PHYS 536

R. J. Wilkes

Session 7

Spherical waves; point sources

Rays and waves

Plane waves and spherical waves

1/29/2019

Course syllabus and schedule – updated

See : http://courses.washington.edu/phys536/syllabus.htm

6	19-Jan	Thu	K. Ch. 5, 6	H: Ch. 1	… of sound; Harmonic plane waves, intensity, impedance. *UPDATED BELOW HERE:*
7	24-Jan	Tue	K. Ch. 5	H: Ch. 2	Spherical waves; Eikonal equation and ray tracing; transmission and reflection at interfaces;
8	26-Jan	Thu	K: Ch. 7	H: Ch. 7	Absorption losses; Pulsating spheres and simple sources and dipoles ← Tonight
9	31-Jan	Tue	K. Ch. 8-10	H: Chs. 13	Near field, far field; Radiation impedance; waves in pipes; rectangular cavities; Helmholtz resonators;
10	2-Feb	Thu	K. Ch 12		Resonant bubbles; Acoustic impedance; physical acoustic filters; Decibels and sound level measurements
11	7-Feb	Tue	K. Ch 12	H: Ch. 28	Environmental acoustics and noise criteria; industrial and community noise regulations; noise mitigation; frequency, pitch and musical tones
12	9-Feb	Thu		H: Chs. 18-19	Interference effects; Musical acoustics; Musical instruments: strings. REPORT 1 PAPER DUE by 7 PM; REPORT 2 PROPOSED TOPIC DUE
13	14-Feb	Tue	K. Ch. 11	H: Ch. 16	Musical instruments: winds; The ear, and human hearing;
14	16-Feb	Thu	K. Ch. 11	H: Chs. 21-22	Human hearing: the inner ear; pitch perception; acoustics of speech
15	21-Feb	Tue	K. Chs. 13-14	H: Ch. 27; Ch. 6	Room acoustics; Transducers for use in air and water: Microphones and loudspeakers; hydrophones and pingers
16	23-Feb	Thu	K. Ch 15		Underwater acoustics; sound absorption underwater, the sonar equation; applications: acoustical positioning, seafloor imaging, sub-bottom profiling; Doppler effect
17	28-Feb	Tue			Course wrapup. Student report 2 presentations
18	2-Mar	Thu			Student report 2 presentations
19	7-Mar	Tue			Student report 2 presentations
20	9-Mar	Thu			Student report 2 presentations. TAKE-HOME FINAL EXAM ISSUED
--	17-Mar	Fri			FINAL EXAM ANSWERS DUE by 5 PM

Class is over after you turn in your take-home exam. No in-person final exam during finals week.

From session 2

Driven damped mechanical oscillator

- Analogy to electrical circuits: mechanical impedance

$$\text{complex impedance } Z_m = R_m + iX_m = |Z_m| e^{i\Theta}$$

$$\mathrm{Re}[Z_m] = R_m; \quad \mathrm{Im}[Z_m] = \text{reactance } X_m = (\omega m - s/\omega)$$

$$|Z_m| = \sqrt{R_m^2 + (\omega m - s/\omega)^2},$$

$$\Theta = \tan^{-1} \frac{X_m}{Z_m}$$

$$\text{Displacement : } x(t) = \frac{Fe^{i\omega t}}{i\omega[R_m + i(\omega m - s/\omega)]}$$

$$\text{Speed : } u(t) = \frac{Fe^{i\omega t}}{[R_m + i(\omega m - s/\omega)]} = \frac{f(t)}{Z_m} \rightarrow Z_m = \frac{f(t)}{u(t)}$$

Z_e = ratio of V to I

Z_m = ratio of driving force to speed

"mechanical ohm" has units of force/speed

From last time

Specific acoustic impedance

- We found acoustic impedance = driving force/resulting speed
- In terms of acoustical pressure p, we get

Specific acoustical impedance $z = p / u$

$$\text{For plane waves } u_{\pm} = \pm \frac{p_+}{\rho_0 c} \rightarrow \quad z = \pm \rho_0 c \text{ (for } +x \text{ or } -x \text{ direction)}$$

$\rho_0 c =$ Characteristic impedance of medium

Remember: lower case p, ρ, u etc are *relative to* baseline values P_0, ρ_0 , etc

- SI units for z : Pa-s/m (1 Pa-s/m =1 *rayl*, named after Rayleigh)

z is real for plane waves, but standing waves or diverging waves have complex values:

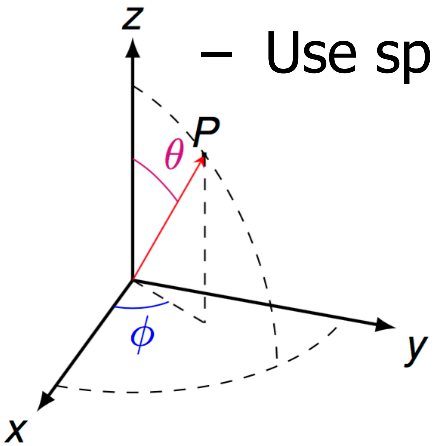
$z = r + i x$

(r, x = specific acoustic resistance, reactance) characteristic of medium for non-planar wave type considered

Material	Sound Velocity (in./sec.)	Density (lbs./In.^{3})	Acoustical Impredance (lbs./In.^{2} sec.)
Steel	206,500	0.283	58,400
Copper	140,400	0.320	45,000
Cast Iron	148,800	0.260	38,700
Lead	49,800	0.411	20,400
Glass	216,000	0.094	20,300
Concrete	198,000	0.072	14,200
Water	56,400	0.036	2,030

Spherical waves

- Common case: small source, large volume for wave expansion
  - Use spherical coordinates (r, θ, φ)



$x = r\sin\theta\cos\phi \qquad r = \sqrt{x^2+y^2+z^2}$

$y = r\sin\theta\sin\phi \qquad \theta = \arccos(z/r)$

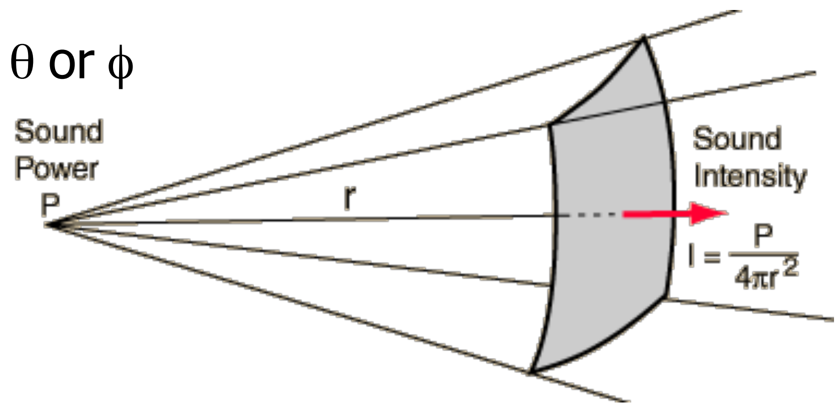
$z = r\cos\theta \qquad \phi = \arctan(y/x)$

acoustic wave equation

$$\nabla^2 p = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}$$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

- Simplest case: spherically symmetric
  - Intensity = f(r only), not dependent on θ or φ



Then $\nabla^2 p$ reduces to r terms only,

$$\nabla^2 p = \frac{\partial^2 p}{\partial r^2} + \frac{2}{r}\frac{\partial p}{\partial r}$$

wave eqn is $\dfrac{\partial^2 p}{\partial r^2} + \dfrac{2}{r}\dfrac{\partial p}{\partial r} = \dfrac{1}{c^2}\dfrac{\partial^2 p}{\partial t^2}$;

since $I = P^2/(2\rho_0 c) = p^2/const$

conservation of energy $\rightarrow$ I should drop off as $1/r^2$

(~ area of spherical surface of radius r)

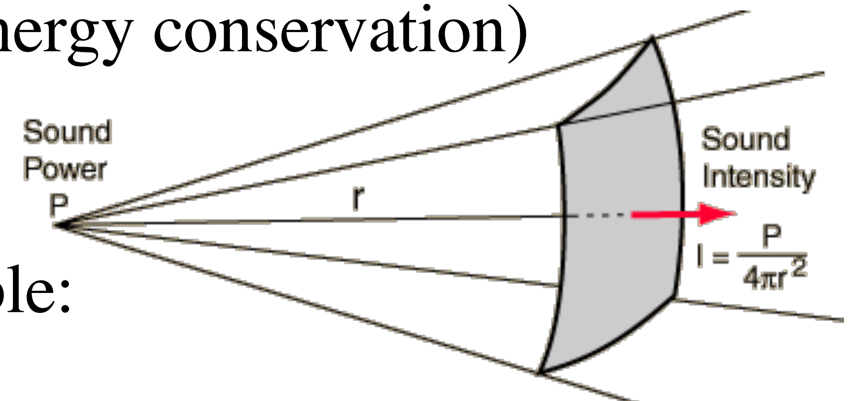
Spherical waves

- Symmetrical, diverging spherical waves

$I \propto p^2$ should drop off as $1/r^2$ (energy conservation)

$\rightarrow p$ should drop off as $1/r$

so $(r\,p)$ should be $\sim const$



$\rightarrow$ use $f = (r\,p)$ as wave eqn variable:

$$\frac{\partial^2 f}{\partial r^2} = \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} \quad \text{(same form as plane wave case)}$$

solution must be of the form $f(r,t) = f_1(ct - r) + f_2(ct + r)$

$$\rightarrow p = \frac{f}{r} = \frac{1}{r} f_1(ct - r) + \frac{1}{r} f_2(ct + r) :$$

f_1 is diverging wave, f_2 converging

Spherical waves

- Symmetrical, diverging spherical waves

Most applications involve only diverging spherical waves:

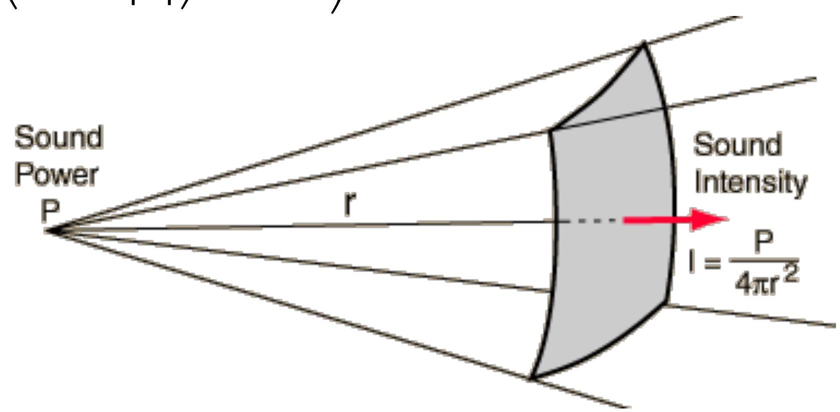
general (complex) form of solution is $p(r,t) = \frac{A}{r}\exp(i\omega t - kr)$

Can use pressure potential $\Phi(r,t)$ to relate p to other acoustic variables:

$$\Phi_+(r,t) = -\frac{\vec{p}}{i\omega\rho_0}, \quad \vec{u} = \nabla\Phi = \left(1 - \frac{i}{k|r|}\right)\frac{\vec{p}}{i\omega\rho_0}$$

Observable quantities are real parts:

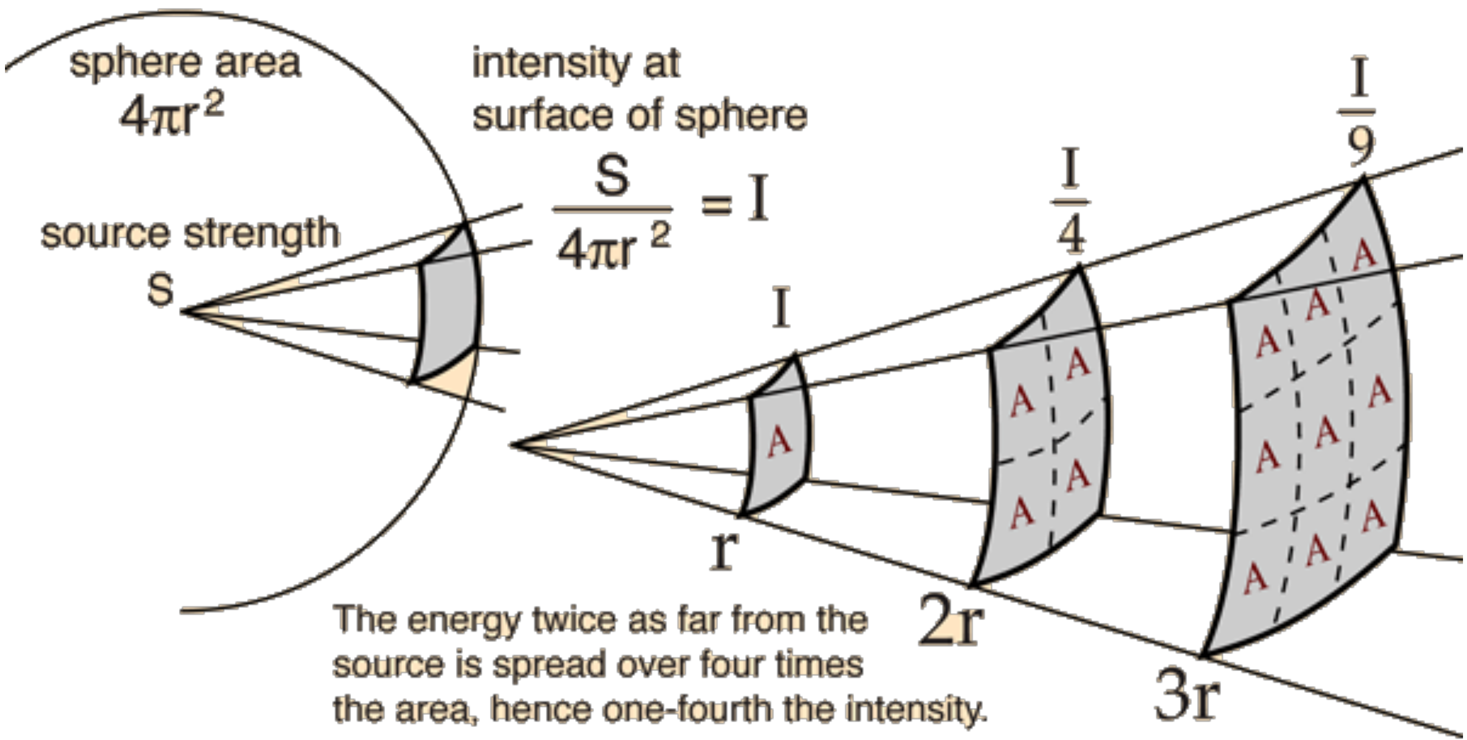
$$p(r,t) = \mathrm{Re}\left(\frac{A}{r}\exp(i\omega t - kr)\right), \quad u(r,t) = \mathrm{Re}\left(=\left(1 - \frac{i}{k|r|}\right)\frac{\vec{p}}{i\omega\rho_0}\right)$$



Visualizing spherical waves

- http://www.falstad.com/ripple/
- http://www.falstad.com/wavebox/fullscreen.html

- Intensity drops off as $1/r^2$ because wave energy is uniformly distributed over a sphere of radius r, with surface area ~ r^2
  - factor of 2 increase in r → factor of 4 decrease in I



Point sources

- Point source = emitter << wavelength of sound radiated
  - Human hearing range: λ=20m ~ 2cm in air, 75m ~7.5 cm in water
  - Symmetrical, diverging spherical waves from a monopole source emitting a single frequency

Solution $p(\vec{r},t) = \frac{A}{r}\exp(i\omega t - kr)$ satisfies the wave eqn with point source at $\vec{r} = 0$:

$$\nabla^2 p - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = -4\pi A\delta(\vec{r})\exp(i\omega t), \quad \left(c = \sqrt{B/\rho_0}\right)$$

$$\left(\text{recall:} \int_V 4\pi A\delta(\vec{r})\,dV = 4\pi A \;\text{ if } V \text{ includes } \vec{r} = 0, \text{ or } 0 \text{ if not}\right)$$

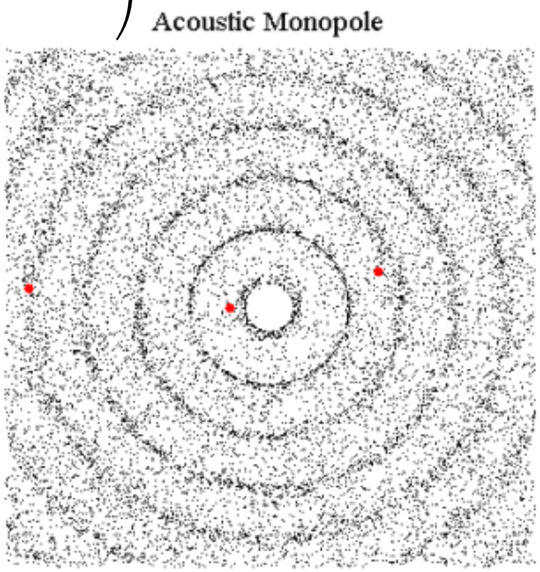
For a point source at $\vec{r} = \vec{r}_0$ instead of 0, the wave equation becomes

$$\nabla^2 p - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = -4\pi A\delta(\vec{r} - \vec{r}_0)\exp(i\omega t)$$

and its solution is

$$p(r,t) = \frac{A}{|\vec{r} - \vec{r}_0|}\exp(i\omega t - k|\vec{r} - \vec{r}_0|)$$

Red dots = particle motion



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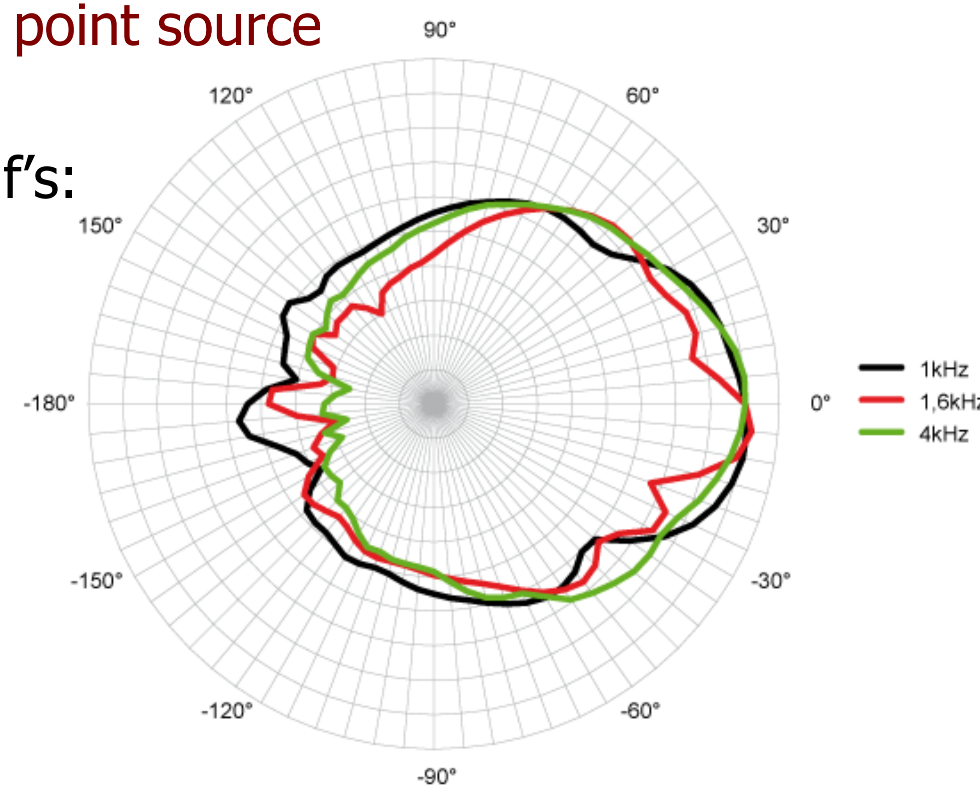
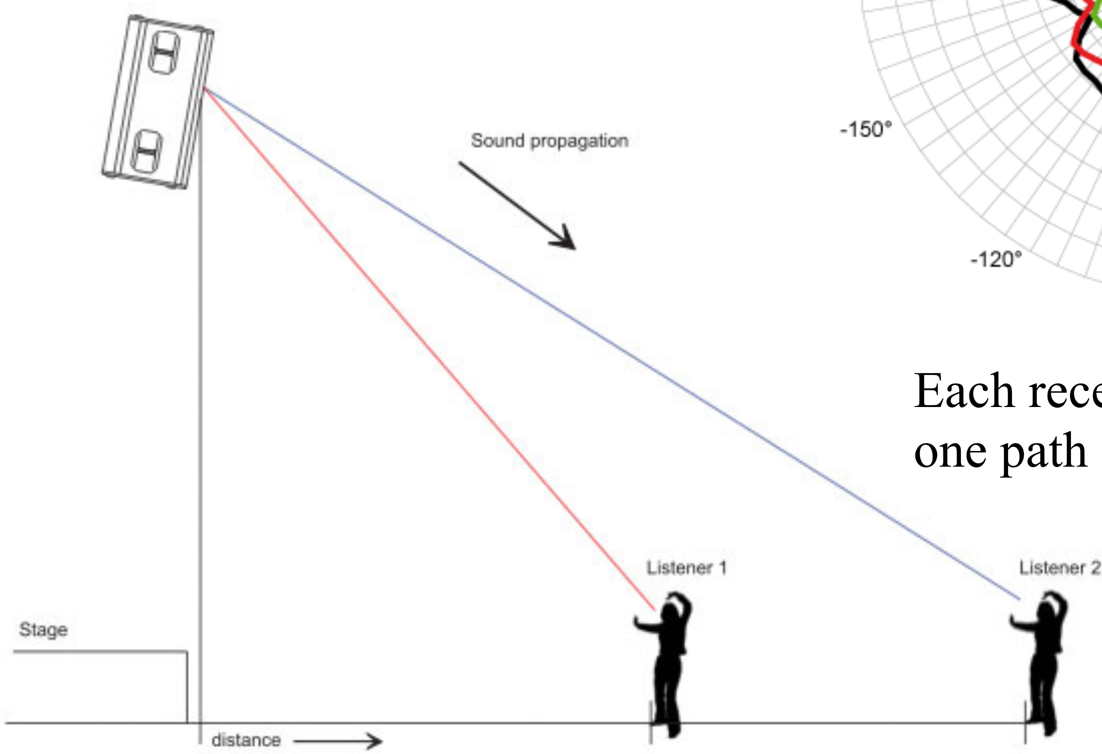
Point source loudspeakers for large halls

Loudspeaker designed to act as a point source

- Angular distribution at various f's:





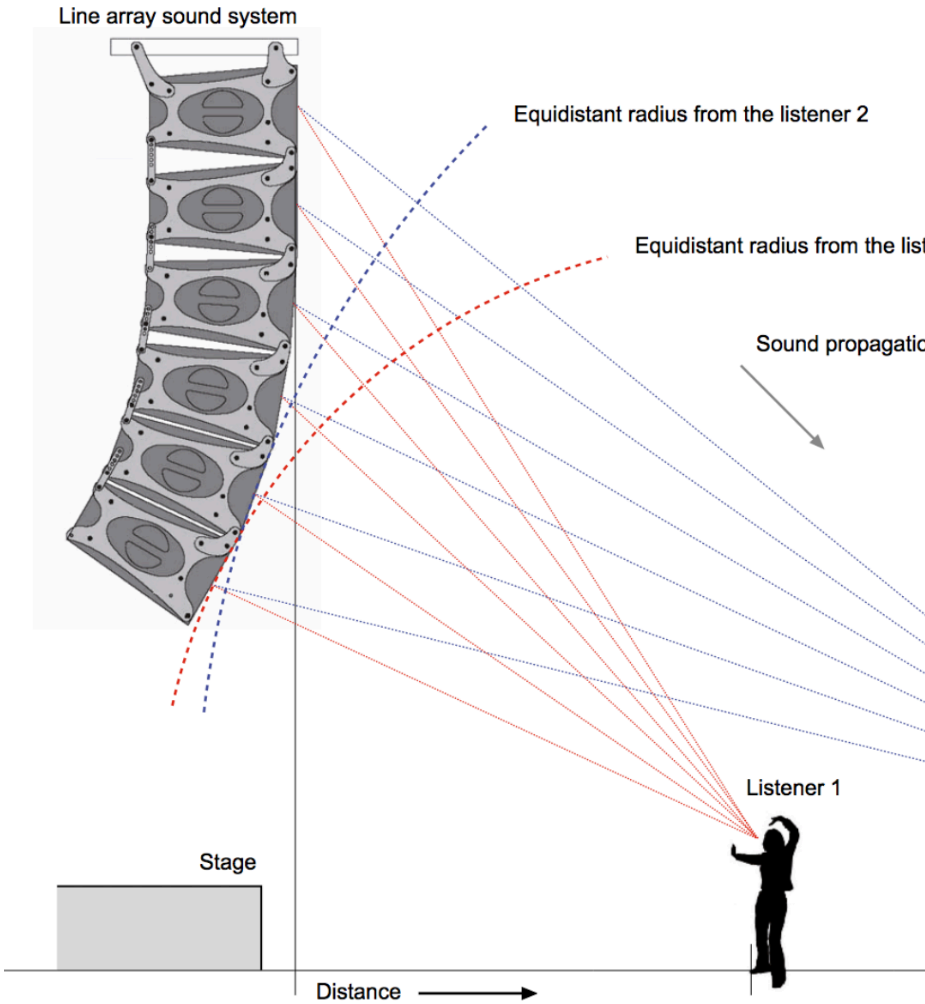
Each receiver has a single direct path:
one path length, no interference effects

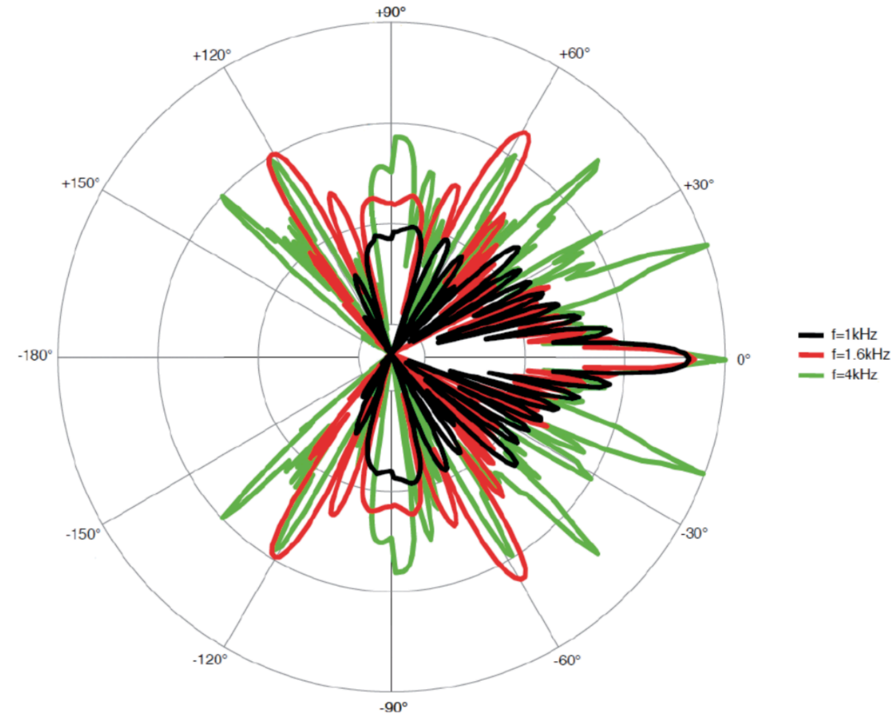
From kv2audio.com

For comparison: line sources

Loudspeakers arrayed in a line:

- Angular distribution at various f's:





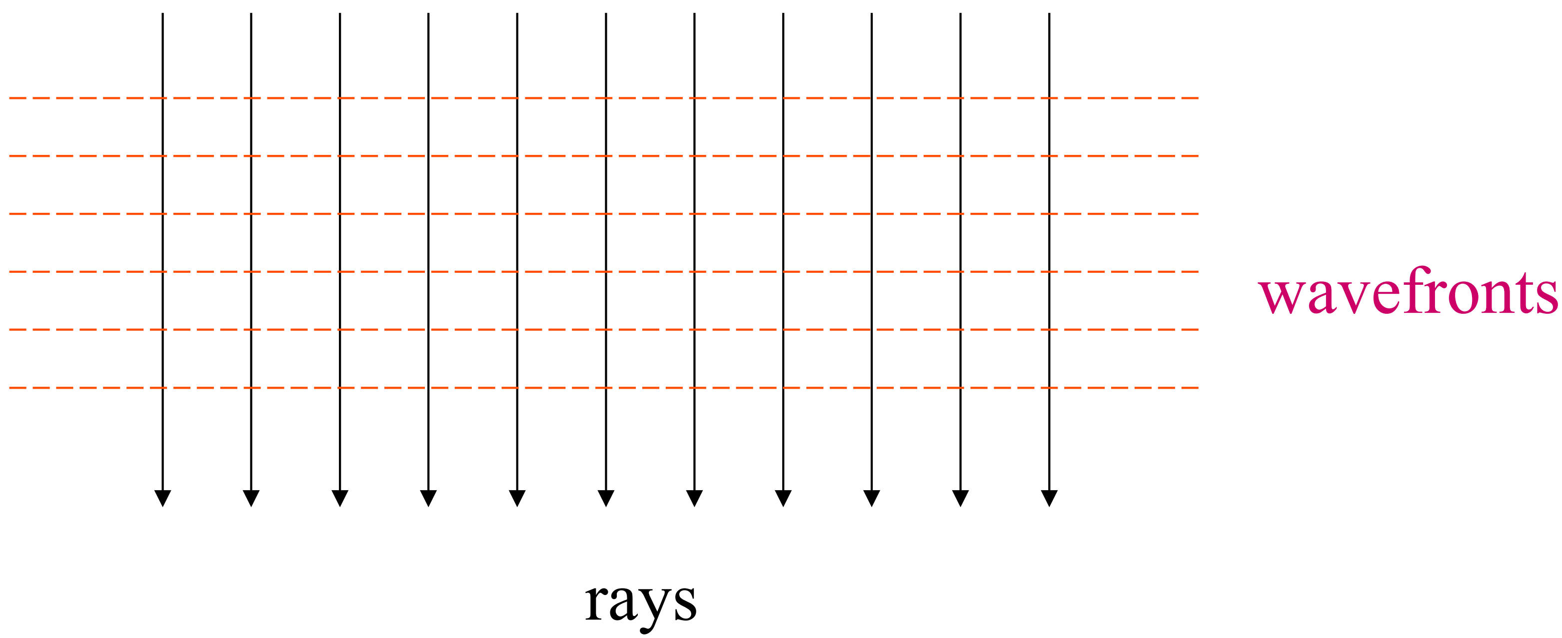
Each receiver has multiple path lengths to loudspeakers: interference causes spikes in angular distribution of sound – dead spots

From kv2audio.com

From 1st session

Rays vs waves: handy fiction

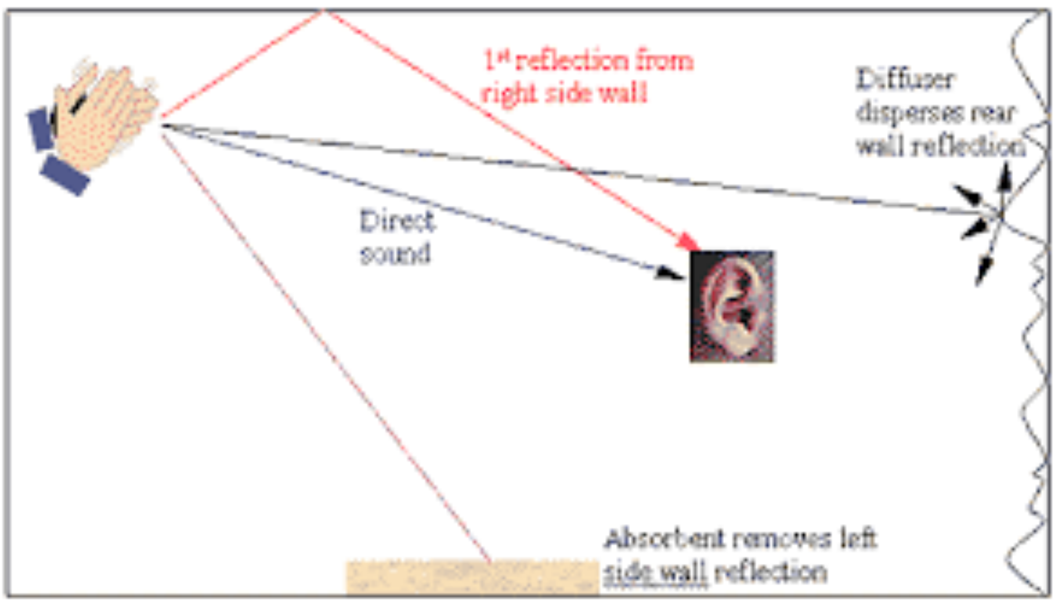
- We know sound travels in the form of waves, but
  - We can picture sound "rays" as lines from source outward
  - We define rays as lines perpendicular to the wavefronts
  - "Ray tracing" is useful for following paths of sound waves



Ray optics

- Use simple geometric rules of ray tracing to analyze sound arriving at any point from a given point source
  - Refraction where sound speed changes
    - Localized angle change at interfaces between media
    - Continuous angle change in media with varying c
  - Reflection off hard surfaces:
    - Specular reflection from smooth surfaces
    - Diffuse reflection from textured surfaces
  - Transmission vs absorption for all media and interfaces

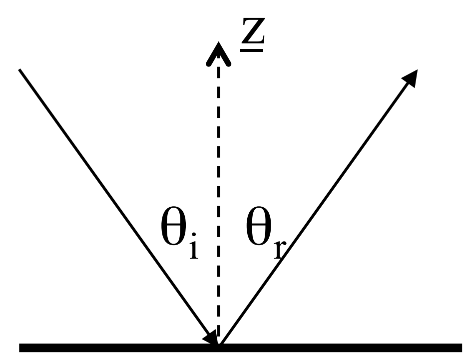
Arrival time depends on path length of ray



Geometrical (Ray) "optics" for acoustics

- Valid when system dimensions >> wavelength λ
  - Neglect wave nature of sound
  - Analyze systems by ray tracing
  - Rays = Lines normal to wavefronts
- Basic Laws:
- Reflection: $\theta_r = \theta_i$

  $\underline{z}$ = normal to surface



- Refraction: $n_1 \sin\theta_1 = n_2 \sin\theta_2$

  Snell's (or Descartes') Law

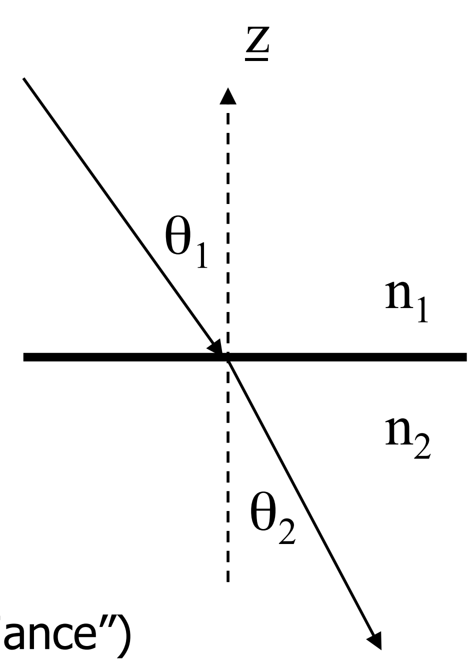
  n=index of refraction = c/v

  c= reference value of sound speed

  v=speed of sound at given location

  Can be derived several ways:
  - Wavelets (Huygens)
  - Variational principle (Fermat)
    - Ray follows fastest path from A to B

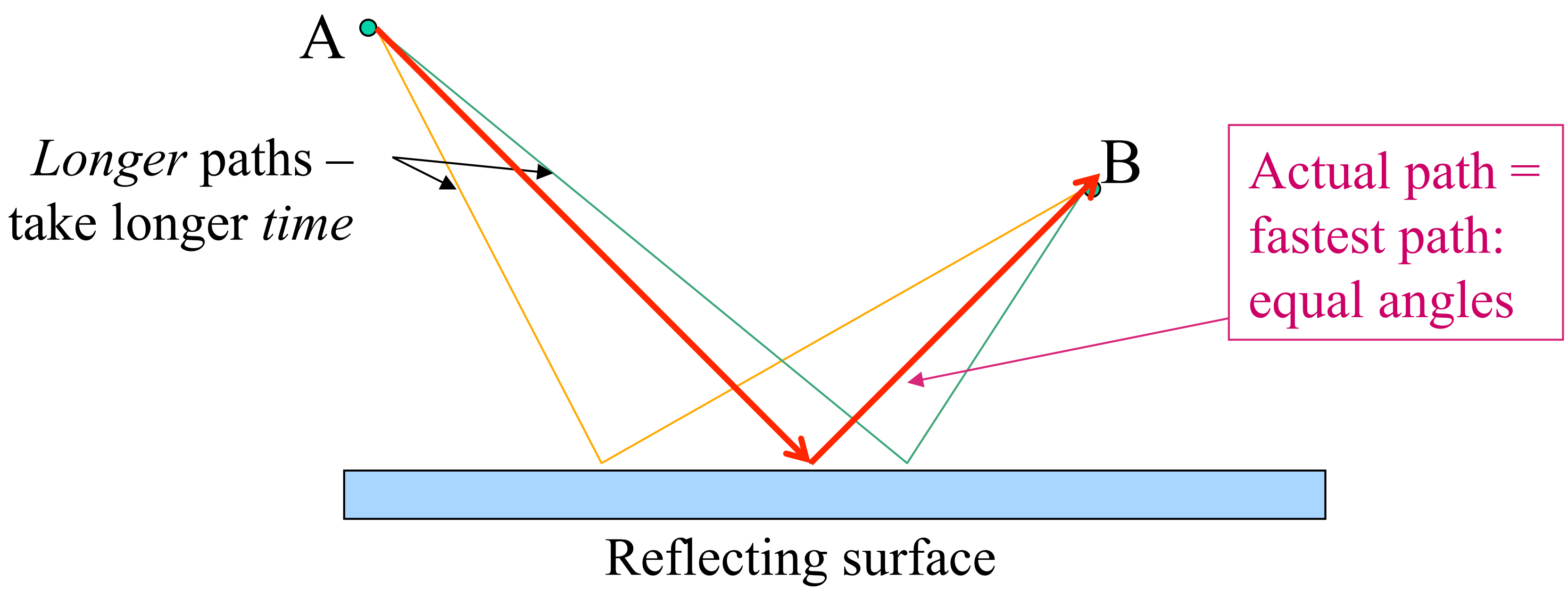
    → reversibility of ray paths ("time-symmetry invariance")



Ray reflection (Deep Thought)

- One important idea you learned in school: "angle of reflection equals angle of incidence"
  - Reflected ray makes same angle with reflector as incoming ray

  Reflection law is also consistent with the "principle of least time"
    - In going from point A to point B, reflecting off a mirror, the ray path actually traveled is also the *fastest* (shortest) route
    - Nature automatically finds the most *economical* path !



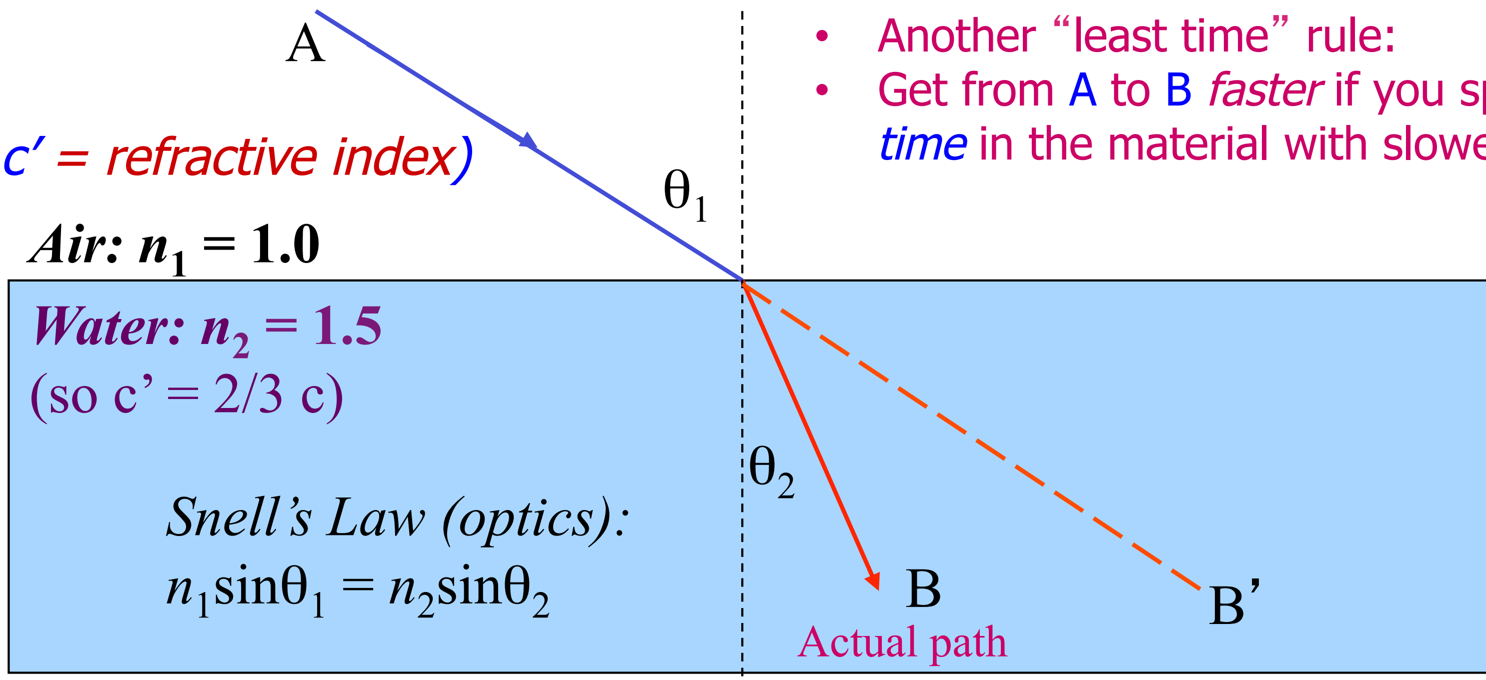
Refraction at a flat interface

- Sound rays *bend* at interfaces between media

  Same idea for acoustics as for refraction of light rays:
  - Universal constant c = 3×10^8 m/s = speed of light *in vacuum*
    - But speed of light in any *material medium* is slower: c' < c
    - In typical glass, c' = (2/3)c, pure water = (3/4)c, air =0.9997c
  - Rays bend more the larger the *difference* in speed between media
  - Object at B *appears* to be at location B' (= our brains' expectation)

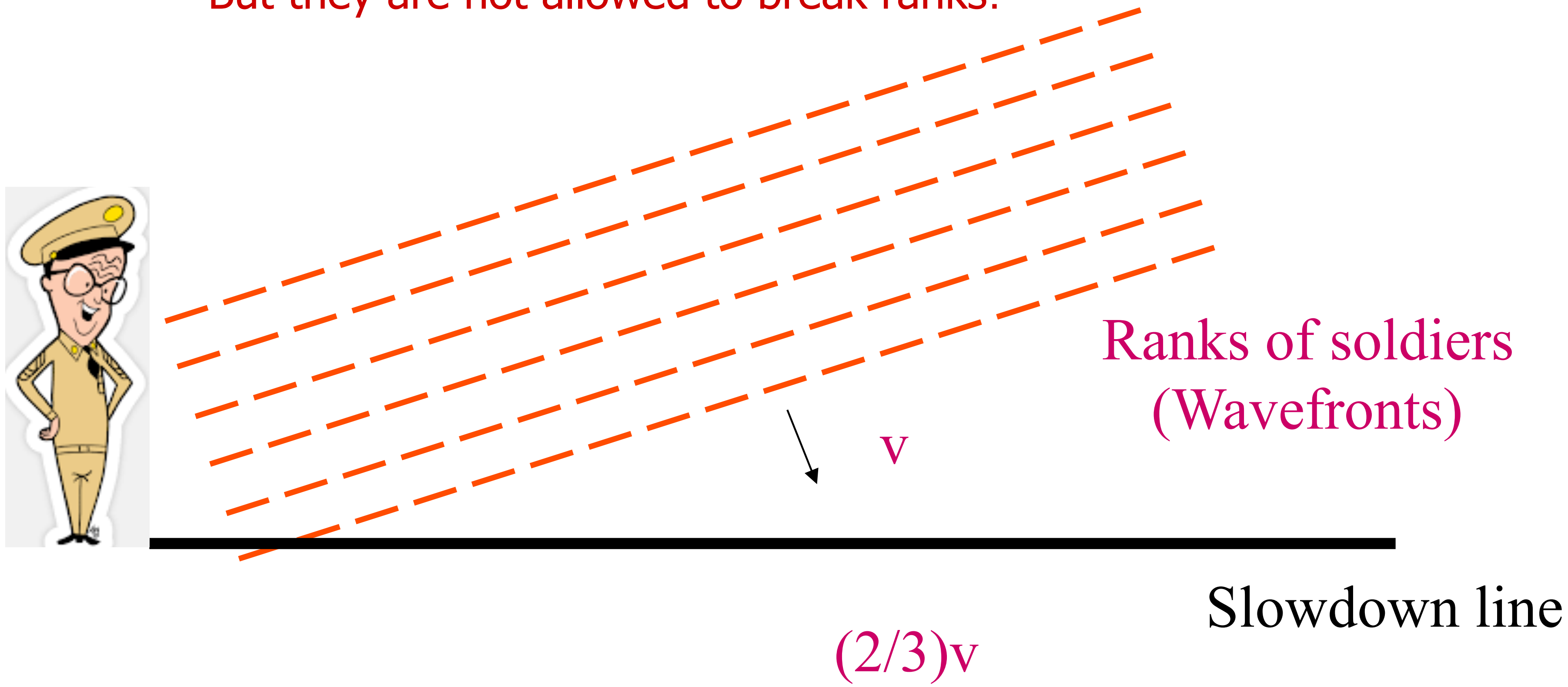
(*n = c / c' = refractive index*)

- Another "least time" rule:
- Get from A to B *faster* if you spend *less time* in the material with slower c'



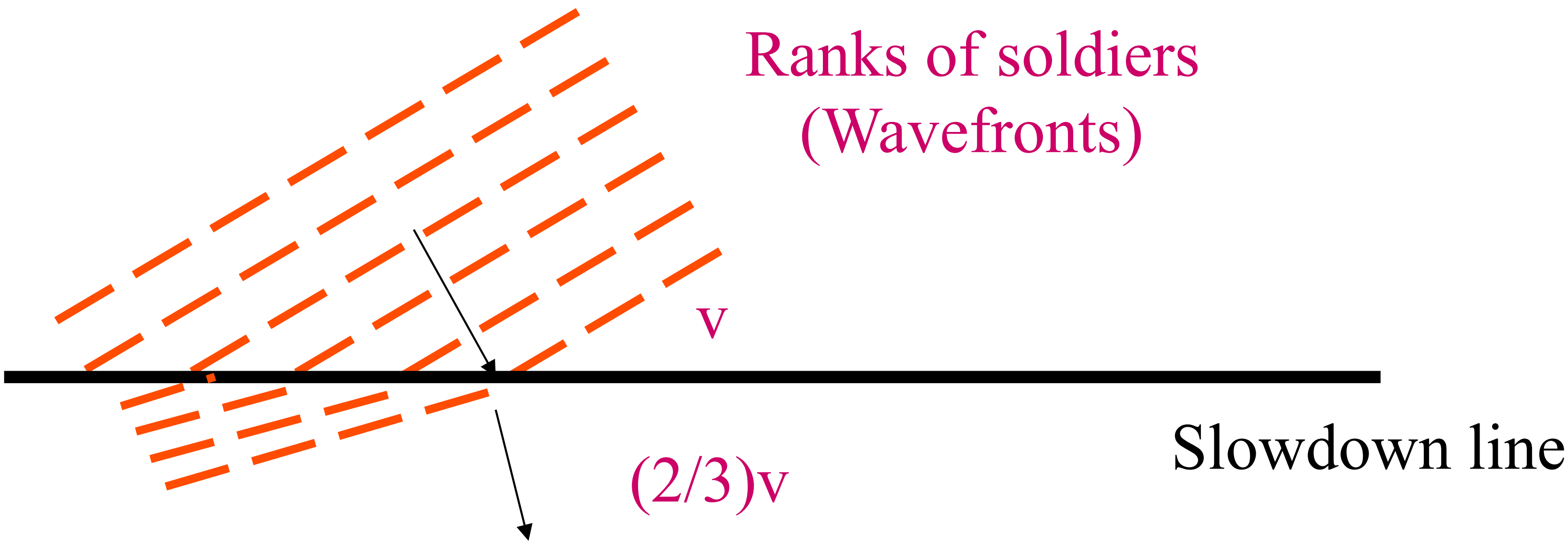
Parade analogy to understand refraction

- Imagine soldiers lined up in ranks, marching at constant speed
- Sgt. Bilko orders them to *slow down* to 2/3 normal speed when they cross a line marked on the parade ground
  - But they are not allowed to break ranks!



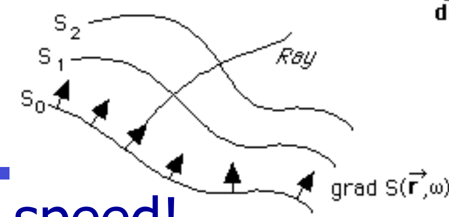
Parade analogy

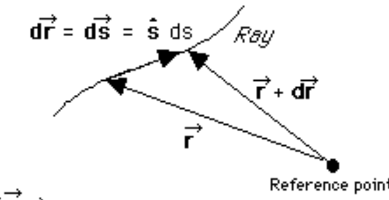
- Here is a picture after a few soldiers have passed the line



  – Notice:
    - if parade had approached at a right angle (***v*** perpendicular to the line), there would be no change of direction, ranks would just get closer
    - This analogy works whether you believe in waves (ranks as a unit) or particles (individual soldiers)
      – Isaac Newton and Thomas Young would agree on this

Eikonal equation





- There is no "universal constant" sound speed!
- Must use an arbitrary reference speed c_0
- The index of refraction is a variable: $n(x,y,z) = c_0 / c(x,y,z)$

For a plane wave $\vec{p} = \vec{A}e^{i(\omega t - \vec{k}\cdot\vec{r})}$, $\vec{k}\cdot\vec{r} =$ constant defines surfaces of constant phase. Rewrite this as

$$\vec{p} = A(x,y,z)\exp\left(i\omega\left[t - \Gamma(x,y,z)/c_0\right]\right) \qquad (\Gamma \text{ has units of length})$$

So places where $\Gamma(x,y,z) =$ constant are surfaces of constant phase.

As $\vec{k} = \nabla(\vec{k}\cdot\vec{r})$ points in the direction of propagation (ray direction), so does $\nabla\Gamma(x,y,z)$

Put this in wave eqn $\nabla^2 p = \dfrac{1}{c^2}\dfrac{\partial^2 p}{\partial t^2} \rightarrow \left(\dfrac{\omega}{c_0}\right)^2 - \left(\dfrac{\omega}{c_0}\right)^2 \nabla\Gamma\cdot\nabla\Gamma = 0$

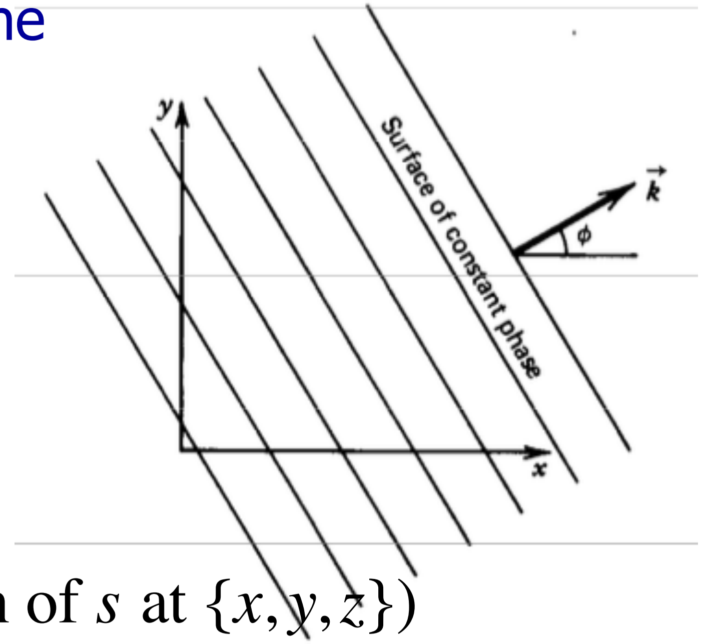
if amplitude A and $\nabla\Gamma$ vary slowly enough (requires A and $c \sim$ const over distances on the order of $\lambda \approx c/f$) $\rightarrow \nabla\Gamma\cdot\nabla\Gamma = const$

let $\nabla\Gamma\cdot\nabla\Gamma = n^2$ (*Eikonal equation* - encountered also in QM)

where $n(x,y,z) = c_0 / c(x,y,z)$

Eikonal equation and ray tracing

- Eikonal equation helps us do ray tracing: ∇Γ is the direction of ray
- Example: ray of sound moving in x-y plane



$\nabla\Gamma = n\cos\phi\,\hat{x} + n\sin\phi\,\hat{y}$ but ϕ will vary with t as the ray propagates.

Small patch on surface of constant phase moves along ray path.

Useful to consider variation of $\nabla\Gamma$ with s = distance along ray path:

$\nabla\Gamma = n(x,y,z)\hat{s}$ ($\hat{s}$ = unit vector in direction of s at $\{x,y,z\}$)

(see Kinsler for details - works for 3 dimensions also)

direction cosines of $\hat{s} = \{\alpha,\beta,\gamma\}$: $\hat{s} = \alpha\hat{x} + \beta\hat{y} + \gamma\hat{z}$, and $\alpha^2 + \beta^2 + \gamma^2 = 1$

rate of change in $\hat{s}$ direction is $\dfrac{d}{ds} = \alpha\dfrac{\partial}{\partial x} + \beta\dfrac{\partial}{\partial y} + \gamma\dfrac{\partial}{\partial z}$

Then $\dfrac{d(n\alpha)}{ds} = \dfrac{\partial n}{\partial x}$, $\dfrac{d(n\beta)}{ds} = \dfrac{\partial n}{\partial y}$, $\dfrac{d(n\gamma)}{ds} = \dfrac{\partial n}{\partial z}$

Ray tracing

- Sound ray moving in a plane – so c=c(z only) – is a common case:

Then $\dfrac{d(n\alpha)}{ds}=0, \quad \dfrac{d(n\beta)}{ds}=0, \quad \dfrac{d(n\gamma)}{ds}=\dfrac{\partial n}{\partial z}$

- *c(x,y,z)=c(x)*

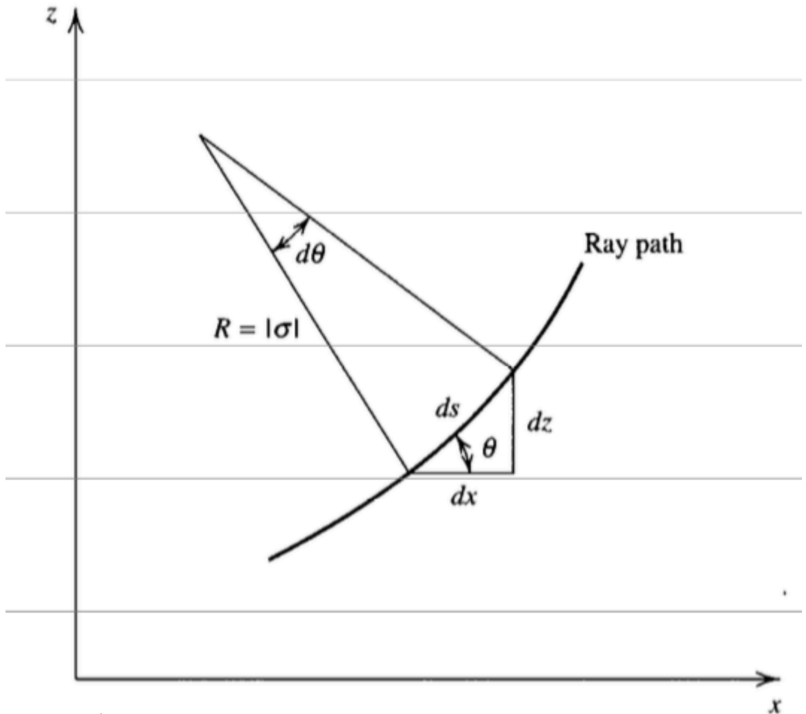
Then $n = n(x) = c_0 / c(x)$

If ray starts in the x-z plane with an angle θ with the x axis, then $\alpha=\cos\theta, \quad \gamma=\sin\theta, \quad \beta=0,$

and $\dfrac{d(n\beta)}{ds}=0 \rightarrow \beta=0$ always;

so $\dfrac{d(n\cos\theta)}{ds}=0 \rightarrow n\cos\theta = const,$

and $\dfrac{d(n\sin\theta)}{ds}=\dfrac{\partial n}{\partial z}$



θ_0 = angle of ray with the x axis at location where $c = c_0$

Then $n(x,y,z)\cos\theta = const = c_0\cos\theta / c(x,y,z)$

$$\rightarrow \frac{\cos\theta}{c(x,y,z)}=\frac{\cos\theta_0}{c_0} \quad \text{(Snell's Law)}$$

Snell's law

- When sound speed c(z) decreases with z, θ increases
- Rays always bend toward a region of lower c:

$$n = \frac{c_0}{c(z)} \rightarrow c(z) = c_0 n^{-1} \rightarrow \ \mathrm{sign}\left(\frac{dc}{dz}\right) = \ -\mathrm{sign}\left(\frac{dn}{dz}\right)$$

$$\frac{dc}{dz} = g, \quad \text{gradient of sound speed}$$

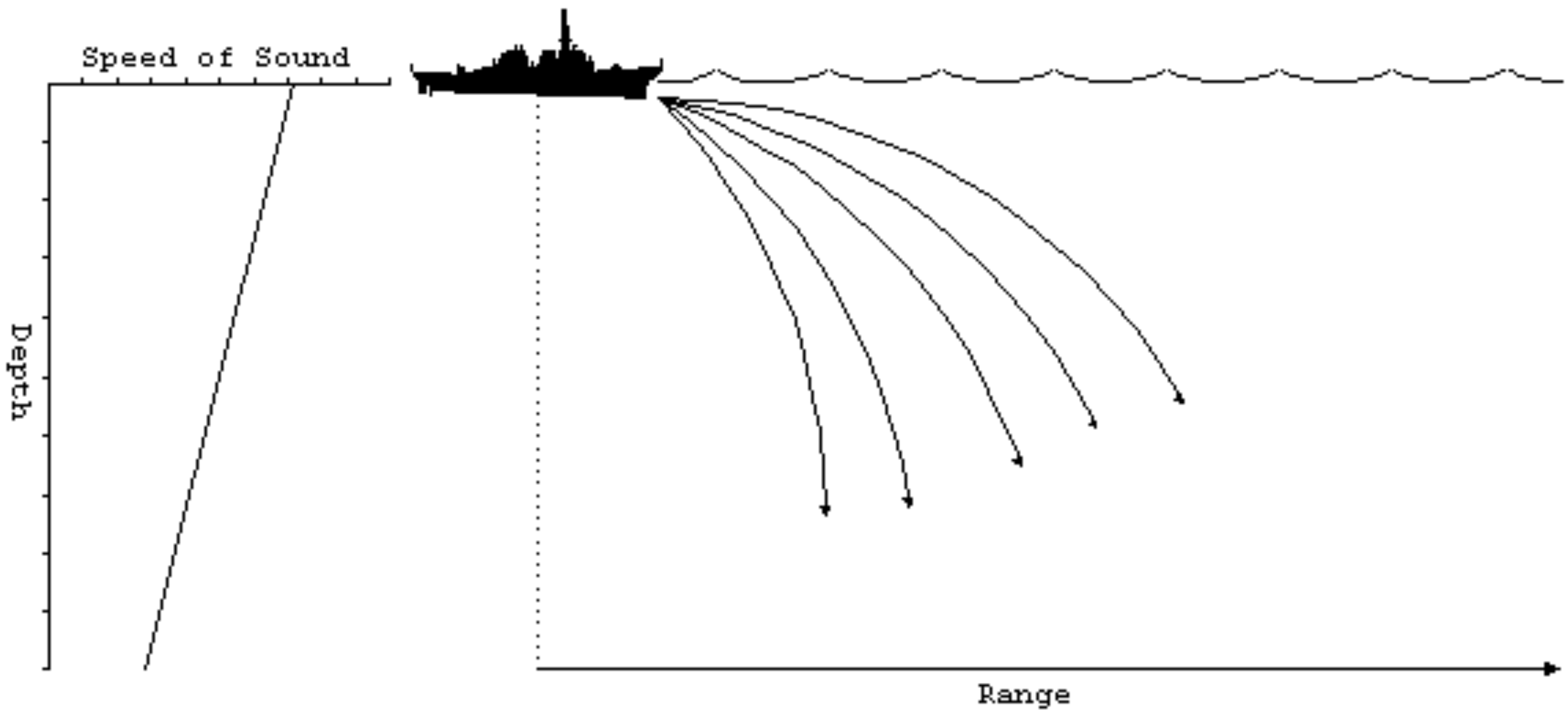
$$dz = \sin\theta ds \quad \text{and} \quad ds = Rd\theta, \quad \text{where } |R| = \text{ radius of curvature of ray}$$

$$R = -\frac{1}{g}\frac{c_0}{\cos\theta_0}$$

R is positive if $d\theta$ increases with s $\left(\dfrac{d^2\theta}{ds^2} > 0\right)$, negative if it decreases

Ray refraction with varying c

- Classic case: sound rays in water- oversimplified: linear c(z) = a - b z
- Sound speed c(z) decreases with depth z, θ (angle wrt x) increases
- Rays always bend downward, *toward* a region of lower c



- More on this in a few weeks…

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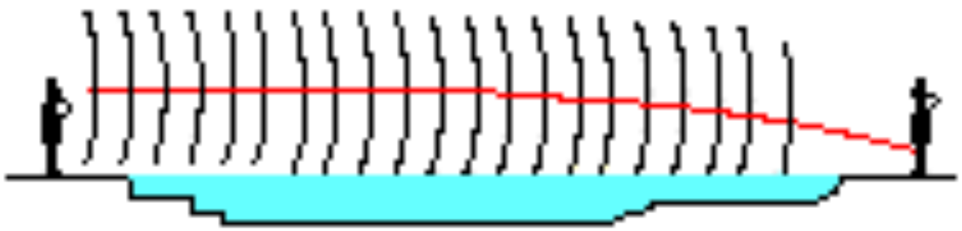
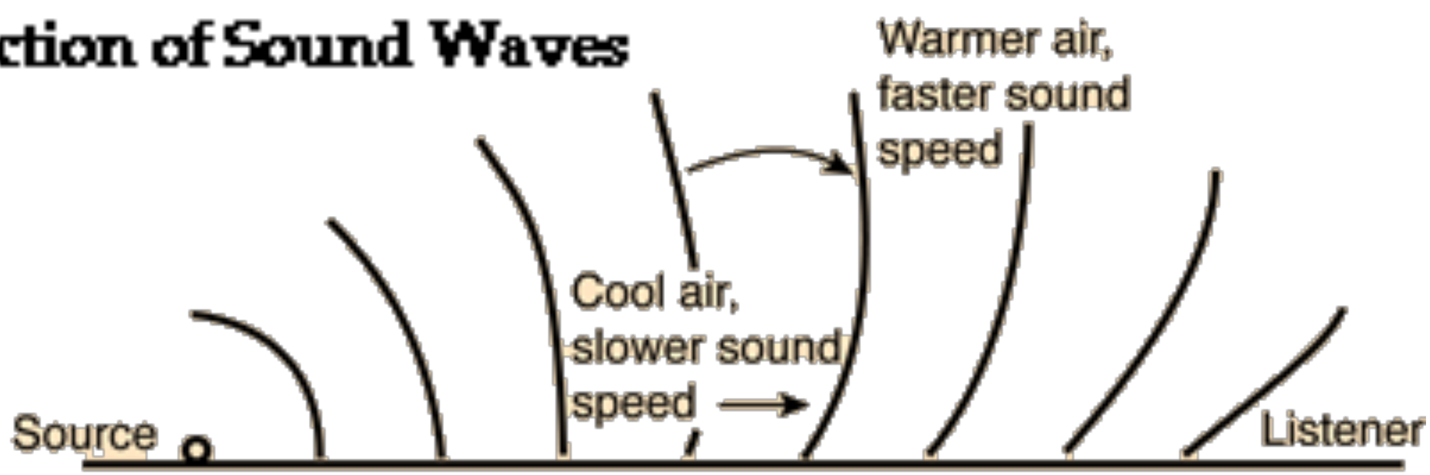
Ray refraction with varying c

- Another classic case: sound rays in air, passing over cool water
- Air near water is cooler, sound speed c(z) decreases with temperature
- Rays always bend downward, *toward* a region of lower c



Refraction of Sound Waves



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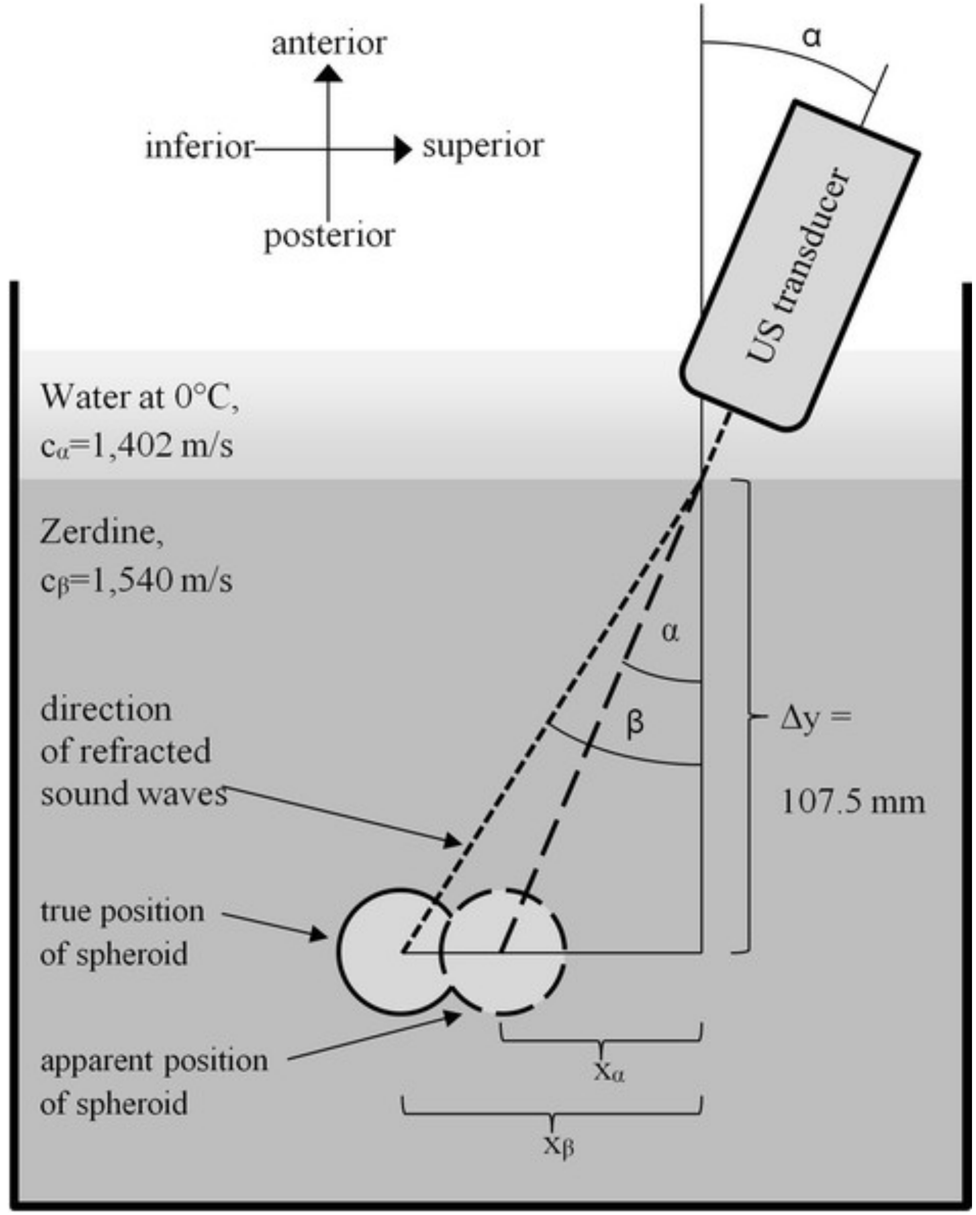
- More on this in a few weeks…

Hazards of ray refraction

Radiation Oncology 10(1):119

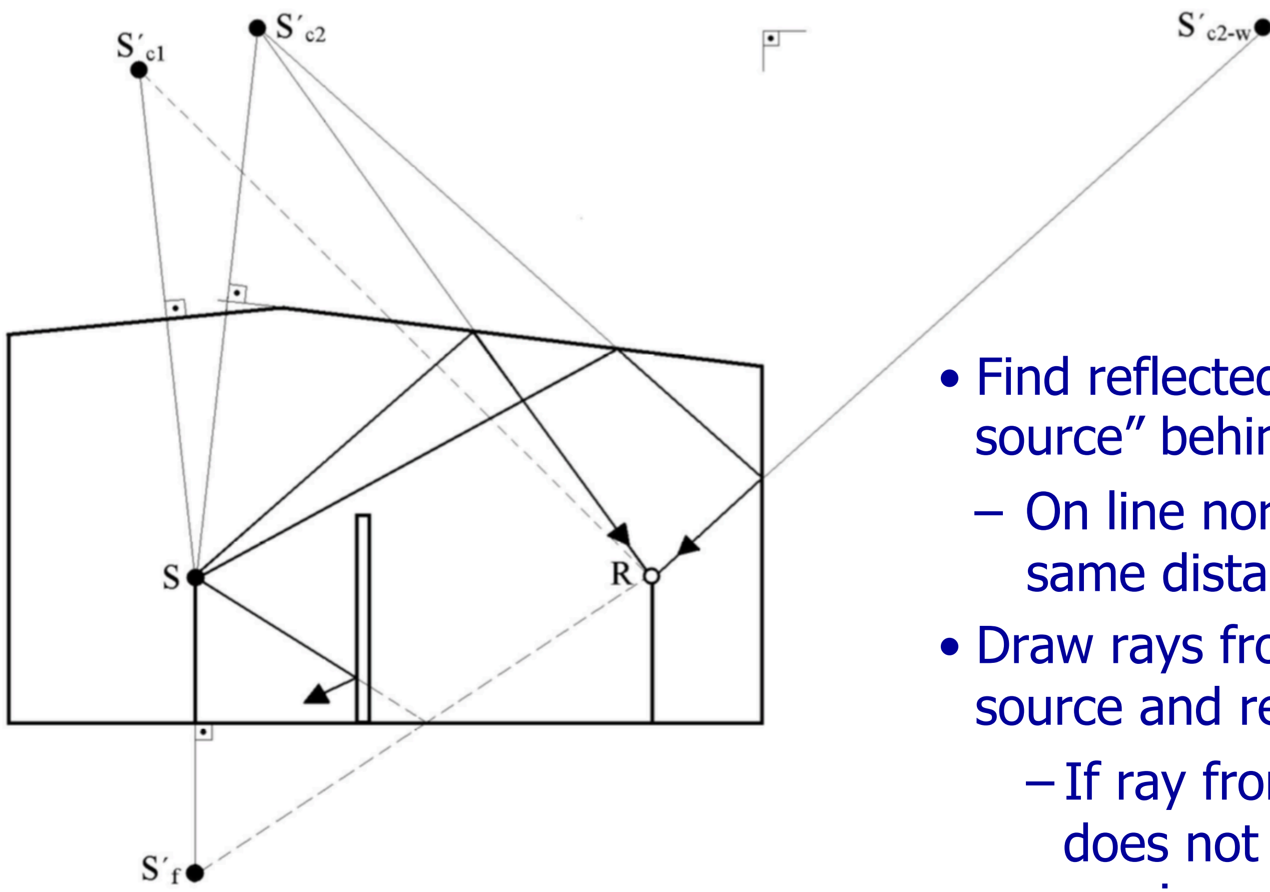
- Ultrasound rays in human body, with varying sound speeds
- Demonstration using Zerdine -= substance used in "phantoms"

  (Phantom = object used in medical imaging calibration and experiments, simulating tissue properties)
- Error in position of target could have serious consequences!



Ray tracing with image-source method

Simple geometric ray tracing tool, from optics of mirrors:



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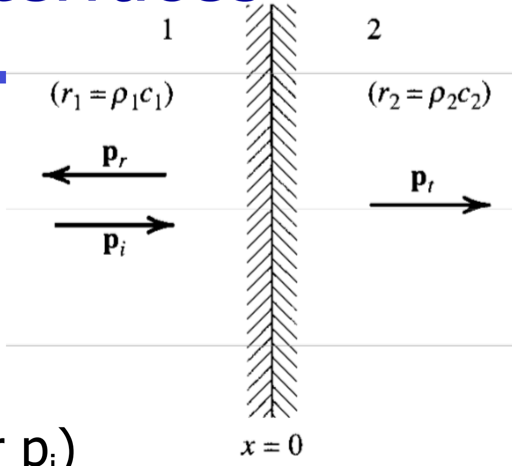
- Find reflected rays via "virtual image source" behind reflector
  - On line normal to surface from source, same distance behind
- Draw rays from image source to real source and receiver
  - If ray from image source to receiver does not intersect same wall, receiver cannot "see" it (dashed line) so ignore
- Secondary images for multi-surface reflection ray paths

Reflection and transmission at interfaces



- Reflections are characterized by reflectance r
  (assume plane waves for now)
  - Reflected intensity $I_R = rI_0$
    $r = R^2$ where
  R=*pressure (amplitude)* reflectance factor: $p_R = Rp_i = \sqrt{r}\, p_i$)
  - Similar coefficients for transmission (T) and absorption (A): $T = p_T \,/\, p_i$
  (Coefficients must be real numbers)
- Pressure reflection coefficients are related to acoustic impedance

Specific impedance $z = p/u$; for plane waves $u = \dfrac{p}{\rho_0 c} \rightarrow \quad z = \rho_0 c$

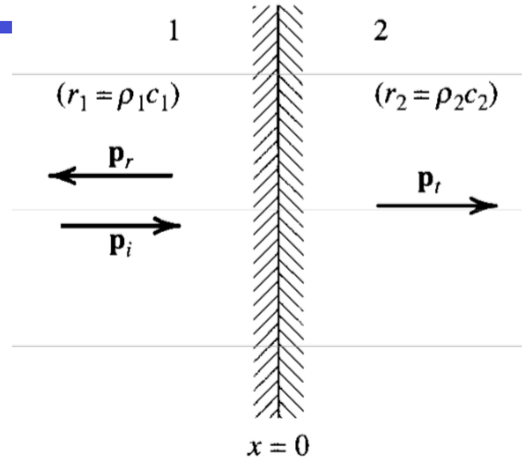
$\rho_0 c =$ Characteristic impedance of medium So $z = \rho c = r + i x \rightarrow \mathrm{Re}(z) = r_i = \rho_i c_i$

Intensity of plane wave is

$$I = \frac{1}{2}\frac{p^2}{\rho c} = \frac{1}{2}\frac{p^2}{r}, \quad so \quad R_I = \frac{I_{refl}}{I_{inc}} = \left(\frac{p_r}{p_i}\right)^2 = |R|^2$$

- Assume for intensity coeffs, T+R+A=1: all energy accounted for
  - If A is small, can assume T+R ~ 1

Reflection and transmission at interfaces



- For a (massless) interface between 2 fluids
  - Must have no net pressure difference at boundary: $p_2=p_1$ (continuity of pressure across boundary)
  - Must have no relative motion at boundary: $u_2=u_1$ (continuity of normal velocity across boundary)

  So at interface,

$$p_i + p_r = p_t \quad \text{and } u_i + u_r = u_t$$

$$\frac{p_i + p_r}{u_i + u_r} = \frac{p_t}{u_t}; \quad \text{specific impedance } z = \frac{p}{u} = \rho_0 c$$

$$\rightarrow z_2 = z_1 \quad \text{continuity of specific impedance in normal direction}$$

…for rays arriving normal to interface

$$\text{for plane waves } \frac{p}{u} = \pm r \text{ (depending on direction of propagation)}$$

$$\rightarrow \frac{p_t}{u_t} = r_2, \ \frac{p_i}{u_i} = +r_1, \text{ and } \frac{p_r}{u_r} = -r_1$$

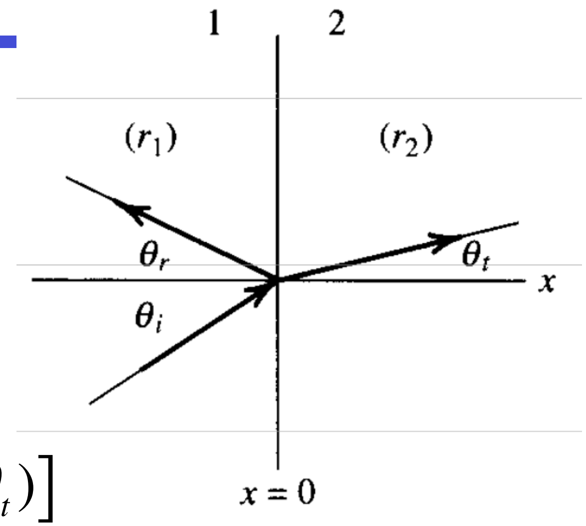
$$\text{So } \frac{p_i + p_r}{u_i + u_r} = \frac{p_t}{u_t} \rightarrow \frac{p_i + p_r}{p_i / r_1 - p_r / r_1} = r_1 \frac{p_i + p_r}{p_i - p_r} = r_2$$

$$r_1 (p_i + p_r) = r_2 (p_i - p_r) \rightarrow (r_1 + r_2) p_r = (r_2 - r_1) p_i$$

$$\rightarrow p_r / p_i = R = \frac{(r_2 - r_1)}{(r_2 + r_1)} = \frac{(1 - r_1 / r_2)}{(1 + r_1 / r_2)}$$

Reflection and transmission at interfaces

- For rays arriving at an angle θ_i to interface:



Plane wave incident with propagation vector $\vec{k}_1$

in the x-y plane: $p_i = Ae^{i(\omega t - \vec{k}\cdot\vec{r})}$

$\rightarrow p_i = P_i \exp\left[i(\omega t - k_1 x\cos\theta_i - k_1 y\sin\theta_i)\right]$

reflected wave is $p_r = P_r \exp\left[i(\omega t + k_1 x\cos\theta_r - k_1 y\sin\theta_r)\right]$

transmitted wave is $p_t = P_t \exp\left[i(\omega t - k_2 x\cos\theta_t - k_2 y\sin\theta_t)\right]$

Continuity of pressure at interface $x = 0$: $P_{y(1)} = P_{y(2)}$

$$P_i \exp(-ik_1 y\sin\theta_i) + P_r \exp(-ik_1 y\sin\theta_r) = P_t \exp(-ik_2 y\sin\theta_t)$$

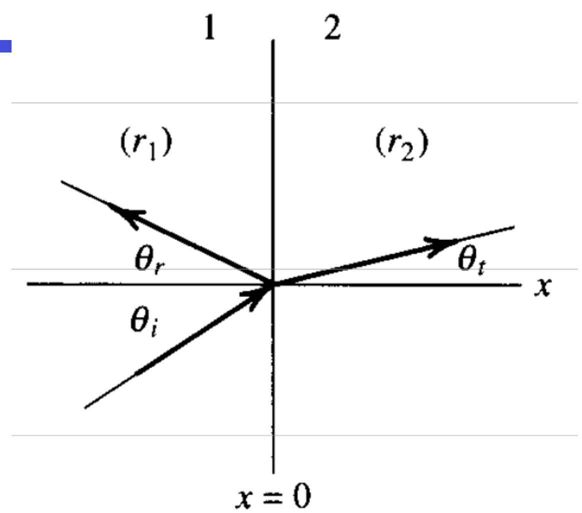
must be true for any y, so must have $k_1 \sin\theta_i = k_1 \sin\theta_r = k_2 \sin\theta_t$

so $\sin\theta_i = \sin\theta_r$ (reflection law) and $\dfrac{\sin\theta_i}{c_1} = \dfrac{\sin\theta_t}{c_2}$ (Snell's law)

With all exponents equal, $P_i + P_r = P_t \rightarrow 1 + R = T$

- We get sines instead of cosines here due to convention of taking angle relative to normal (from optics)
  - Underwater acoustics uses convention of measuring θs from interface surface direction, so they use cosines

Reflections at surfaces



- "It can be shown" (see Kinsler)
  - For reflection at an interface

$$R = \frac{(r_2/\cos\theta_t - r_1/\cos\theta_i)}{(r_2/\cos\theta_t + r_1/\cos\theta_i)} \quad \text{for plane wave,}$$

$r = p/u = z$ (Rayleigh reflection coefficient)

For $c_1 > c_2$, $\theta_t < \theta_i$ and is a real number.

For $c_2 > c_1$ and $\theta_i < \theta_{crit}$, $\theta_t > \theta_i$ and is a real number.

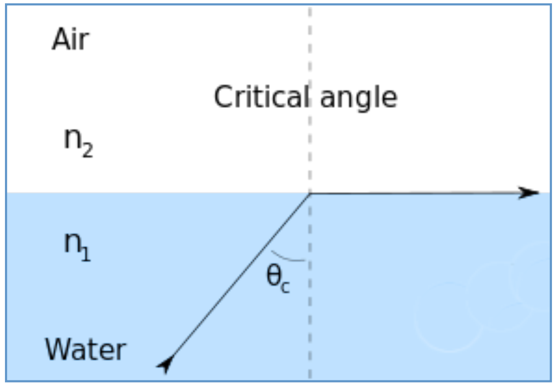
if $\theta_i > \theta_{crit}$, $\sin\theta_t > 1$, so $\cos\theta_t$ is imaginary.

The transmitted pressure is

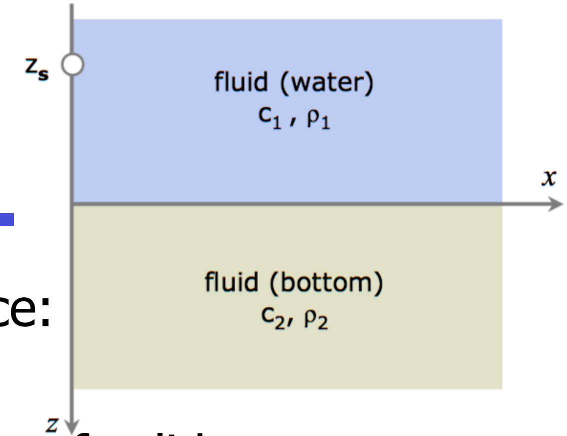
$$p_t = P_t \exp(-\gamma x)\exp\left[i(\omega t - k_1 y \sin\theta_t)\right], \quad \gamma = k_2\sqrt{(c_2/c_1)^2 \sin^2\theta_i - 1}$$

At critical angle, amplitude *decays* in x direction but wave *propagates* in the y direction (parallel to interface)

At critical angle, refracted ray lies in plane of interface:



Reflections at solid(ish) surfaces



- For reflection on a (hard) solid surface, with normal incidence:
  - Ignore transmission: consider only reflected wave
  - Describe behavior in terms of *normal acoustic impedance* of solid

$$z_n = \frac{p}{\vec{u}\cdot\hat{n}} \quad , \quad \hat{n} = \text{ unit vector normal to surface}$$

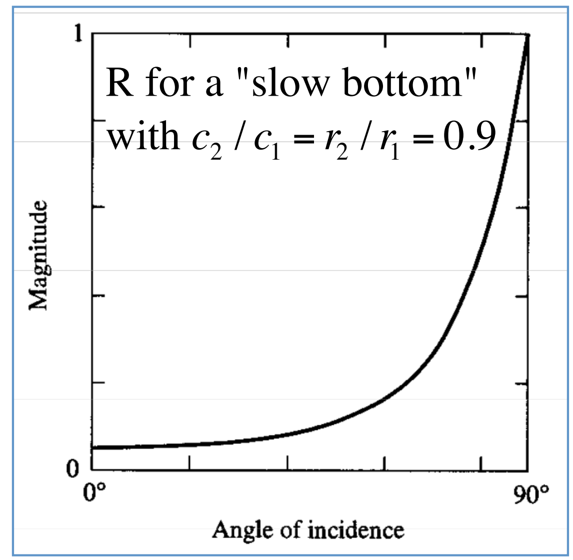
For normal-incidence wave, $\vec{u}\cdot\hat{n} = u_x$ the particle speed

$$z_n = r_n + i\,x_n$$

continuity of p and u at $x = 0$: $p_i + p_r = p_t, \quad u_i + u_r = u_t$

$$\rightarrow z_n = \frac{p_i + p_r}{u_i + u_r} \rightarrow z_n = r_1 \frac{1+R}{1-R} \rightarrow R = \frac{(r_n - r_1) + i\,x_n}{(r_n + r_1) + i\,x_n}$$

For $x_n = 0$, this is same as for normal incidence on a fluid-fluid interface.



  - For a semi-solid surface (eg seabottom), transmitted power might be significant, in both transverse and shear waves – sound speed must include shear modulus:

$$c = \sqrt{B/\rho_0} \rightarrow \sqrt{(B + (4/3)G)/\rho_0}, \quad G = \text{shear modulus}$$