(k_n x)$$

$$\frac{1}{c}\frac{\partial y}{\partial t} = (\omega_n / c)\left(-A_n \sin(\omega_n t) + B_n \cos(\omega_n t) \right) \sin(k_n x)$$

$$E_n = \int_L \frac{1}{2} \rho_L c^2 \left[\left(\frac{\partial y}{\partial x} \right)^2 + \left(\frac{1}{c}\frac{\partial y}{\partial t} \right)^2 \right] dx = \frac{1}{4} \rho_L L\, \omega_n^2 \left(A_n^2 + B_n^2 \right)$$

$$= \frac{1}{4} m\, \omega_n^2 \left(A_n^2 + B_n^2 \right)$$

m = mass of string, $\sqrt{A_n^2 + B_n^2}$ =max displacement amplitude

– Amplitude U_n of speed of motion in nth mode is

$$U_n = \left| \frac{\partial y}{\partial t} \right| = \omega_n \sqrt{A_n^2 + B_n^2} \rightarrow \quad E_n = \frac{1}{4} m\, U_n^2 \rightarrow \quad E_{TOT} = \sum_n E_n$$

"It can be shown" that for a string plucked in the center by

$\delta y = h,$ the amplitude of motion $A_n = \dfrac{1}{n^2} \dfrac{8h}{\pi^2} \sin\left(\dfrac{n\pi}{2} \right)$ (only odd-n harmonics for string plucked at center)

$A_n \sim \dfrac{1}{n^2} \rightarrow$ 3rd harmonic has 1/9 amplitude of fundamental, 5th harmonic has 1/25, etc

Motion of a plucked string

When string is plucked, each allowed mode oscillates at its own natural frequency f_n .
The sum changes with time as the individual modes add.

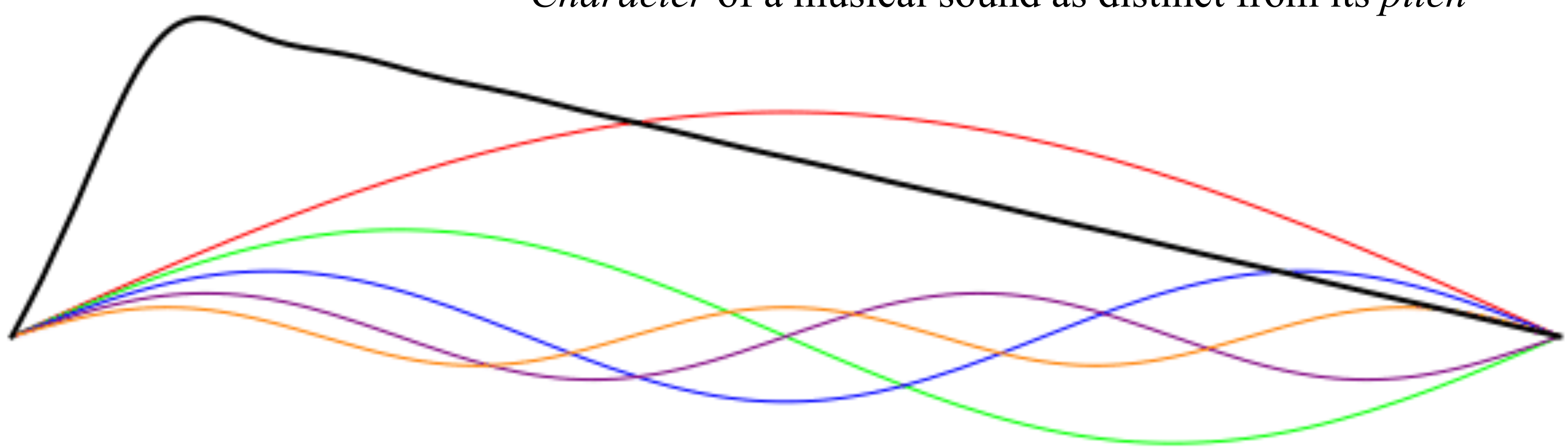
- Thick black curve = the actual string (the sum of all of the individual modes)
- Colors = individual modes each with its own f_n and maximum amplitude.

The f associated with the total (sum) motion = frequency of the fundamental mode.
A string tuned to f Hz will repeat complete cycle of motion f times per second.

- Plucking at different locations enhances different harmonics.

Relative amplitudes of harmonic components determines *timbre**, so affects the perceived sound.

* *Character* of a musical sound as distinct from its *pitch*



https://www.acs.psu.edu/drussell/Demos/waves/wavemotion.html

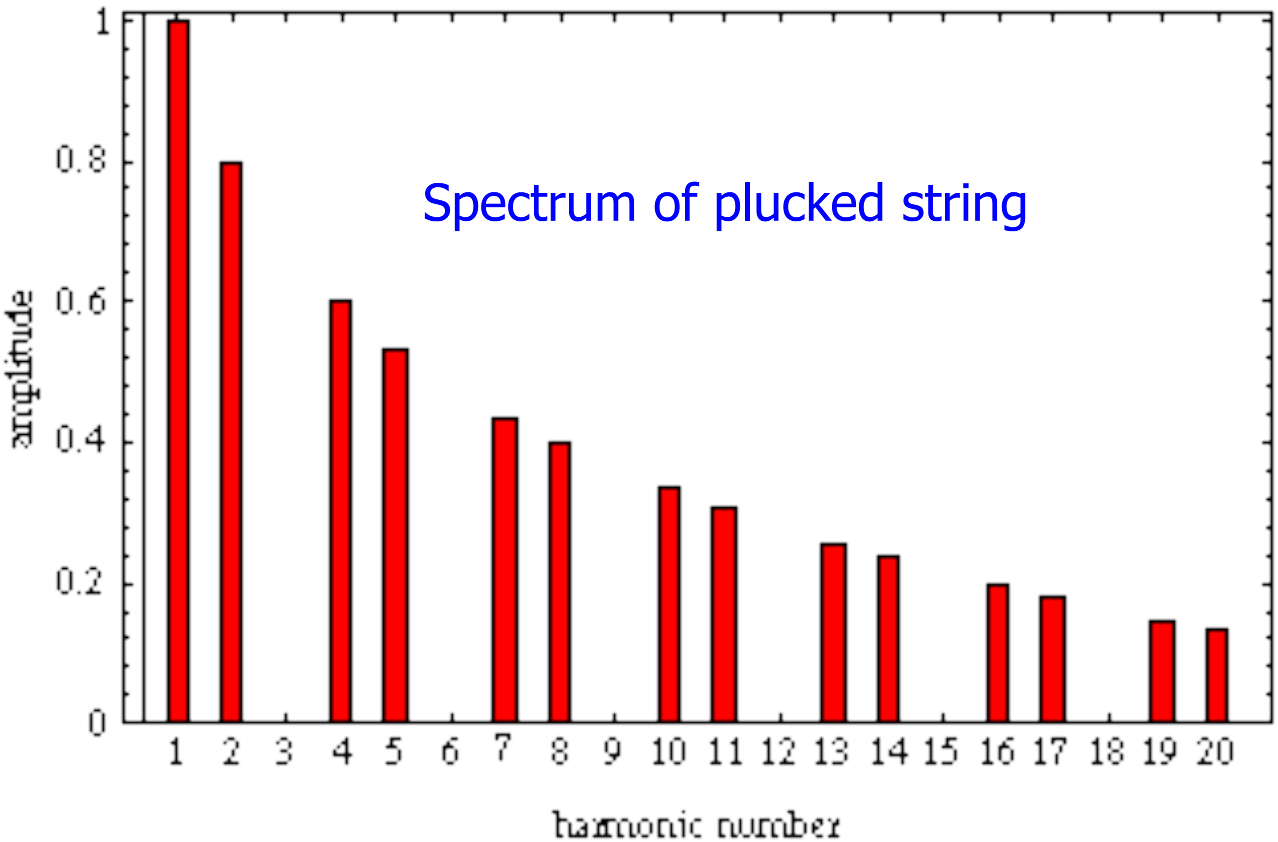
Motion of plucked string

Video:

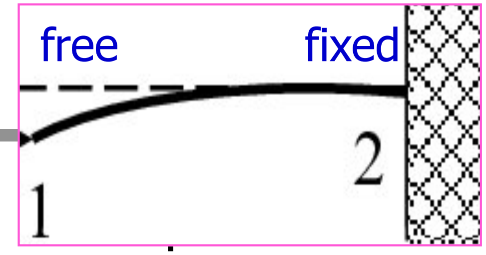
https://youtube.com/watch?v=_X72on6CSL0&si=EnSIkaIECMiOmarE

Play 24—45 sec, muted

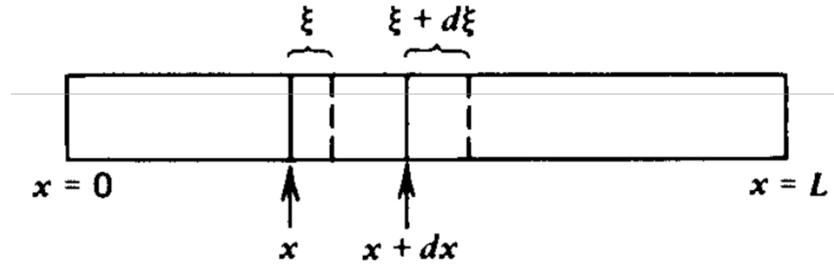
Motion of Plucked String Dan Russell, @DanRussellPSU



Vibrations in solid bars: Longitudinal waves



- Compression waves in "slender" bars with fixed ends
  - For long, thin bars, each slice of the bar can be treated as moving as a unit
  - Longitudinal displacement of a slice at position x along bar is



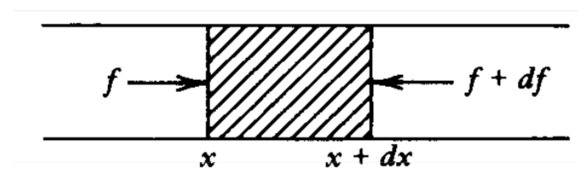
$$\delta l = \xi(x,t) \to \xi(x+dx,t) - \xi(x,t) = \left(\frac{\partial \xi}{\partial x}\right)dx$$

$$Strain \equiv \frac{\left(\frac{\partial \xi}{\partial x}\right)dx}{dx} = \left(\frac{\partial \xi}{\partial x}\right); \quad Stress \equiv \frac{f}{S}; \quad Hooke's\ Law:\ Stress \propto Strain$$

$$\to \frac{f}{S} = -Y\left(\frac{\partial \xi}{\partial x}\right); \quad (\text{convention:} \ \ +f = \text{compression}, \ -f = \text{stretching})$$

$$f = -SY\left(\frac{\partial \xi}{\partial x}\right); \quad \text{The net force on slice (positive = +x direction) is}$$

$$df = f(x) - f(x+dx) = f - \left(f + \frac{\partial f}{\partial x}dx\right) = \frac{\partial f}{\partial x}dx = -SY\left(\frac{\partial^2 \xi}{\partial x^2}\right)dx$$



$$F = ma \to \rho\left(S\,dx\right)\frac{\partial^2 \xi}{\partial t^2} = -SY\left(\frac{\partial^2 \xi}{\partial x^2}\right)dx; \quad S\,dx = \text{volume of slice}; \rho = \text{volume density}$$

Longitudinal waves in fixed-end bar

- So *F=ma* leads to

$$\rho S \frac{\partial^2 \xi}{\partial t^2} = -SY\left(\frac{\partial^2 \xi}{\partial x^2}\right) \rightarrow \text{looks like 1D wave equation: } \frac{\partial^2 \xi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \xi}{\partial t^2}, \quad \text{with } c^2 = \frac{Y}{\rho}$$

  - General solution of wave equation = some function of *(ct ± x)*

$$\xi(x,t) = \xi_1(ct - x) + \xi_2(ct + x), \quad \text{with phase speed } c = \sqrt{Y/\rho}$$

Complex harmonic solution is

$$\xi(x,t) = Ae^{i(\omega t - kx)} + Be^{i(\omega t + kx)} \quad \text{with wave number } k = \omega / c$$

  - Approximation only works for L >> diameter << wavelength
  - Follow usual path: apply boundary conditions

$$\xi(0,t) = 0 \rightarrow A + B = 0, \quad B = -A$$

$$\xi(x,t) = Ae^{i\omega t}\left(e^{-ikx} - e^{+ikx}\right) = -\left(Ae^{i\omega t}\right)2i\sin(kx)$$

$$\boxed{\xi(L,t) = 0 \rightarrow \sin(kL) \rightarrow k_n L = n\pi,} \quad n = 1,2,3...$$

$$\omega_n = k_n c = \frac{n\pi c}{L} \rightarrow f_n = \frac{nc}{2L}$$

(Same as for fixed-fixed string)

$$\text{Re}\left(\zeta(x,t)\right) = \left(A_n \cos\omega_n t + B_n \sin\omega_n t\right)\sin k_n x$$

Longitudinal waves in free-end bar

- If end of bar is free, must have *F=0* at ends, so

$$f = -SY\left(\frac{\partial \xi}{\partial x}\right) = 0 \rightarrow \frac{\partial \xi}{\partial x} = 0; \quad \text{Applied at } x = 0:$$

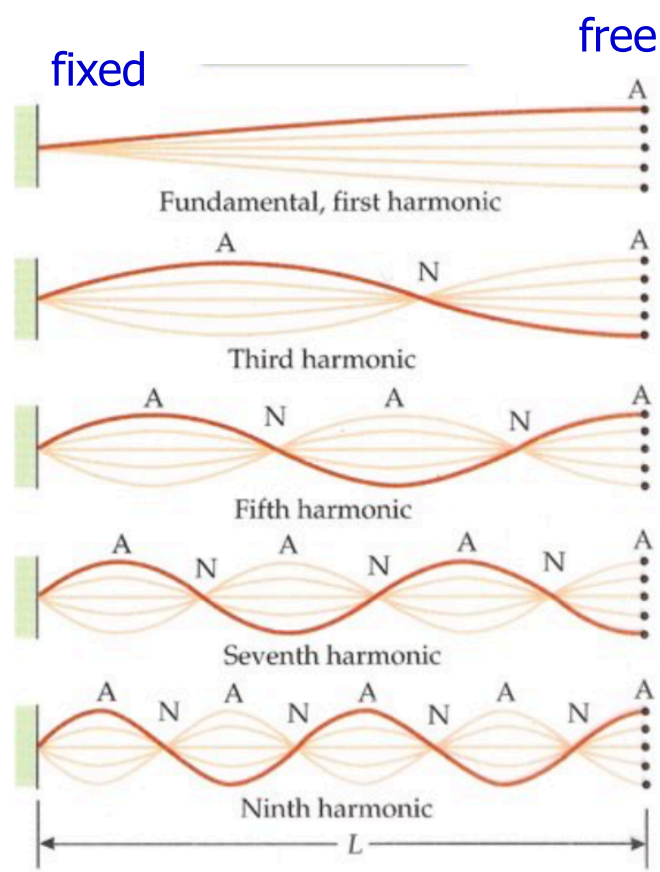
$$\frac{\partial}{\partial x}\left(Ae^{i(\omega t - kx)} + Be^{i(\omega t + kx)}\right) = 0 \rightarrow -A + B = 0 \rightarrow B = A$$

$$\xi(x,t) = Ae^{i\omega t}\left(e^{-ikx} + e^{+ikx}\right) = 2Ae^{i\omega t}\cos(kx)$$

Applied at $x = L$:

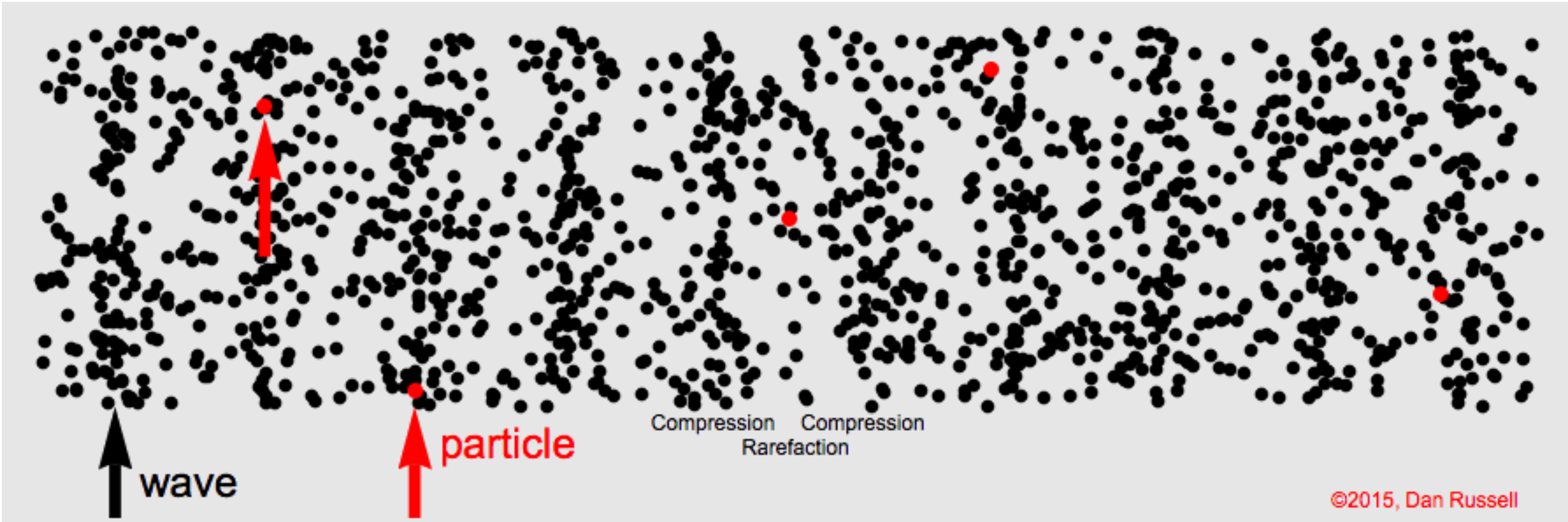
$$\sin(kL) = 0 \rightarrow \omega_n = \frac{n\pi c}{L} \rightarrow f_n = \frac{nc}{2L}$$

(Same as for fixed-end bar)

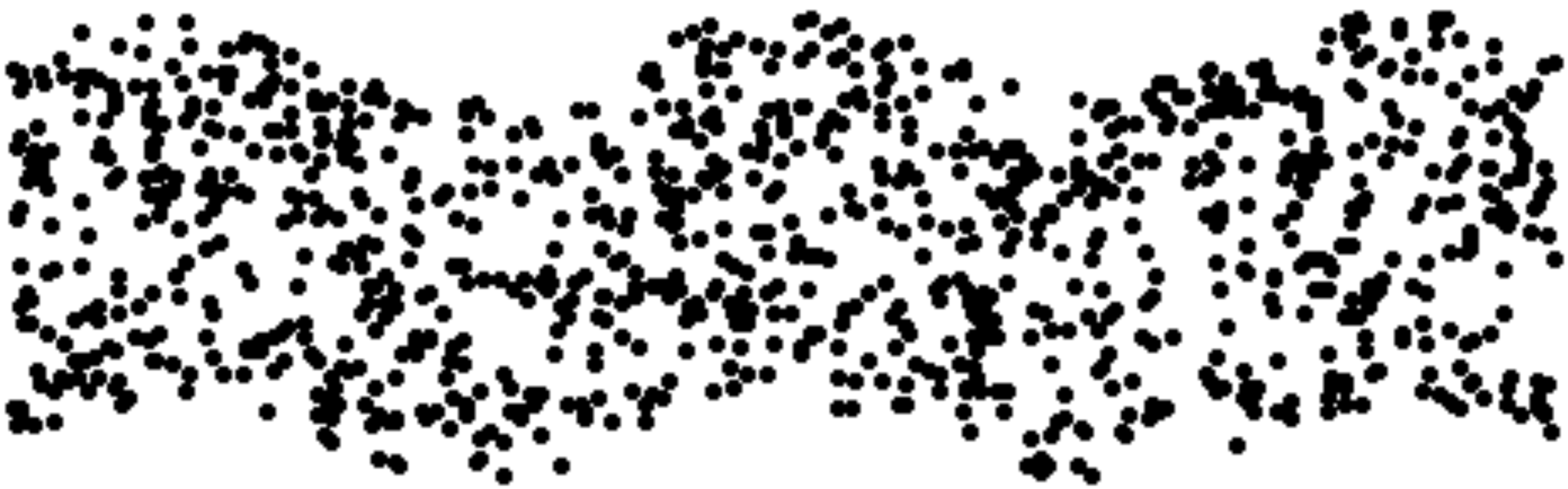


- Fixed ends must be nodes
- Free ends must be antinodes (maxima)
- If bar is clamped at x, must be node there
    - Other modes will be suppressed

Longitudinal and transverse waves in bars



https://www.acs.psu.edu/drussell/Demos/waves/wavemotion.html



Transverse vibrations of a bar

- When a uniform straight bar (length L, cross-section S) is bent, the lower part is compressed and the upper part stretched, but there will be a neutral axis at the center:

$$\delta x_r = \frac{\partial \xi}{\partial x} dx = \text{ increment in length of bar at radius r from neutral axis}$$

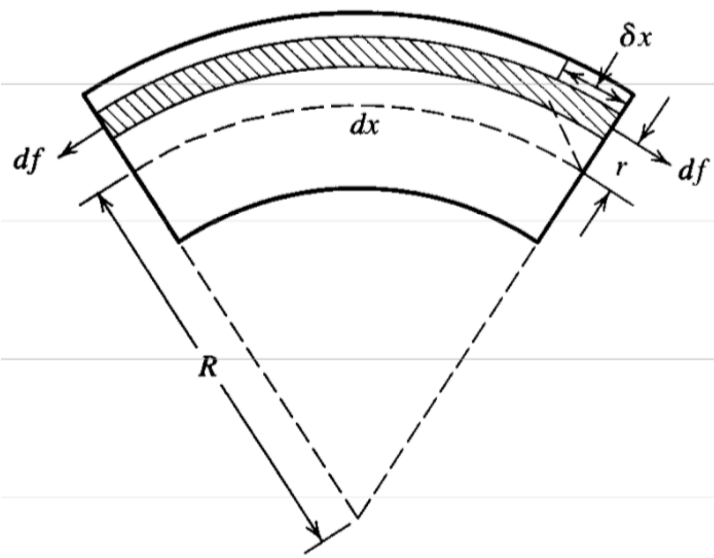
Y = Young's modulus = slope of stress vs strain curve

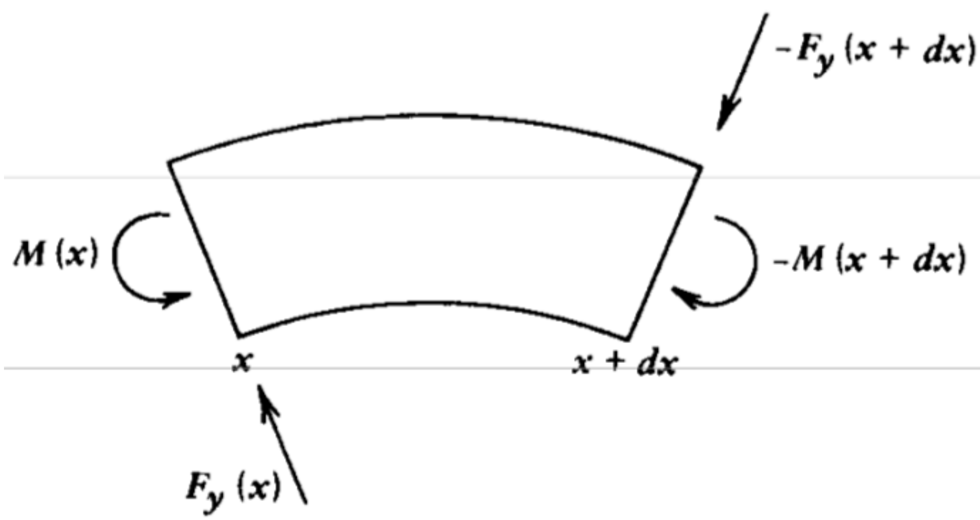
$$df = -YdS\frac{\delta x_r}{dx} = -YdS\left(\frac{\partial \xi}{\partial x}\right) = 0 \quad \text{(sign: } - \text{for tension, + for compression)}$$

$$dx / R = (dx + \delta x_r)/(R + r) \quad \text{(same opening angle) so } \delta x_r / dx = r / R$$

$$\rightarrow df = -Y\frac{r}{R}dS$$

net force is 0, but there is a bending moment M





Transverse vibrations of a bar

- Bending bar produces shear forces as well as bending moment
  - Equilibrium → no net force or torque on bar as a whole

bending moment:

$$M = \int r\,df = -\frac{Y}{R}\int r^2\,dS \qquad \text{let } \kappa = \frac{\int r^2\,dS}{S} \quad (\sim \text{radius of gyration of bar})$$

(for rectangular bar, $\kappa = \text{thickness}/\sqrt{12}$; for circular rod, $\kappa = \text{radius}/2$)

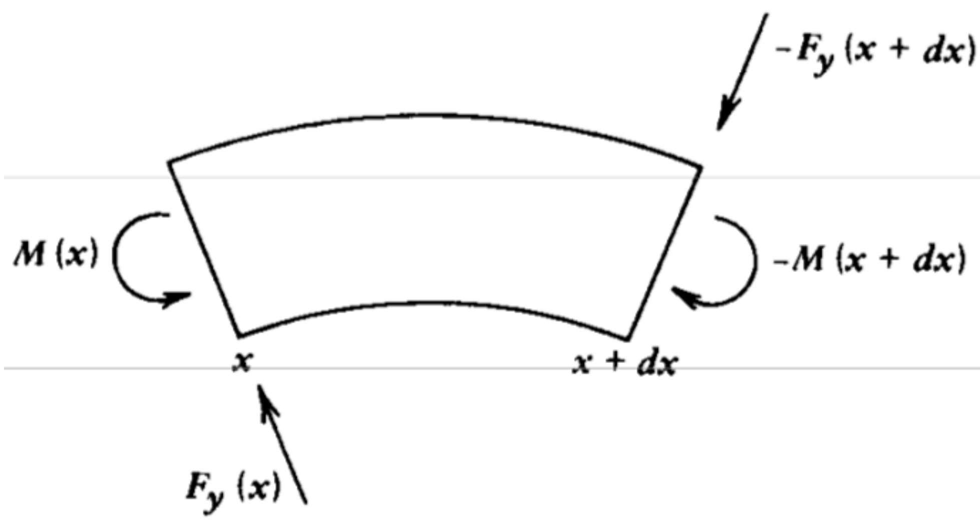
$$M = -\frac{YS\kappa^2}{R} \qquad R \text{ depends on position } x\text{: for small displacements in } y$$

$$R \approx 1/\left(\frac{\partial^2 y}{\partial x^2}\right) \rightarrow M = -YS\kappa^2\left(\frac{\partial^2 y}{\partial x^2}\right)$$

bending moment related to shear:

$$F_y \approx -\frac{\partial M}{\partial x} = -YS\kappa^2\left(\frac{\partial^3 y}{\partial x^3}\right)$$

When we get into 3rd derivatives, math is getting too messy...
Let's just quote results
(see Kinsler for details)



Transverse vibrations of a bar

- Net force on a small segment dx (negative = downward)

$$dF_y = F_y(x) - F_y(x+dx) = -\frac{\partial F_y}{\partial x}dx = -YS\kappa^2\left(\frac{\partial^4 y}{\partial x^4}\right)dx$$

$$F = ma \to -YS\kappa^2\left(\frac{\partial^4 y}{\partial x^4}\right)dx = \rho S dx\left(\frac{\partial^2 y}{\partial t^2}\right) \to = \frac{\partial^2 y}{\partial t^2} = -\kappa^2 c^2\left(\frac{\partial^4 y}{\partial x^4}\right), \quad \boxed{c = \sqrt{Y/\rho}}$$

$$y(x,t) = \Psi(x)e^{i\omega t} \to \frac{\partial^2 \Psi}{\partial t^2}\omega^2 e^{i\omega t} = \kappa^2 c^2\left(\frac{\partial^4 \Psi}{\partial x^4}\right)e^{i\omega t} \to \frac{\partial^4 \Psi}{\partial x^4} = \frac{\omega^2}{\kappa^2 c^2}\frac{\partial^2 y}{\partial t^2}$$

$$\boxed{v = \sqrt{\omega\kappa c}} \to \frac{\partial^4 \Psi}{\partial x^4} = \frac{\omega^4}{v^4}\frac{\partial^2 y}{\partial t^2}; \text{ try } \Psi(x) = Ae^{\gamma x} \to \gamma^4 = \frac{\omega^4}{v^4} \to \gamma = \pm\frac{\omega}{v} \text{ or } \pm i\frac{\omega}{v}$$

$$\Psi(x) = Ae^{(\omega/v)x} + Be^{-(\omega/v)x} + Ce^{i(\omega/v)x} + De^{-i(\omega/v)x}$$

$$\to \mathrm{Re}\, y = \mathrm{Re}\left(\Psi e^{i\omega t}\right) = \cos(\omega t + \phi)\left[A\cosh(\omega x/v) + B\sinh(\omega x/v) + C\cos(\omega x/v) + D\sin(\omega x/v)\right]$$

- Notice nothing here is wave motion at speed *c*
- Wave moves to the right with speed *v* (phase velocity)
- But *v* is frequency dependent: $v = \sqrt{\omega\kappa c}$ Higher frequency → higher *v*
  - Bar is a dispersive medium for transverse vibrations – different frequencies present spread out spatially, altering wave shape

Transverse vibrations of a bar

- Example: bar clamped on one end (x=0), free on the other (L)

Fixed end: $y = 0, \text{ and } \frac{\partial y}{\partial x} = 0$

Free end: $M = 0 \rightarrow \frac{\partial^2 y}{\partial x^2} = 0, \text{ and } F = 0 \rightarrow \frac{\partial^3 y}{\partial x^3} = 0$

Applying these at $x = 0$ and $x = L$ respectively, we get (skipping many steps!):

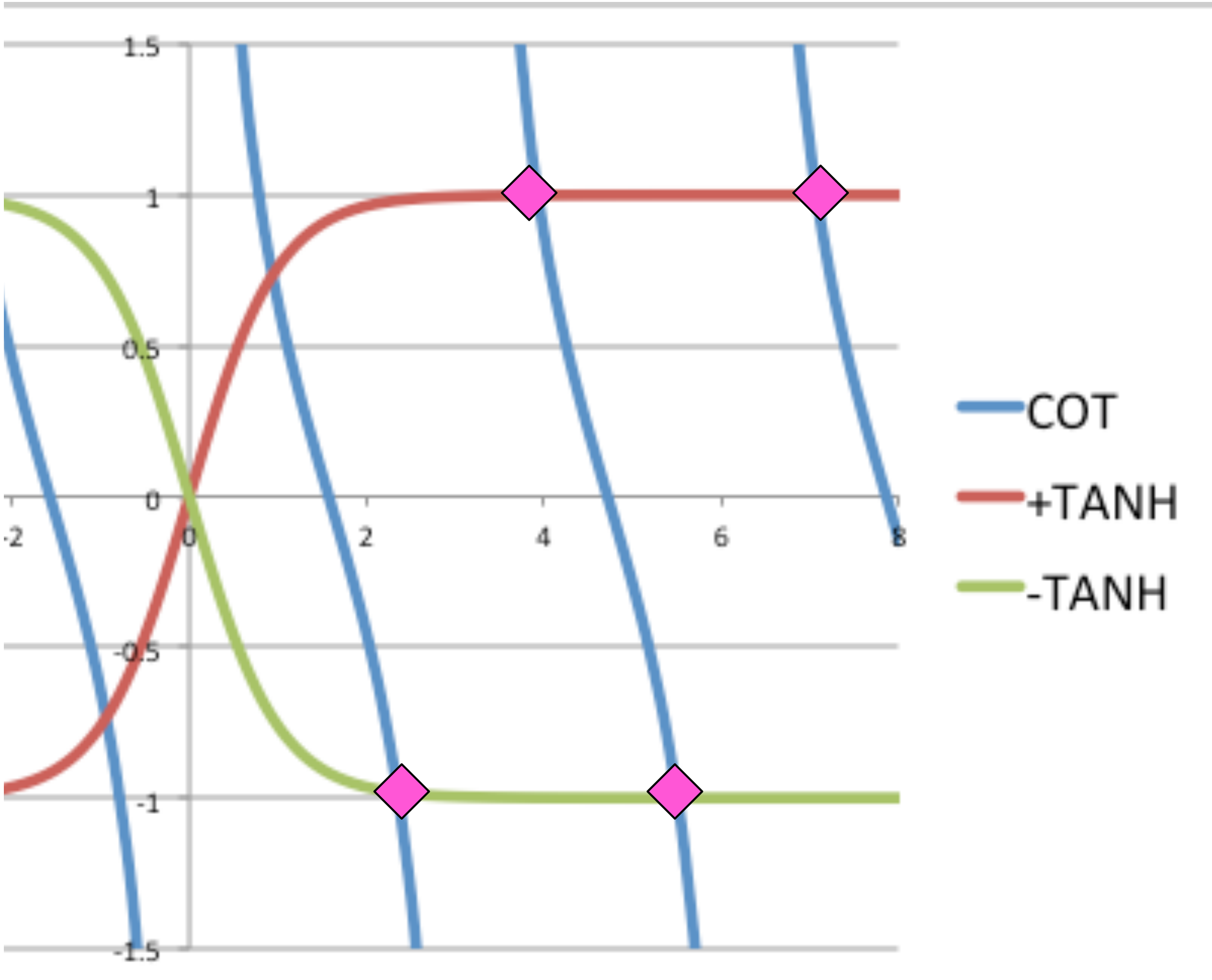
$\cot(\omega L / 2v) = \pm\tanh(\omega L / 2v)$; Solve graphically $\rightarrow \omega L / 2v \approx (2n-1)\pi / 4$ (except for n=1)

Put in $v = \sqrt{\omega \kappa c}$, $f_n = \omega / 2\pi \rightarrow f_n = \frac{\pi \kappa c}{8L^2}\left(1.19^2, 3^2, 5^2 ...\right)$

- For Al bar 1 m long, with circular cross section 0.01 m radius, we get c_{AL-VIB} = sqrt(Y/ρ) =5055 m/s, κ = r/2=0.005, → f_1 = 1509 Hz, v_1 = 490 m/s
- Notice overtones are not harmonics (integer multiples) of f_1

Frequency	Phase Speed	Wavelength (cm)	Nodal Positions (cm from clamped end)
f_1	v_1	335.0	0
$6.267f_1$	$2.50v_1$	133.4	0, 78.3
$17.55f_1$	$4.18v_1$	80.0	0, 50.4, 86.8
$34.39f_1$	$5.87v_1$	57.2	0, 35.8, 64.4, 90.6

Graphical solutions for cot(x)=tanh(x)



n	$2(n-1)\pi/4 = \omega L/2v$
2	2.36
3	3.93
4	5.50
5	7.07