Solutions for Homework set 1 – chapters 1—5 in Kinsler

1. (a) $\mathbf{AB} = AB\exp[j(2\omega t+\theta+\phi)]$ and $\mathrm{Re}\{\mathbf{AB}\} = \underline{AB\cos(2\omega t+\theta+\phi)}$

   (b) $\mathbf{A/B} = (A/B)\exp[j(\theta-\phi)]$ and $\mathrm{Re}\{\mathbf{A/B}\} = \underline{(A/B)\cos(\theta-\phi)}$

   (c) $\mathrm{Re}\{\mathbf{A}\}\mathrm{Re}\{\mathbf{B}\} = \underline{AB\cos(\omega t+\theta)\cos(\omega t+\phi)}$

   (d) From (a), phase $= \underline{2\omega t+\theta+\phi}$

   (e) From (b), phase $= \underline{\theta-\phi}$

2. $m = 0.5$ kg $\quad m' = 0.2$ kg $\quad \tau = 1$ s $\quad g = 9.8$ m/s^2 $\quad \Delta x = 0.04$ m

$$\Delta F = s'\Delta x = m'g$$

$$s = 0.2\cdot 9.8/0.04 = 49 \text{ N/m}$$

$$\omega_0 = \sqrt{s/m} = \sqrt{49/0.5} = 9.90 \text{ rad/s}$$

$$\beta = 1/\tau = 1 \text{ s}^{-1}$$

$$R_m = 2m\beta = \underline{1.0 \text{ kg/s} \quad \text{or} \quad 1.0 \text{ N}\cdot\text{s/m}}$$

$$\omega_d = \sqrt{\omega_0^2-\beta^2} = \sqrt{98-1} = \underline{9.85 \text{ rad/s}}$$ or, alternatively,

$$\omega_d = \omega_0\sqrt{1-(\beta/\omega)^2} \approx \omega_0\left[1-\tfrac{1}{2}(\beta/\omega)^2\right] \approx 9.90[1-1/(2\cdot 9.8)]$$

$$\approx 9.90(1-0.05) \approx \underline{9.85 \text{ rad/s}}$$

$$x = Ae^{-\beta t}\cos(\omega_d t+\phi)$$

$$\dot{x} = -A\omega_d e^{-\beta t}\sin(\omega_d t+\phi) - A\beta e^{-\beta t}\cos(\omega_d t+\phi)$$

$$x(0) = A\cos\phi = 0.04$$

$$\dot{x}(0) = -A\omega_d\sin\phi - \beta A\cos\phi = 0$$

$$\therefore \tan\phi = -\beta/\omega_d = -0.1015$$

$$\phi = \underline{-5.8^\circ}$$

$$A = 0.04/\cos\phi = \underline{0.0402 \text{ m}}$$

3. $f = F\sin\omega_0 t \quad t \geq 0$ and $\beta \ll \omega_0$

$\mathbf{f} = -jF e^{j\omega_0 t}$

In this approximation $\omega_d = \omega_0\sqrt{1-(\beta/\omega_0)^2} \approx \omega_0 - \frac{1}{2}\beta^2/\omega_0$

The steady state (homogeneous) solution is $\mathbf{u}_h = \mathbf{f}/\mathbf{Z}_m = j\,\omega_0\mathbf{x}_h$ whence

$\mathbf{x}_h = \dfrac{1}{j\omega_0}\dfrac{-jF e^{j\omega_0 t}}{\mathbf{Z}_m} = -\dfrac{F}{\omega_0\mathbf{Z}_m}e^{j\omega_0 t}$ But $\mathbf{Z}_m = R_m$ for $\omega = \omega_0$ so

$$x_h = -\frac{F}{\omega_0 R_m}\cos\omega_0 t$$

Substituting into (1.8.1) yields the complete solution

$x = A e^{-\beta t}\cos(\omega_d t + \phi) - \dfrac{F}{\omega_0 R_m}\cos\omega_0 t$ with constants to be determined

Applying the initial conditions $x(0) = 0$ and $\left.\dfrac{dx}{dt}\right|_{t=0} = 0$ gives

$$A\cos\phi - \frac{F}{\omega_0 R_m} = 0$$

$$-\beta A\cos\phi - \omega_d A\sin\phi = 0$$

From the second equation $\dfrac{\sin\phi}{\cos\phi} = -\dfrac{\beta}{\omega_d}$ but $\dfrac{\beta}{\omega_d} \ll 1$ so that $\phi \approx -\dfrac{\beta}{\omega_d}$

and then the first gives $A \approx \dfrac{F}{\omega_0 R_m\cos(\beta/\omega_d)} \approx \dfrac{F}{\omega_0 R_m}$

Substitution into the complete solution gives

$$x \approx \frac{F}{\omega_0 R_m}\left[e^{-\beta t}\cos(\omega_d t - \beta/\omega_d) - \cos\omega_0 t\right]$$

$$\approx \frac{F}{\omega_0 R_m}\left[e^{-\beta t}\left(\cos\omega_d t + \frac{\beta}{\omega_d}\sin\omega_d t\right) - \cos\omega_0 t\right]$$

Now for $\omega_0 t < 1$ with $\beta/\omega_0 \ll 1$ this becomes

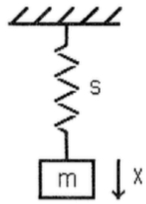
$$x \approx \frac{F}{\omega_0 R_m}(e^{-\beta t} - 1)\cos\omega_0 t$$

and for $\omega_0 t \geq 1$ the exp($-\beta t$) makes the first term in the complete solution negligibly small, so the above approximation is valid for all t.

5.

$$\mathbf{U}(t) = \int_{-\infty}^{\infty} \frac{\delta(w-\omega)}{\mathbf{Z}(w)} F e^{jwt}\, dw = \underline{\frac{F}{\mathbf{Z}} e^{j\omega t}}$$

6.



f $= mg \cdot 1(t)$

$\mathbf{Z}(w) = j(wm - w/s)$

$\omega_0 = (s/m)^{1/2}$

With the help of Table 1.15.1, we have

$$\mathbf{G}(w) = \frac{mg}{2\pi}\frac{1}{jw}$$

Then, again usingTable 1.15.1

$$\mathbf{U}(t) = \int_{-\infty}^{\infty} \frac{mg}{2\pi}\frac{1}{jw}\frac{e^{jwt}}{j(wm - s/w)}\, dt = \frac{-mg}{2\pi m}\int_{-\infty}^{\infty} \frac{e^{jwt}}{w^2 - \omega_0^2}\, dt$$

$$= g\int_{-\infty}^{\infty} \frac{1}{2\pi}\frac{1}{\omega_0^2 - w^2} e^{jwt}\, dt = \frac{g}{\omega_0}\sin\omega_0 t \cdot 1(t)$$

and integration over time yields the displacement

$$X(t) = \int_0^t U(t)\, dt = \frac{g}{\omega_0}\frac{-1}{\omega_0}\cos\omega_0 t\Big|_0^t = \underline{\frac{mg}{s}(1 - \cos\omega_0 t)}$$

8. $y = 4\cos(3t - 2x) \qquad \rho_L = 0.1$ g/cm

(a) $A = \underline{4 \text{ cm}} \qquad c = \omega/k = 3/2 = \underline{1.5 \text{ cm}} \qquad f = \omega/2\pi = 3/2\pi = \underline{0.48 \text{ Hz}}$

$\lambda = 2\pi/k = \pi = \underline{3.1 \text{ cm}} \qquad k = \underline{2 \text{ cm}^{-1}}$

(b) $u = \partial y/\partial t = -12\sin(3t - 2x)$ so $u(0,0) = \underline{0 \text{ m/s}}$

9. $m_L = 0.2$ kg, $m_s = 0.05$ kg, $L = 1.0$ m, so that $\rho_L = 0.05$ kg/m

(a) $c^2 = T/\rho_L = 0.2 \cdot 9.8/0.05 = 39.2$ so $c = \underline{6.26 \text{ m/s}}$

(b) $$\cot kL = \frac{m_L}{m_s} kL = \frac{0.2}{0.05 \cdot 1} kL = 4kL$$

By trial and error, or tables, $kL = 0.48$ so $f = 0.48c/2\pi L = \underline{0.48 \text{ Hz}}$

The first overtone has $kL = 3.219$ so $f = \underline{3.21 \text{ Hz}}$

(c) $\mathbf{y} = A \sin kL\, e^{j\omega t}$ from which we obtain $\dfrac{A}{|\mathbf{y}(L)|} = \dfrac{1}{|\sin 3.219|} = \underline{12.9}$

10. $y = 2\sin(x/5)\cos 3t$ with all units in cgs

(a) $c = \omega/k = 3/(1/5) = \underline{15 \text{ cm/s}}$

$f = 3/(2\pi) = \underline{0.48 \text{ Hz}}$

$k = 1/5 = \underline{0.2 \text{ cm}^{-1}}$

(b) $x = L/2 = 31.4/2 \approx 5\pi$

$y = 2\sin\pi\cos\omega t = 0$

amplitude $= \underline{0 \text{ cm}}$ speed $= \underline{0 \text{ cm/s}}$

$x = L/4 = 31.4/4 \approx 5\pi/2$

$y = 2\cos 3t = 0$

amplitude $= \underline{2 \text{ cm}}$ speed $= \underline{6 \text{ cm/s}}$

(c)
$$\frac{dE}{dx} = \frac{1}{2}\rho_L\left[c^2\left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial t}\right)^2\right]$$

$$= \frac{1}{2}\rho_L\left(\frac{225 \cdot 4}{25}\cos^2\frac{x}{5}\cos^2 3t + 36\sin^2\frac{x}{5}\sin^2 3t\right)$$

$$= 1.8\left(\cos^2\frac{x}{5}\cos^2 3t + \sin^2\frac{x}{5}\sin^2 3t\right)$$

at $x = L/2 = 5\pi \qquad \dfrac{dE}{dx} = \underline{-1.8\cos^2 3t}$

at $x = L/4 = 5\pi/2 \qquad \dfrac{dE}{dx} = \underline{1.8\sin^2 3t}$

10. cont'd (*d*) Since the total energy remains constant, we can choose any time for the integration over x. For convenience, choose $3t = \pi/2$.

$$E = 1.8\int_0^L \sin^2\frac{x}{5}\,dx = 1.8\int_0^{10\pi} \sin^2\frac{x}{5}\,dx$$

$$= 1.8\cdot 5\left[\frac{1}{2}\frac{x}{5} - \frac{1}{4}\sin\frac{2x}{5}\right]_0^{10\pi} = 9\pi$$

$$= 28.3 \text{ ergs} = \underline{2.83\times 10^{-6}\text{ J}}$$

11.

$S = 1\times 10^{-4}\text{ m}^2$ free at $x = 0$

$L = 0.25$ m loaded with 0.15 kg at L

$m = 0.15$ kg

$m_b = 7700\times 10^{-4}\cdot 0.25 = 0.1925$ kg

(*a*)

$$\tan kL = -\frac{m}{m_b}kL = -0.78\,kL \quad \text{so that} \quad kL = 2.12 = \frac{2\pi f}{c}L$$

$$f = 2.12\frac{5050}{2\pi\cdot 0.25} = \underline{6.8\text{ kHz}}$$

(*b*) $\xi = 2A\cos kx\, e^{j\omega t}$ and clamp the bar where $\cos kx = 0$

$$kx = \frac{\pi}{2} \qquad x = \frac{\pi\cdot 0.25}{2\cdot 2.12} = \underline{0.185\text{ m}}$$

(*c*)

$$\frac{\cos 0}{\cos kL} = \frac{1}{\cos 2.12} = \underline{1.91}$$

(*d*) $\tan kL = -0.78\,kL \qquad kL = 4.965 \qquad f = 6800\dfrac{4.965}{2.12} = \underline{15.9\text{ kHz}}$

Alternate 11. *A steel bar of cross section 0.0001m2 and 0.25m length is clamped at both ends. a) what is its fundamental frequency for longitudinal vibrations? b) what is the fundamental frequency for the same bar but free at both ends?*

c=sqrt(y/rho); for common steel we can take Y=200 GPa = 2x10^11 Pa, density=7900 kg/m^3, so c = sqrt(2x10^11/7900)=5030 m/s.
For longitudinal waves, cross-section does not matter as long as bar is "slender" (L>>diameter) as in this case. For ends clamped,
f_1 = c / 2L = 5030 m/s / 0.5 m= 10 kHz
for ends free, result is same - difference is that clamped ends must be nodes, free ends must be anti-nodes, but fundamental f has L = lambda/2.

12. $\omega c\kappa$ = [rad/s][m/s][m] = $[\text{m}]^2/[\text{s}]^2$ so $\sqrt{\omega c\kappa}$ has the dimensions of speed

$v = \sqrt{\omega c \kappa}$ and if $v = c$ then $\omega = \dfrac{c}{\kappa}$

A circular rod of radius a has $\kappa = a/2$ so that $f = c/\pi a$ and for this aluminum rod $f = \dfrac{5.15\times10^3}{\pi\cdot 0.005} = \underline{328\ \text{kHz}}$

13. Let $L_x = a$ and $L_y = 2a$ Then $f_{nm} = \dfrac{c}{2}\sqrt{\left(\dfrac{n}{a}\right)^2 + \left(\dfrac{m}{2a}\right)^2} = \dfrac{c}{2a}\sqrt{n^2 + \dfrac{m^2}{4}}$

$f_{11} = \dfrac{c}{2a}\sqrt{1.25}$

$f_{11}/f_{11} = \sqrt{1.25/1.25} = 1.000$ $\quad f_{21}/f_{11} = \sqrt{4.25/1.25} = 1.844$

$f_{12}/f_{11} = \sqrt{2/1.25} = 1.265$ $\quad f_{14}/f_{11} = \sqrt{5/1.25} = 2.000$

$f_{13}/f_{11} = \sqrt{3.25/1.25} = 1.612$ $\quad f_{22}/f11 = \sqrt{5/1.25} = 2.000$

14. $a = 2.0$ cm $\quad d = 0.02$ cm $\quad \rho = 7700\ \text{kg/m}^3$

$Y = 19.5\times10^{10}\ \text{N/m}^2 \quad \sigma = 0.28$

(*a*)

$$f_1 = 0.47\frac{d}{a^2}\sqrt{\frac{Y}{\rho(1-\sigma^2)}} = 0.47\frac{2\times10^{-4}}{0.02^2}\sqrt{\frac{19.5\times10^{10}}{7700\cdot(1-0.28^2)}} = \underline{1230\ \text{kHz}}$$

(*b*) If $d = 0.04$ cm then $\underline{f_1 = 2460\ \text{Hz}}$

(*c*) If $a = 4$ cm then $\underline{f_1 = 307\ \text{Hz}}$

15. $\mathbf{p} = \dfrac{A}{r}\cos kr\, e^{j\omega t}$ and so $\mathbf{\Phi} = -\dfrac{A}{j\omega\rho_0 r}\cos kr\, e^{j\omega t}$

(*a*) $\mathbf{u} = \nabla\mathbf{\Phi} = \dfrac{\partial\mathbf{\Phi}}{\partial r} = \underline{-j\dfrac{A}{\rho_0 c r}\left[\sin kr + \dfrac{\cos kr}{kr}\right]e^{j\omega t}}$

(*b*)

$$\mathbf{z} = \frac{\mathbf{p}}{\mathbf{u}} = \underline{j\rho_0 c\frac{\cos kr}{\left[\sin kr + \dfrac{\cos kr}{kr}\right]}}$$

(*c*)

$$I(t) = pu = \frac{A}{r}\cos kr\cos\omega t\,\frac{A}{r}\frac{1}{\rho_0 c}\left[\sin kr + \frac{\cos kr}{kr}\right]\sin\omega t$$

$$= \underline{\left(\frac{A}{r}\right)^2\frac{1}{\rho_0 c}\cos kr\left[\sin kr + \frac{\cos kr}{kr}\right]\cos\omega t\sin\omega t}$$

(*d*) Because $\dfrac{1}{T}\displaystyle\int_0^T \cos\omega t\sin\omega t\, dt = 0$ we have $\langle I(t)\rangle_T = \underline{I = 0}$

16. $P = 2$ Pa $\quad f = 100$ Hz $\quad$ in air

(*a*) $I = \dfrac{P^2}{2\rho_0 c} = \dfrac{2^2}{2 \cdot 415} = \underline{4.8 \times 10^{-3}\ \text{W/m}^2}$

$IL = 10\log(4.8 \times 10^{-3}/10^{-12}) = \underline{96.8\ \text{dB}\ re\ 10^{-12}\ \text{W/m}^2}$

(*b*) $U/\omega = P/(\rho_0 c\,\omega) = 2/(415 \cdot 2\pi \cdot 100) = \underline{7.7 \times 10^{-6}\ \text{m}}$

(*c*) $U = P/(\rho_0 c) = 2/415 = \underline{4.82 \times 10^{-3}\ \text{m/s}}$

(*d*) $P_e = P/\sqrt{2} = \underline{1.41\ \text{Pa}} = 14.1\ \mu\text{bar}$

(*e*) $SPL = 20\log(14.1/0.0002) = \underline{97\ \text{dB}\ re\ 20\ \mu\text{bar}}$

17. (*a*) $\mathcal{ML} = -80$ dB *re* 1 V/μbar $= 20\log[\mathcal{M}/(1\ \text{V}/\mu\text{bar})]$

$\mathcal{M} = 10^{-4}\ \text{V}/\mu\text{bar} = 10^{-9}\ \text{V}/\mu\text{Pa}$

$\mathcal{ML} = 20\log[10^{-9}/(1\ \text{V}/\mu\text{Pa})] = \underline{-180\ \text{dB}\ re\ 1\ \text{V}/\mu\text{Pa}}$

(*b*) $SPL = 80$ dB *re* 1 μbar $= 20\log(P/1\ \mu\text{bar})$

$P = 10^4\ \mu\text{bar}$

$\mathcal{ML} = -80 = 20\log[(V/10^4)/(1\ \text{V}/\mu\text{bar})] = 20\log(V/10^4)$

$V = \underline{1\ \text{V}}$