PHYS 536

R. J. Wilkes

Session 6

Waves in membranes and sheets;

Acoustic wave equation;

speed of sound;

Harmonic plane waves

1/19/2023

Course syllabus and schedule – first part…

See : http://courses.washington.edu/phys536/syllabus.htm

Session	date	Day	Readings:	K=Kinsler, H=Heller	Topic
1	3-Jan	Tue	K ch. 1	H: Ch. 1, 2	Course intro, acoustics topics, overview of wave properties; pulses, transverse and longitudinal waves, overview of sound speeds
2	5-Jan	Thu	K ch. 1	H: Ch. 9, 10	harmonic oscillators: simple, damped, driven; complex exponential solutions, electrical circuit analogy, resonance, Q factor
3	10-Jan	Tue	K ch. 1	H: Ch. 3	Fourier methods: Fourier series, integrals, Fourier transforms, discrete FTs, sampling and aliasing
4	12-Jan	Thu	K. chs 10	H: Ch. 4, 11	Frequencies and aliasing; convolution and correlation; discrete convolution; digital filtering, optimal filters, FIR filters, noise spectra; power spectra. REPORT 1 PROPOSED TOPIC DUE
5	17-Jan	Tue	K. ch. 2, 3, 4	H: Ch. 13, 15	waves in strings, bars and membranes; Acoustic wave equation; speed of sound; Harmonic plane waves, intensity, impedance.
6	19-Jan	Thu	K. Ch. 5, 6	H: Ch. 1	Spherical waves; transmission and reflection at inter Tonight
7	24-Jan	Tue	K. Ch. 8	H: Ch. 7	Radiation from small sources; Baffled simple source, piston radiation, pulsating sphere;
8	26-Jan	Thu	K: Ch. 10	H: Chs. 13-15	Near field, far field; Radiation impedance; resonators, filters
9	31-Jan	Tue	K. Ch. 9-10	H: Chs. 16-19	Musical instruments: wind, string, percussion
10	2-Feb	Thu	K. Ch 14		Transducers for use in air: Microphones and loudspeakers
11	7-Feb	Tue	K. Ch 11	H: Chs. 21-22	The ear, hearing and detection
12	9-Feb	Thu	K. Chs 5,11		Decibels, sound level, dB examples, acoustic intensity; noise, detection thresholds. REPORT 1 PAPER DUE by 7 PM; REPORT 2 PROPOSED TOPIC DUE

…2nd part of class schedule

Session	date	Day	Readings:	K=Kinsler, H=Heller	Topic
13	14-Feb	Tue	K. Ch. 12	H: Ch. 28	Environmental acoustics and noise criteria; industrial and community noise regulations
14	16-Feb	Thu	K. Ch. 15		Underwater acoustics; sound speed in seawater; undersea sound propagation.
15	21-Feb	Tue	K. Ch. 15		Sonar equations, undersea noise; transducers for use in water (hydrophones and pingers), sonar and positioning systems
16	23-Feb	Thu	Notes		Applications: acoustical positioning, seafloor imaging, sub-bottom profiling;
17	28-Feb	Tue			Course wrapup. Student report 2 presentations
18	2-Mar	Thu			Student report 2 presentations
19	7-Mar	Tue			Student report 2 presentations
20	9-Mar	Thu			Student report 2 presentations. TAKE-HOME FINAL EXAM ISSUED
--	17-Mar	Fri			FINAL EXAM ANSWERS DUE by 5 PM

Class is over after you turn in your take-home exam. No in-person final exam during finals week.

Announcements

- Problem set 1 solutions will be posted on Feb 7 – try them before then …
  - As mentioned, prob 11 as originally posted was not covered in class (too complicated) so you can replace it with:

11. A steel bar of cross section $0.0001m^2$ and 0.25m length is clamped at both ends. a) what is its fundamental frequency for longitudinal vibrations? b) what is the fundamental frequency for the same bar but free at both ends?

- You can access most scientific journals, many popular journals, and books online via the UW library – no need to be on campus
  - See http://www.lib.washington.edu/help/connect
  - See also http://www.lib.washington.edu/help/connect/husky-onnet for how to VPN onto campus network

From last time

Transverse vibrations of a bar

- Net force on a small segment dx (negative = downward)

$$dF_y = F_y(x) - F_y(x+dx) = -\frac{\partial F_y}{\partial x}dx = -YS\kappa^2\left(\frac{\partial^4 y}{\partial x^4}\right)dx$$

$$F = ma \rightarrow -YS\kappa^2\left(\frac{\partial^4 y}{\partial x^4}\right)dx = \rho S dx\left(\frac{\partial^2 y}{\partial t^2}\right) \rightarrow = \frac{\partial^2 y}{\partial t^2} = -\kappa^2 c^2\left(\frac{\partial^4 y}{\partial x^4}\right), \quad c = \sqrt{Y/\rho}$$

$$y(x,t) = \Psi(x)e^{i\omega t} \rightarrow \frac{\partial^2 \Psi}{\partial t^2}\omega^2 e^{i\omega t} = \kappa^2 c^2\left(\frac{\partial^4 \Psi}{\partial x^4}\right)e^{i\omega t} \rightarrow \frac{\partial^4 \Psi}{\partial x^4} = \frac{\omega^2}{\kappa^2 c^2}\frac{\partial^2 y}{\partial t^2}$$

$$v = \sqrt{\omega\kappa c} \rightarrow \frac{\partial^4 \Psi}{\partial x^4} = \frac{\omega^4}{v^4}\frac{\partial^2 y}{\partial t^2}; \quad \text{try} \ \ \Psi(x) = Ae^{\gamma x} \rightarrow \gamma^4 = \frac{\omega^4}{v^4} \rightarrow \gamma = \pm\frac{\omega}{v} \ \text{or} \ \pm i\frac{\omega}{v}$$

$$\Psi(x) = Ae^{(\omega/v)x} + Be^{-(\omega/v)x} + Ce^{i(\omega/v)x} + De^{-i(\omega/v)x}$$

$$\rightarrow \mathrm{Re}\, y = \mathrm{Re}\left(\Psi e^{i\omega t}\right) = \cos(\omega t + \phi)\left[A\cosh(\omega x/v) + B\sinh(\omega x/v) + C\cos(\omega x/v) + D\sin(\omega x/v)\right]$$

- Notice nothing here is wave motion at speed *c*
- Wave moves to the right with speed *v* (phase velocity)
- But *v* is frequency dependent: $v = \sqrt{\omega\kappa c}$ Higher frequency → higher *v*
  - Bar is a dispersive medium for transverse vibrations – different frequencies present spread out spatially, altering wave shape

Transverse vibrations of a bar

- Example: bar clamped on one end (x=0), free on the other (L)

Fixed end: $y = 0, \text{ and } \dfrac{\partial y}{\partial x} = 0$

Free end: $M = 0 \rightarrow \dfrac{\partial^2 y}{\partial x^2} = 0, \text{ and } F = 0 \rightarrow \dfrac{\partial^3 y}{\partial x^3} = 0$

Applying these at $x = 0$ and $x = L$ respectively, we get (skipping many steps!):

$\cot(\omega L / 2v) = \pm \tanh(\omega L / 2v)$; Solve graphically $\rightarrow \omega L / 2v \approx (2n-1)\pi / 4$

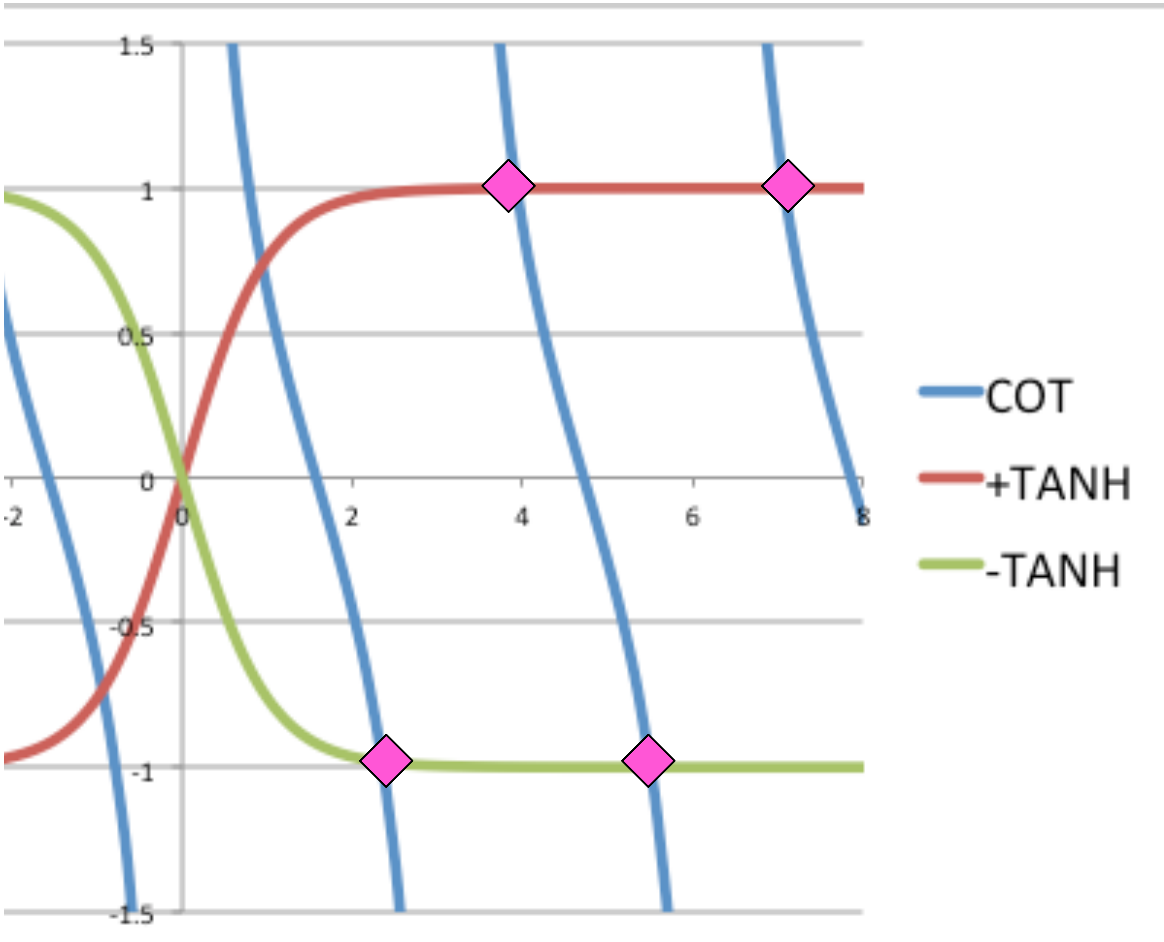
(except for n=1)

Put in $v = \sqrt{\omega \kappa c}$, $f_n = \omega / 2\pi \rightarrow f_n = \dfrac{\pi \kappa c}{8L^2}\left(1.19^2, 3^2, 5^2 ...\right)$

- For Al bar 1 m long, with circular cross section 0.01 m radius, we get $c_{AL\text{-}VIB}$ = sqrt(Y/ρ) =5055 m/s, κ = r/2=0.005, → f_1 = 1509 Hz, v_1 = 490 m/s
- Notice overtones are not harmonics (integer multiples) of f_1

Frequency	Phase Speed	Wavelength (cm)	Nodal Positions (cm from clamped end)
f_1	v_1	335.0	0
$6.267f_1$	$2.50v_1$	133.4	0, 78.3
$17.55f_1$	$4.18v_1$	80.0	0, 50.4, 86.8
$34.39f_1$	$5.87v_1$	57.2	0, 35.8, 64.4, 90.6

Graphical solutions for cot(x)=tanh(x)



n	$2(n-1)\pi/4 = \omega L/2v$
2	2.36
3	3.93
4	5.50
5	7.07

Vibrations in strings and bars: summary

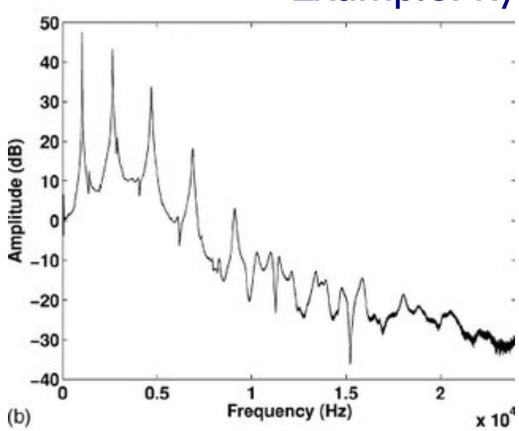
- Transverse vibration frequency for fixed-end strings: $f_n = \frac{nc}{2L}$, n = 1,2,3… $c = \sqrt{T/\rho_L}$
- Longitudinal vibration of fixed- or free-end bar

$$f_n = \frac{nc}{2L}, \quad c = \sqrt{Y/\rho}$$

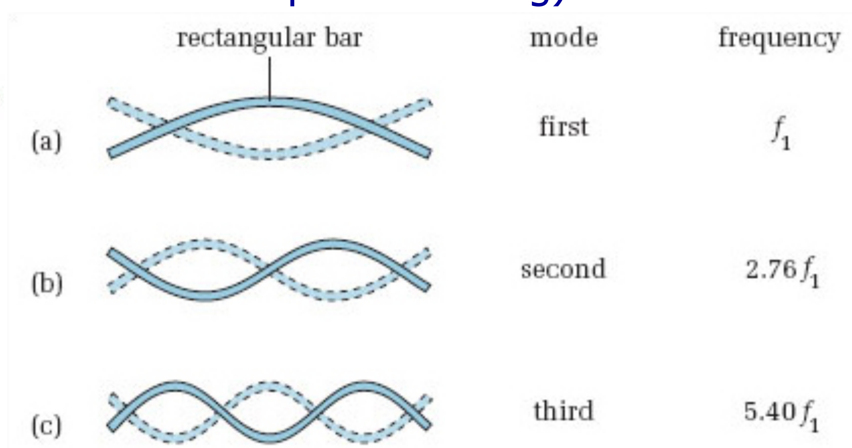
- Transverse vibration frequency for fixed-end bar

$$f_n = \frac{\pi \kappa c}{8L^2}\left(1.19^2, 3^2, 5^2 \ldots\right), \text{ with } \kappa = \text{radius of gyration, speed } v = \sqrt{\omega \kappa c}$$

  - Not harmonics!
  - Example: xylophone bar (not uniform bar – shaped for tuning)



x 10^4



Waves in membranes and sheets

- Use the same approach, but now in 2D: y(x) → y(x,z)
  - Define equations of motion in terms of forces, material density, etc
  - Insert general solution y(x,,z, t) = Ψ(x,z) exp(iωt)
  - Apply boundary conditions: fixed, free, or mixed
- Equations of motion
  - use Laplacian scalar differential operator:

For rectangular sheets, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}$;

For circular sheets, $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$, with $x = r\cos\theta$ and $z = r\sin\theta$

Solutions:

For rectangular sheets, $y(x,z,t) = \Psi(x,z)e^{i\omega t}$

Spatial part of equations of motion → $\nabla^2\Psi + k^2\Psi = 0$ $\quad k^2 = \omega^2 / c^2 = \omega^2\left(\frac{\rho}{T}\right)$

This is the time-independent wave (Helmholtz) equation

Assume Ψ is factorizable into independent 1D functions, $\Psi(x,z) = X(x)Z(z)$

Results: $k^2 = k_x^2 + k_z^2$ and $\Psi(x,z) = A\sin(k_x x + \phi_x)\sin(k_z z + \phi_z)$

Free vibrations in membranes with fixed edges

- For rectangular sheet with clamped edges, boundary conditions give normal modes of vibration with

$$k_x = \frac{n\pi}{L_x}, \quad k_z = \frac{m\pi}{L_z}, \quad n,m = 1,2,3\ldots \quad \rightarrow \quad f_{nm} = \omega_{nm}/2\pi = \frac{c}{2}\sqrt{\left(\frac{n}{L_x}\right)^2 + \left(\frac{m}{L_z}\right)^2}$$

- For circular sheet with clamped edges, similar solution (but in r,θ coordinates) plus boundary conditions give results:

$$\text{Separation of variables } y(r,\theta) = R(r)\Theta(\theta) \;\rightarrow\; \frac{r^2}{R}\left(\frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\frac{\partial R}{\partial r}\right) + k^2 r^2 = -\frac{1}{\Theta}\left(\frac{\partial^2 \Theta}{\partial \theta^2}\right)$$

two sides have different variables - cannot be equal unless both = constant: m^2

$$\left(\frac{\partial^2 \Theta}{\partial \theta^2}\right) = -m^2\Theta \rightarrow \Theta(\theta) = \cos(m\theta + \phi), \quad (\phi = \text{ initial phase})$$

unlike r, θ is periodic: $\theta + 2n\pi = \theta, \quad n = 1,2\ldots \rightarrow$ can only have $m = 0,1,2\ldots$

$$R(r) \text{ equation of motion } \rightarrow \quad \left(\frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\frac{\partial R}{\partial r}\right) + \left(k^2 - \frac{m^2}{r^2}\right)R = 0, \quad m = 0,1,2\ldots$$

- This is the Bessel equation: solutions are Bessel functions

Free vibrations in circular sheets with fixed edges

- General solution for circular sheet of radius $r = a$ is

$$R(r) = AJ_m(kr) + BY_m(kr), \quad k = \omega / c, \quad m = 1,2,3...$$

$J_m(kr), Y_m(kr) =$ Bessel functions of 1st and 2nd kind, respectively

Need $R(r)$ finite at $r = 0$, so must have $B = 0 \;\rightarrow\; R(r) = AJ_m(kr)$

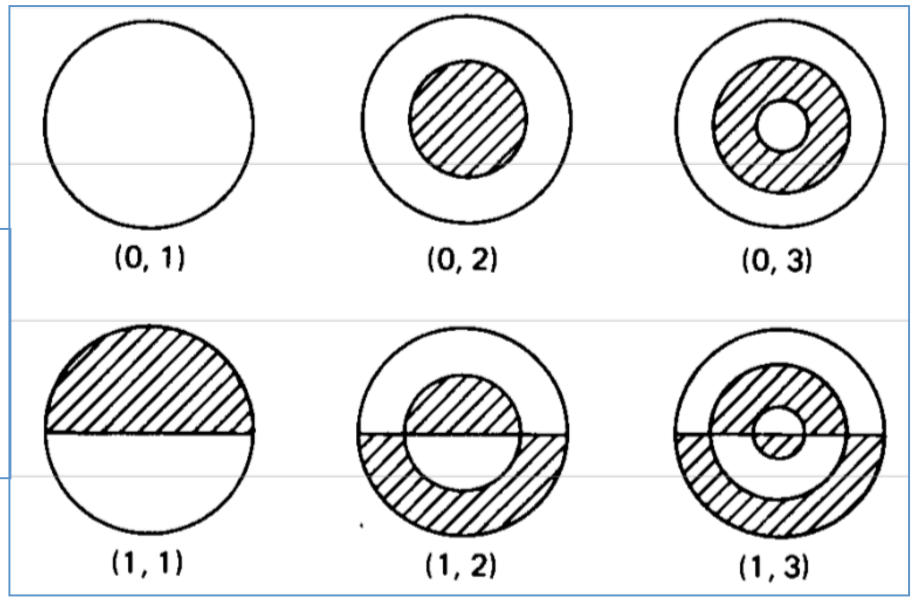
Boundary condition $R(r) = 0$ at $r = a \rightarrow J_m(k_{mn}a) = 0$

$$y(r,\theta,t) = A_{mn} J_m(k_{mn} r) \cos(m\theta + \phi_{mn}) e^{i\omega_{mn} t}$$

Let j_{mn} = values of kr where $J_m(j_{mn}) = 0$ then $k_{mn} = j_{mn} / a$

$$\rightarrow \quad f_{nm} = \omega_{nm} / 2\pi = \frac{1}{2\pi} \frac{j_{mn}}{a}$$

Normal modes (m,n) of fixed-rim circular membrane: Shaded areas are 180 deg out of phase with clear areas, separated by nodal lines



Bessel functions

- ... Arise in problems with cylindrical symmetry (“cylindrical harmonics”)
- $J_\alpha(\rho) =$ *Bessel function* of the 1st kind, order α

For integer or any positive α, $J_\alpha(x)$ are finite at the origin (x = 0).

For negative non-integer α, they diverge at x = 0.

Power-series representation: $$J_\alpha(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\alpha+1)} \left(\frac{x}{2}\right)^{2m+\alpha},$$

where Γ(z) = gamma function (generalization of factorial)

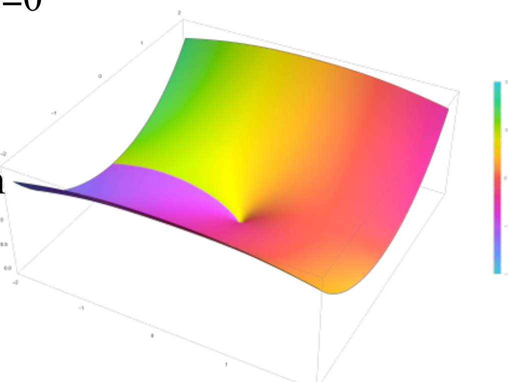
For integer α = n,

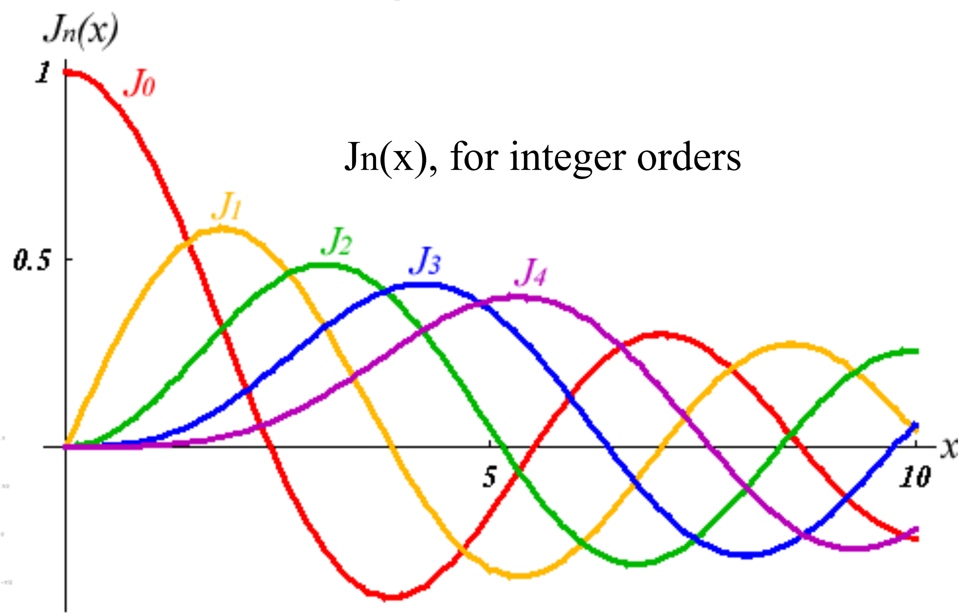
$$J_{-n}(x) = (-1)^n J_n(x).$$

And an integral representation exists:

$$J_n(x) = \frac{1}{2\pi}\int_{\tau=0}^{\pi} \exp[i(n\tau - x\sin(r)\,d\tau$$

Jn(z) for n = 0.5 in the complex plane from -2-2i to 2+2i





Bessel functions

- $Y_\alpha(\rho) =$ *Bessel function* of the 2nd kind, order α

have a singularity at x = 0

For non-integer α, $Y_\alpha(x)$ is connected with $J_\alpha(x)$ by

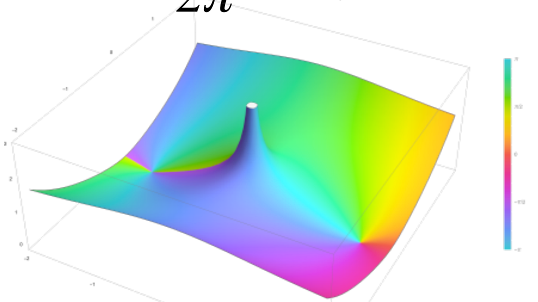
$$Y_\alpha(x) = \frac{J_\alpha(x)\cos(\alpha\pi) - J_{-\alpha}(x)}{\sin(\alpha\pi)}.$$

For integer order α = n, $Y_\alpha(x)$ is defined by the limit as non-integer $\alpha \rightarrow$ n:

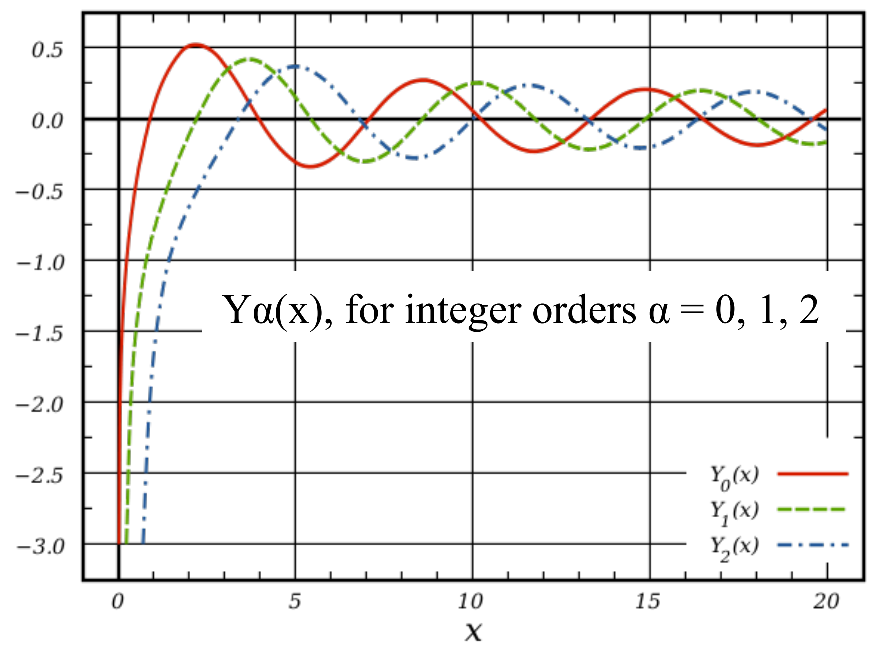
$$Y_n(x) = \lim_{\alpha\to n} Y_\alpha(x). \quad \text{and} \quad Y_{-n}(x) = (-1)^n Y_n(x).$$

And an integral representation exists:

$$J_n(x) = \frac{1}{2\pi}\int_{\tau=0}^{\pi} \exp[i(n\tau - x\sin(r)\,d\tau$$



$Y_n(z)$ with n = 0.5 in the complex plane from -2-2i to 2+2i

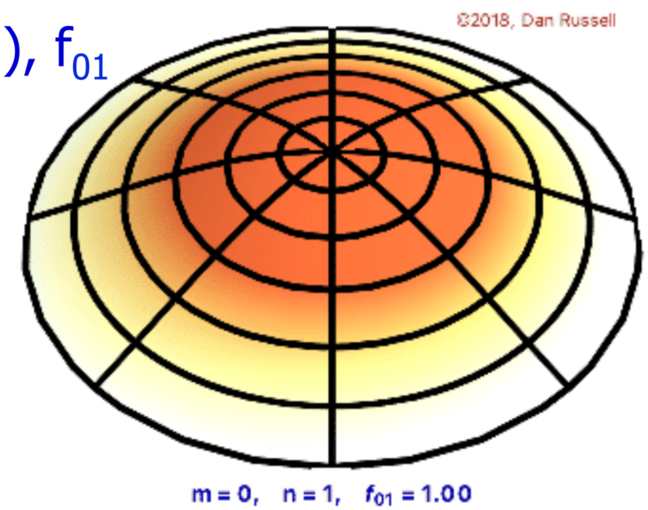


Free vibrations in circular sheets with fixed edges

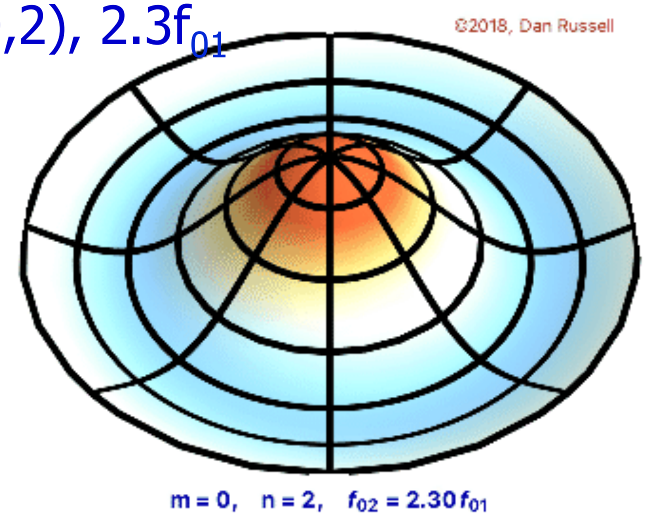
- Examples of motion of first few normal modes (m,n) and corresponding frequencies

m=number of diameter nodes
n=number of circular nodes

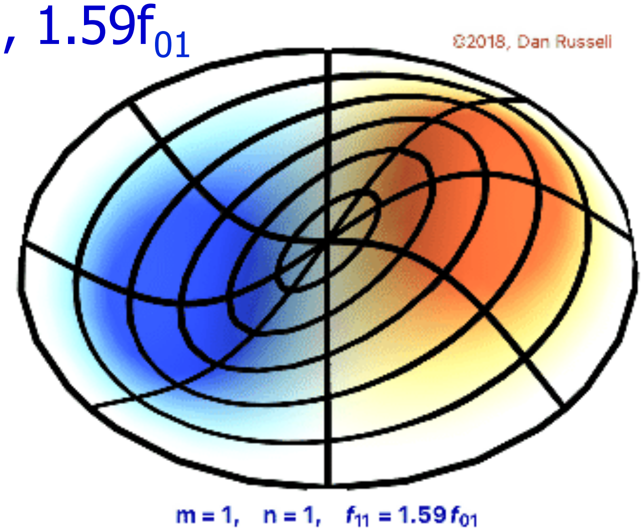
(0,1), f_{01}



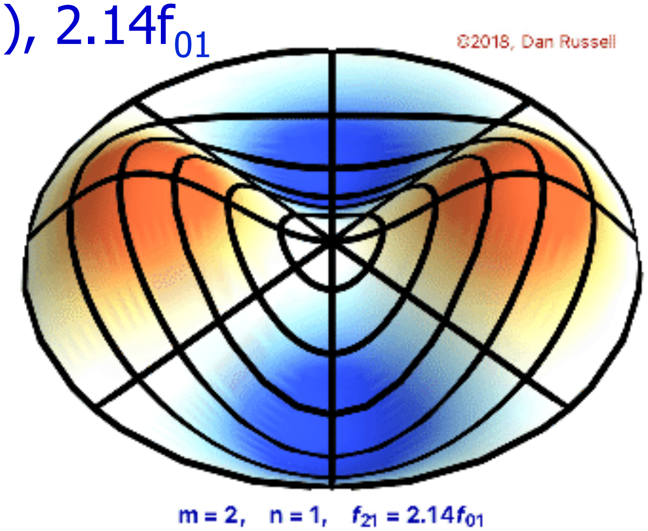
(0,2), $2.3f_{01}$



(1,1), $1.59f_{01}$



(2,1), $2.14f_{01}$



Fundamental mode (0,1) of a drum

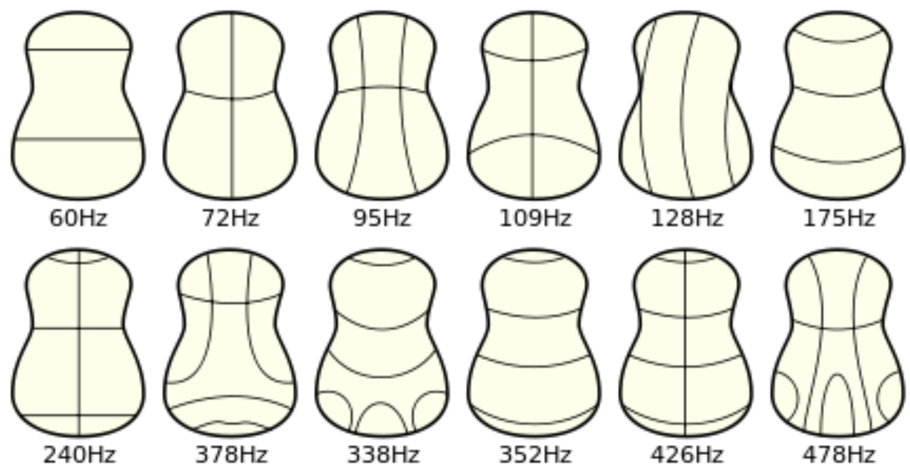
- (0,1) mode of a drum, such as a tympani, is excited when the drumhead (membrane) is struck at the center.
  - In (0,1) mode, membrane acts like a monopole source, and radiates energy very efficiently.
  - The membrane quickly transfers its vibrational energy into radiated sound
  - The short duration (typically ~ fraction of a second) means this mode does not contribute greatly to the musical tone quality of a drum - a "thump" which decays quickly, with no definite pitch.
- When struck somewhere between the center and outer edge, the (1,1) mode is excited.
  - This mode takes longer to decay and drum “rings” for a while
  - Sound has a definite pitch which lingers for a several seconds. This mode is what we hear as the **“principal tone”** of the timpano. only five or six of
- **Preferred modes** that mainly contribute to the sound spectrum and give the tympani its pitch are in the lower diametric modes (1,1), (2,1), (3,1), (4,1), (5,1).
- https://youtu.be/UYCC8xvTBck preferred modes 1—6
- Applet to demonstrate combining modes of circular membrane

  http://www.falstad.com/circosc/fullscreen.html

Chladni plates

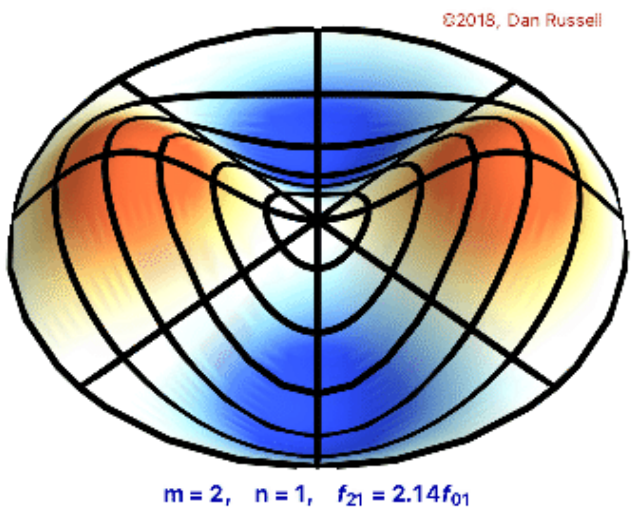
- Chladni (c. 1790--1800) showed nodal lines by putting sand on vibrating membranes and plates
  - Sand accumulates where $u_y=0$ (nodes)

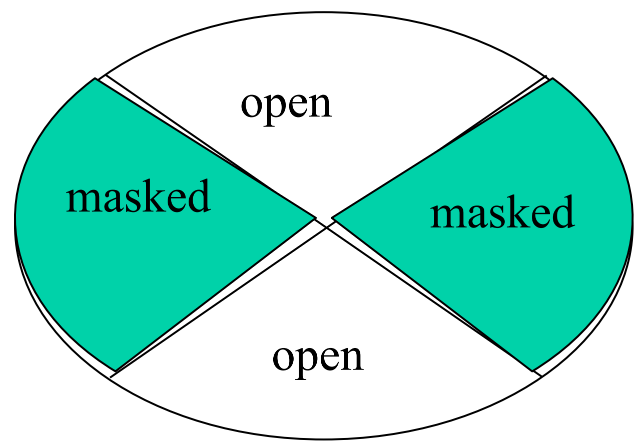
  (Robert Hooke first did this with flour in 1680)
  - Napoleon was fascinated – supported his research
  - Public was fascinated – charlatans sold “vibration science” cures
  - Application: testing design of guitar backs and violin plates



Chladni plates as sound sources

- Speed of vibration in Chladni plate is slower than speed of sound in air
- Sound emitted by segment reaches opposite phase segment quickly
  - Cancellation due to phase difference – sound emitted is diminished
  - Place a mask over one phase's segments to baffle sound – overall volume is greatly increased





  - Demonstration of Chladni plates excited with violin bow

https://youtu.be/IRFysSAxWxI

brass plates on a wooden platform: circular (8in , 10in D), square (10in), and "stadium" (a 8in square with 8in D semicircle at each end); plates are 1/16in thick, and bolted through the center to the wooden base by a 10cm brass rod.

Acoustic wave equation

- Acoustic waves propagating in a compressible fluid (air, water)
- Need to develop equations of motion in 3D
- Many new quantities to define along the way
  - "Particle" of fluid: dV small enough so acoustic variables are ~ constant but large enough to contain billions of molecules →molecular-scale physics ignored
  - Waves are pressure disturbances: small enough amplitude so no significant change in fluid density or other properties
  - Equation of state relates forces to deformations, as in solids

$$P = \rho r T$$

$P =$ pressure (Pa), $\rho =$ density (kg/m^3), $T =$ temperature (K)

$r =$ constant depending upon fluid properties

(eg, for ideal gas: $r = R/M$, $R =$ universal gas constant, $M =$ molecular weight)

  - Typically acoustical disturbances are ~adiabatic processes

Adiabatic eqn of state: $\dfrac{P}{P_0} = \left(\dfrac{\rho}{\rho_0}\right)^{\gamma}$, $\gamma = \left(\dfrac{C_P}{C_V}\right)$ ratio of specific heats

Assume $\Delta P \propto \Delta\rho \rightarrow p \equiv P - P_0 = B\left(\dfrac{\rho - \rho_0}{\rho_0}\right) \rightarrow p = Bs,\quad s =$ density fluctuation

$B =$ adiabatic bulk modulus, $p =$ "acoustic pressure"

Acoustic wave equation

- Continuity equation: fluid can't just disappear!
  - Fluid into volume dV (fixed in space) = fluid out

$$\frac{dm_x}{dt} = \rho u_x \, dA, \quad m = \text{ mass}, \quad \rho = \text{ density}, \quad u = \text{ fluid speed}, \; dA = dydz$$

$$\left\{ \rho u_x - \left[\rho u_x + \frac{\partial(\rho u_x)}{\partial x} dx \right] \right\} dA = -\frac{\partial(\rho u_x)}{\partial x} dV, \quad \text{same for flow in } y \text{ and } z$$

$$\text{so net rate of change of mass inside } dV \text{ is } \; \frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho u)$$

(Non-linear! ρ and u are both variables)

For acoustic pressure waves, $\Delta\rho$ is very small, so assume $\rho = \rho_0(1+s)$

$$\frac{\partial s}{\partial t} + \nabla \cdot u = 0 \quad \text{(linearized continuity equation)}$$

Combine equations of state and continuity: $\int_t \left(\frac{\partial s}{\partial t} + \nabla \cdot u \right) dt = 0$

$$u = \frac{\partial \xi}{\partial t}, \quad \xi = \text{ fluid particle displacement}$$

$$\int_t \nabla \cdot u \, dt = \nabla \cdot \int_t u \, dt = \nabla \cdot \int_t \frac{\partial \xi}{\partial t} dt = \nabla \cdot \xi \rightarrow s = -\nabla \cdot \xi \rightarrow \boxed{p = -B \nabla \cdot \xi}$$

Acoustic wave equation

- Another justifiable simplification: neglect viscosity
  - Already non-adiabaticity, now assume no viscosity: both reasonable assumptions, effects can be added later

Net force on a 'particle of fluid' *dV* :

$$df_x = \left\{ P - \left[P + \frac{\partial P}{\partial x} dx \right] \right\} dA = -\frac{\partial P}{\partial x} dV \quad \text{where dV moves with the fluid}$$

same for y and z forces $\rightarrow \; d\vec{f} = -\nabla P dV$

Velocity is function of time and position, $\vec{u}(x,y,z,t)$

acceleration is $a(x,y,z) = \frac{\partial \vec{u}}{\partial t} + u_x \frac{\partial \vec{u}}{\partial x} + u_y \frac{\partial \vec{u}}{\partial y} + u_z \frac{\partial \vec{u}}{\partial z} = \frac{\partial \vec{u}}{\partial t} + \left(\vec{u} \cdot \nabla\right)\vec{u}$

$$F = ma \rightarrow df = adm \rightarrow -\nabla P dV = \left(\frac{\partial \vec{u}}{\partial t} + \left(\vec{u} \cdot \nabla\right)\vec{u} \right) \rho \, dV$$

$\rightarrow$ Euler's equation: $-\nabla P = \left(\frac{\partial \vec{u}}{\partial t} + \left(\vec{u} \cdot \nabla\right)\vec{u} \right) \rho$

for small-amplitude disturbances $(s << 1)$, $\left(\vec{u} \cdot \nabla\right)\vec{u} << \frac{\partial \vec{u}}{\partial t}$ $\rho \approx \rho_0$ and $\nabla P \approx p$

Euler's equation $\rightarrow$ $-\nabla p \approx \rho_0 \frac{\partial \vec{u}}{\partial t}$ "Linear inviscid force eqn"

Acoustic wave equation

- We have eqn of state, adiabatic continuity, and inviscid force eqns
  - Combine to give a linearized 3D wave equation

1. $p = Bs$, equation of state: s = density fluctuation $\left(\dfrac{\rho - \rho_0}{\rho_0}\right)$, B = bulk modulus

2. $\dfrac{\partial s}{\partial t} + \nabla \cdot u = 0$ linearized continuity equation

$$\text{div } \vec{F} = \nabla \bullet \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

where $u = \dfrac{\partial \xi}{\partial t}$, ξ = fluid particle displacement $\rightarrow p = -B\nabla \cdot \xi$

3. $-\nabla p \approx \rho_0 \dfrac{\partial \vec{u}}{\partial t}$ "Linear inviscid force eqn"

$$\text{grad f} = \nabla f = \frac{\partial f_x}{\partial x}\mathbf{i} + \frac{\partial f_y}{\partial y}\mathbf{j} + \frac{\partial f_z}{\partial z}\mathbf{k}$$

Divergence of (3): $\rho_0 \nabla \cdot \left(\dfrac{\partial \vec{u}}{\partial t}\right) = -\nabla \cdot \nabla p = -\nabla^2 p$

Time derivative of (2): $\dfrac{\partial^2 s}{\partial t^2} + \nabla \cdot \left(\dfrac{\partial \vec{u}}{\partial t}\right) = 0$ note $\dfrac{\partial \nabla \cdot \vec{u}}{\partial t} = \nabla \cdot \dfrac{\partial \vec{u}}{\partial t}$

$\rightarrow \nabla^2 p = \rho_0 \dfrac{\partial^2 s}{\partial t^2}$; use (1) to eliminate $s = p / B$: $\nabla^2 p = \dfrac{1}{c^2}\dfrac{\partial^2 p}{\partial t^2}$

where $c = \sqrt{B / \rho_0}$ $\rightarrow$ can rewrite (1) as: $p = \rho_0 c^2 s$

  - Since *p* and *s* are proportional, *s* also satisfies the wave equation

Acoustic wave equation

$*$ curl $\mathbf{F} = \boldsymbol{\nabla} \times \mathbf{F} =$
$$\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{\imath} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\hat{\jmath} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{\mathbf{k}}$$

- So far, we have

$$\nabla^2 p = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}, \qquad \rho_0\left(\frac{\partial \vec{u}}{\partial t}\right) = -\nabla p, \qquad \text{and} \quad p = \rho_0 c^2 s$$

where $c = \sqrt{B/\rho_0}$, and s = density fluctuation $\left(\dfrac{\rho - \rho_0}{\rho_0}\right)$

recall* that $\nabla \times \nabla f = 0$ for any f $\;\rightarrow\; \rho_0 \dfrac{\partial}{\partial t}\left(\nabla \times \vec{u}\right) = -\nabla \times \nabla p = 0$

so $\nabla \times \vec{u} = 0$ ($\vec{u}$ must be *irrotational*)

$\rightarrow$ $\vec{u}$ can be expressed as gradient of a scalar function Φ

$\vec{u} = \nabla \Phi$ where Φ is a scalar *velocity potential*

- Adiabatic, inviscid fluid has no rotational flow → no turbulence, shear waves, or boundary layers
  - Real fluids have viscosity, velocity u is not curl-free everywhere, but small rotational effects at boundaries do not affect propagation of sound waves

Force eqn $\rightarrow$ $\rho_0 \dfrac{\partial(\nabla\Phi)}{\partial t} + \nabla p = 0 \rightarrow \nabla\left(\rho_0 \dfrac{\partial \Phi}{\partial t} + p\right) = 0 \rightarrow$ true if $p = -\rho_0 \dfrac{\partial \Phi}{\partial t}$

…so Φ also satisfies the wave eqn

Speed of sound in more detail

- Speed of sound is related to thermodynamic variables:

$$p = \rho_0 c^2 s \quad \text{where } s = \left(\frac{\rho - \rho_0}{\rho_0}\right) \rightarrow c^2 = \left.\frac{\partial P}{\partial \rho}\right|_{ADIABATIC}$$

Meaning: evaluate derivative at equilibrium in terms of P and ρ

- For ideal gas,

$$\frac{P}{P_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma}, \quad \gamma = \left(\frac{C_P}{C_V}\right) \text{ ratio of specific heats} \rightarrow \left.\frac{\partial P}{\partial \rho}\right|_{ADIAB} = \gamma \frac{P}{\rho}$$

$$\rightarrow \text{evaluated at equilibrium pressure and density, } c^2 = \gamma \frac{P_0}{\rho_0}$$

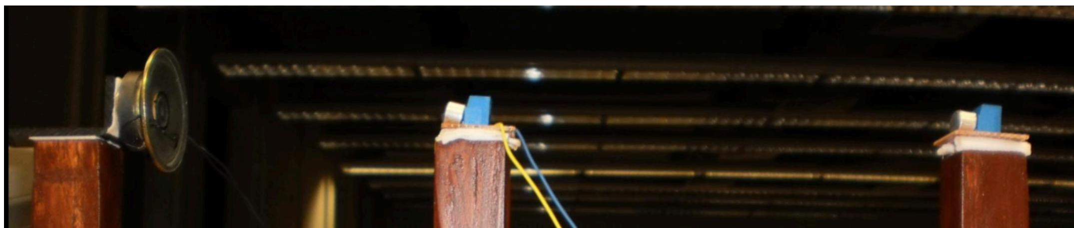
Example: for air at $0^{\circ}\mathrm{C}$ and $P_0 = 1$ atm $= 1.013\text{x}10^5$ Pa,

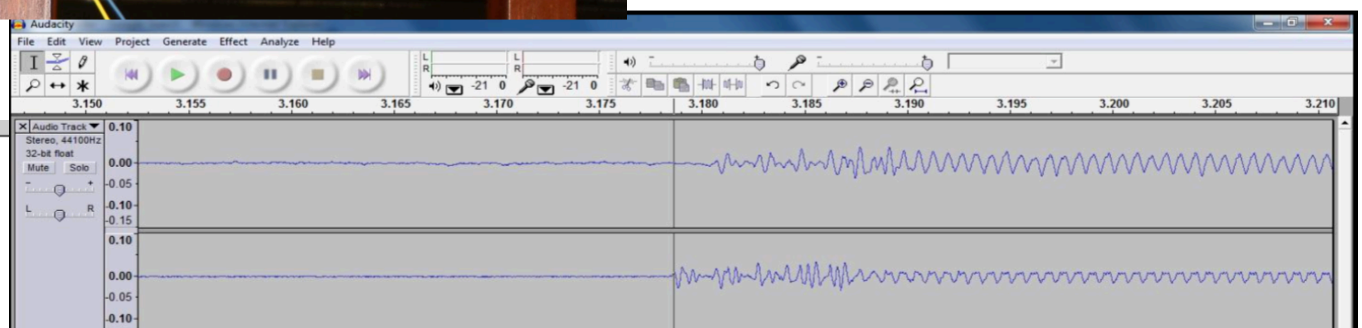
$$\rho_0 = 1.293 \text{ kg/m}^3, \ \gamma = 1.402, \ \sqrt{\gamma \frac{P_0}{\rho_0}} = 331.6 \text{ m/s}$$

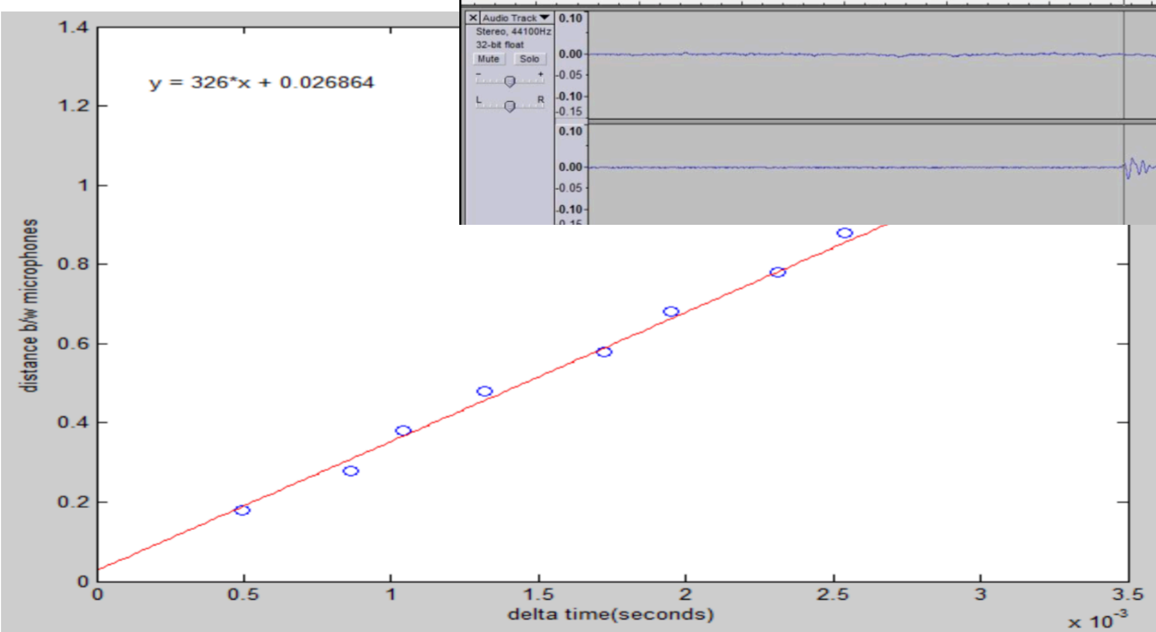
- See if you can google an actually measured value: I couldn't!
  - Everyone just quotes this theoretical value…
  - I found a student lab report that gave c=326m/s – but they forgot to report temperature !

Sound speed measurement @ home

- The student report* described using a loudspeaker aimed at two microphones with measured separation
- They used Audacity freeware (see Heller book's website for link); has 2 inputs with 20 microsec resolution, so they used these to measure Δt







* www.physlab.org/wp-content/uploads/2016/04/Speed_sound.pdf

Harmonic plane waves

- Harmonic plane waves = sin/cos waves in 1 direction:

  p(x,y,z,t)=p(x,t) → *y-z* planes have constant p value

1D wave equation $\frac{\partial^2 p(x,t)}{\partial x^2} = \frac{1}{c^2}\left(\frac{\partial^2 p}{\partial t^2}\right)$ with $c^2 = \mathrm{B}/\rho_0$, B=bulk modulus of fluid

$p = Bs = \rho_0 c^2 s$, s = density fluctuation relative to ρ_0

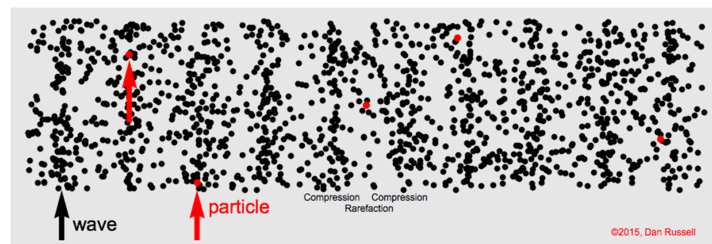
adiabatic B of air = 1.42×10^5 Pa
→ c = 331 m/s, same as before

wave equation has complex solution $p = p_+ + p_-$

with wave in +x direction $p_+ = Ae^{i(\omega t - kx)}$, and wave in -x direction $p_- = Be^{i(\omega t + kx)}$

particle speeds are $u_+ = +\frac{p_+}{\rho_0 c}$, $u_- = -\frac{p_-}{\rho_0 c}$

density fluctuation relative to ρ_0 are $s_\pm = \pm\frac{p_+}{\rho_0 c^2}$



plane wave in arbitrary direction $\vec{p} = \vec{A}e^{i(\omega t - k_x x - k_y y - k_z z)}$

3D wave equation $\nabla^2 p = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}$ is satisfied if $\sqrt{k_x^2 + k_y^2 + k_z^2} = \omega/c$

$\vec{k} = k_x\hat{x} + k_y\hat{y} + k_z\hat{z}$ propagation vector and $\vec{r} = x\,\hat{x} + y\,\hat{y} + z\,\hat{z}$ position vector

$\rightarrow \vec{p} = \vec{A}e^{i(\omega t - \vec{k}\cdot\vec{r})}$ plane wave, and surfaces of constant phase have $\vec{k}\cdot\vec{r}$ = constant

$\vec{k} = \nabla(\vec{k}\cdot\vec{r})$ points in the direction of propagation: $\hat{k} = \vec{k}/\left|\vec{k}\right|$ is the ray direction

Energy transported by plane waves

- Energy carried by small parcel V_0 that moves with fluid:

KE and PE: $E_K = \frac{1}{2}\rho_0 V_0 u^2$, and $E_P = -\int_{V_0}^{V} p\, dV$ (compression from V_0 to V)

must have $\rho_0 V_0 = \rho V \rightarrow dV = -\frac{V_0}{\rho_0} d\rho$

$p = \rho_0 c^2 s$ and $s = (\rho - \rho_0)/\rho_0 \rightarrow d\rho = dp/\rho_0 c^2 \rightarrow dV = -\frac{V_0}{\rho_0 c^2} dp$

$E_P = -\int_0^p p \frac{-V_0}{\rho_0 c^2} dp = \frac{p^2 V_0}{2\rho_0 c^2} \rightarrow$ total acoustic energy in $V_0 = E_{TOT} = \frac{1}{2}\rho_0 V_0 \left(u^2 + \frac{p^2}{\rho_0^2 c^2} \right)$

instantaneous energy density $\mathrm{E}_i = E_{TOT}/V_0 = \frac{1}{2}\rho_0 \left(u^2 + \frac{p^2}{\rho_0^2 c^2} \right)$ J/m^3

time average gives energy density $\mathrm{E} = \langle \mathrm{E}_i \rangle_T = \frac{1}{T}\int_T \mathrm{E}_i\, dt$, T = period of wave

so for plane harmonic wave in the +x direction $p = \rho_0 c\, u \rightarrow \mathrm{E}_i = \rho_0 u^2 = pu/c$

for $p = Pe^{i(\omega t - kx)}$ and $u = Ue^{i(\omega t - kx)}$, $\mathrm{E} = \langle \mathrm{E}_i \rangle_T = \frac{1}{2} PU/c = \frac{1}{2}\frac{P^2}{\rho_0 c} = \frac{1}{2}\rho_0 U^2$

(These simple results only valid for plane waves)

Intensity of acoustic waves

- Intensity of acoustic wave = time rate of energy flow (power) through a unit area normal to propagation direction

$$E_i = pu/c \rightarrow I = \langle pu \rangle_T = \frac{1}{T}\int_T pu\,dt$$

Amplitude = max Δp in wave
Intensity = power/area ~Δp^2

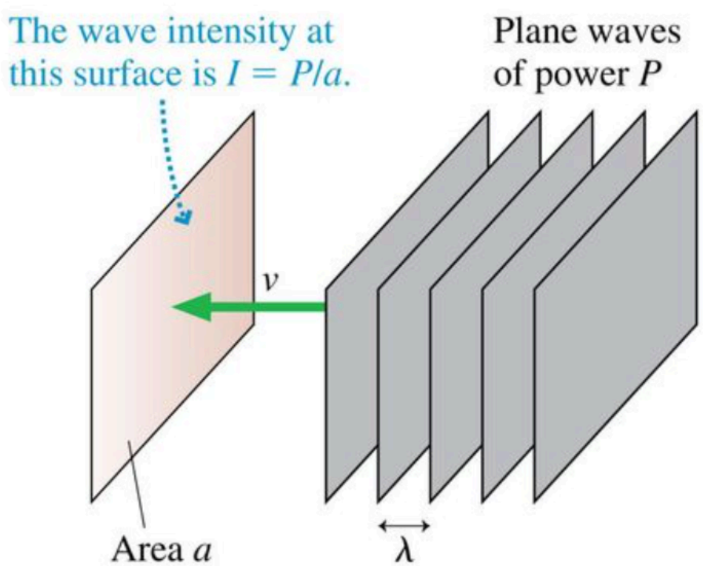
$$\text{for plane wave in +x direction, } I = \frac{1}{2}P_+U_+ = \frac{1}{2}\frac{P_+^2}{\rho_0 c}$$

$$\text{for plane wave in -x direction, } I = -\frac{1}{2}P_-U_- = -\frac{1}{2}\frac{P_-^2}{\rho_0 c}$$

RMS values $\sqrt{\frac{1}{T}\int_T f^2\,dt}$

are useful time averages providing effective intensities:

$$P_{\pm RMS} = P_\pm/\sqrt{2} \rightarrow I_{\pm RMS} = \frac{P_+U_+}{2} = \frac{P_+^2}{2\rho_0 c}$$



Specific acoustic impedance

- We found acoustic impedance = driving force/resulting speed
- In terms of acoustical pressure p, we get

Specific acoustical impedance $z = p / u$

For plane waves $u_{\pm} = \pm \dfrac{p_+}{\rho_0 c} \rightarrow \quad z = \pm \rho_0 c$ (for $+x$ or $-x$ direction)

$\rho_0 c =$ Characteristic impedance of medium

Remember: lower case p, ρ, u etc are *relative to* baseline values P_0, ρ_0 , etc

- SI units for z : Pa-s/m (1 Pa-s/m =1 *rayl*, named after Rayleigh)

z is real for plane waves, but standing waves or diverging waves have complex values:

$z = r + i\,x$

(r, x = specific acoustic resistance, reactance)

characteristic of medium for non-planar wave type considered

Material	Sound Velocity (in./sec.)	Density (lbs./In.^{3})	Acoustical Impredance (lbs./In.^{2} sec.)
Steel	206,500	0.283	58,400
Copper	140,400	0.320	45,000
Cast Iron	148,800	0.260	38,700
Lead	49,800	0.411	20,400
Glass	216,000	0.094	20,300
Concrete	198,000	0.072	14,200
Water	56,400	0.036	2,030